Consistent Vertex-Signed Graphs

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Outline

1. Edge-Signed and Vertex-Signed Graphs
2. Balance and Switching
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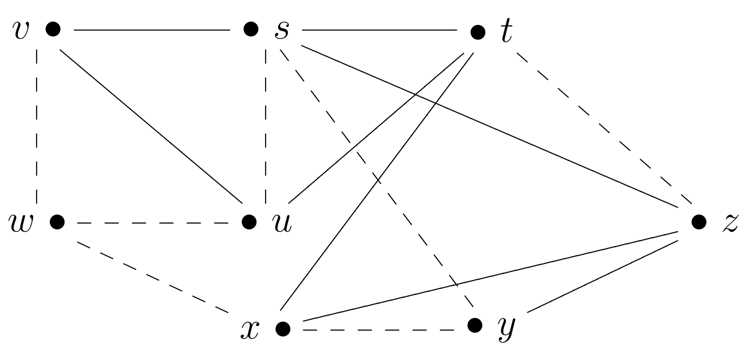
5. Consistency in the Line Graph of a Signed Graph

1. Edge-Signed and Vertex-Signed Graphs

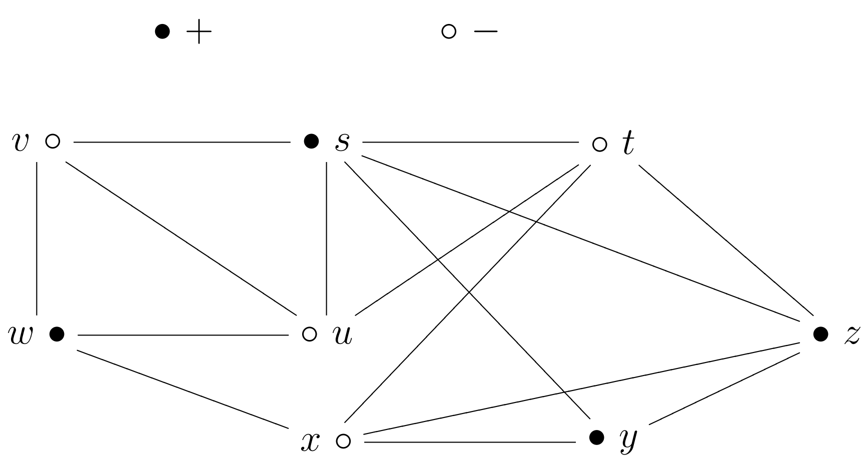
$\Sigma = (\Gamma, \sigma)$ is an **(edge-)signed graph**:

$\Gamma = (V, E)$ is the underlying graph: vertex set V, edge set E.
$\sigma : E \to \{+, -\}$ is the edge signature.



$\mathrm{M} = (\Gamma, \mu)$ is a **vertex-signed** or *marked graph*:

$\Gamma = (V, E)$ is the underlying graph: vertex set V, edge set E.
$\mu : V \to \{+, -\}$ is the vertex signature.



2. Balance and Switching

In Σ the sign of a circle C is $\sigma(C) :=$ product of edge signs.
Σ is **balanced** if every circle is positive.

Switching:

Switching function $\zeta : V \to \{+,-\}$: $\Sigma^\zeta := (|\Sigma|, \sigma^\zeta)$ defined by

$$\sigma^\zeta(vw) := \zeta(v)\sigma(vw)\zeta(w).$$

Switching set $X \subseteq V$: $\Sigma^X := (|\Sigma|, \sigma^X)$ defined by

$$\sigma^X(vw) := \begin{cases} \sigma(vw) \text{ if } v, w \in X \text{ or } v, w \notin X, \\ -\sigma(vw) \text{ if } v \in X,\ w \notin X \text{ or } v \notin X,\ w \in X. \end{cases}$$

Theorem 2.1. *The following statements are equivalent:*

(i) Σ *is balanced.*
(ii) (Harary's Balance Theorem) $V = V_1 \cup V_2$ *where* V_1, V_2 *are disjoint, and every positive edge is within* V_1 *or* V_2 *while every negative edge has one endpoint in each.*
(iii) Σ *switches to an all-positive signature.*

Proof. (ii) $\implies$ (i): The negative edges form a cut, so every circle has an even number of negative edges.

(iii) $\implies$ (ii): If Σ^X is all positive, let $V_1 = X$ and $V_2 = V \setminus X$.

(i) $\implies$ (iii): Choose a spanning tree T and a root r. Define $\zeta(v) := \sigma(T_{rv})$. In Σ^ζ, T is all positive, so there is a negative circle in $\Sigma \iff$ there is a negative edge in Σ^ζ. □

Algorithm to Detect Balance:

(1) Choose T and r, and construct ζ.
(2) Switch to Σ^ζ.
(3) Check the sign of each edge, looking for negative edges.

Complexity: Let $n := |V|$. Time: n^2 (fast).

(1) Find T. Time n^2(?)
(2) Choose r. Time n^0.
(3) Construct ζ. Time n^1.
(4) Switch. Time n^1.
(5) Find a negative edge, if one exists. Time $O(n^2)$.

3. CONSISTENCY

In M the sign of a circle C is $\mu(C) :=$ product of vertex signs.
M is **consistent** if every circle is positive.

Problem (Beineke and Harary 1978). Characterize consistent vertex-signed graphs.

Problem. Algorithm to Detect Balance?

The history, in order of increasing strength:

(1) Beineke & Harary (1974).
Pose the problem. The first step towards understanding.

(2) B.D. Acharya (1983–84).
Converts the problem to a complicated question about an associated signed graph.

(3) S.B. Rao (<1984).
A complicated solution with a polynomial-time algorithm.

(4) Cornelis Hoede (1992).
A moderately complicated solution with a simpler polynomial-time algorithm.

(5) T. Zaslavsky (2010) ??
A perhaps simple description based on Tutte's 3-decomposition theorem, with a perhaps not-so-simple polynomial-time algorithm.

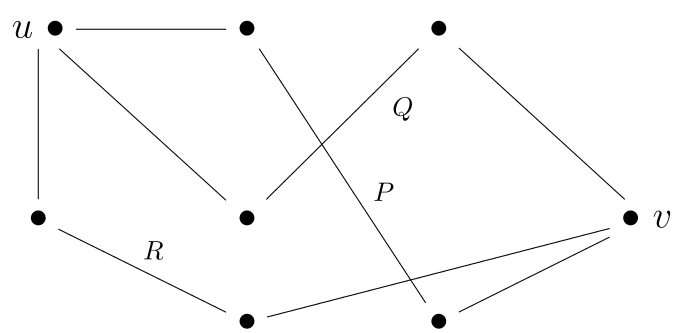
Step 1.

Beineke and Harary observed:

Lemma 3.1. *If two vertices have 3 internally disjoint paths, they must have the same sign.*

Proof.

$$+ = \mu(uPvQu) = \mu(u)\mu(v)\mu(P^\circ)\mu(Q^\circ),$$
$$+ = \mu(uPvRu) = \mu(u)\mu(v)\mu(P^\circ)\mu(R^\circ),$$
$$+ = \mu(uQvRu) = \mu(u)\mu(v)\mu(Q^\circ)\mu(R^\circ),$$



Multiply:

$$+ = \big[\mu(u)\mu(v)\big]^3 \ \mu(P^\circ)\mu(Q^\circ) \ \mu(P^\circ)\mu(R^\circ) \ \mu(Q^\circ)\mu(R^\circ) = \mu(u)\mu(v).$$

□

Corollary 3.2. *If* M *is consistent and* 3*-connected, then all vertices have the same sign.*

Step 2.

B.D. Acharya re-encoded the problem as a problem of edge-signed graphs.

$$\mathrm{M} = (\Gamma, \mu) \mapsto \Sigma.$$

Theorem 3.3. M *is consistent iff every circle in* Σ *is all negative or has an even number of (nontrivial) all-negative components.*

Equivalently, the circle is all positive or all negative or is made up of an even number of all-negative paths and an even number of all-negative paths.

Since I do not have the paper and I don't know the construction, I will have to leave out all the details.

This test for consistency is not simple and does not seem to yield a polynomial-time algorithm.

Step 3.

S.B. Rao, independently, analysed the problem carefully and came up with a complicated algorithm for testing for consistency, that takes polynomial time.

Since I do not have the paper and I don't know the construction, and neither does S.B. Rao, I will leave out all the details.

Step 4.

Cornelis Hoede found a relatively simple test.

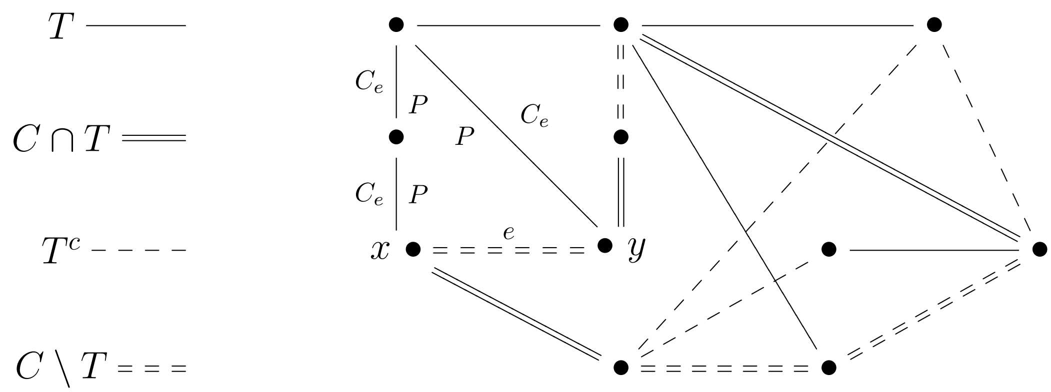
Theorem 3.4. *Take a spanning tree* T *in* $\mathrm{M} = (\Gamma, \mu)$. M *is consistent iff*

(1) *every fundamental cycle of* T *is positive, and*
(2) *whenever two fundamental cycles intersect in a path, the endpoints of that path have the same sign.*

Proof. Necessity is obvious from Lemma 3.1.

Sufficiency is by induction on $k = |C \setminus T|$.

$k = 1$ is by assumption (1).



$k > 1$: $C' := C \triangle C_e$ where $e \in C \setminus T$. $C' \cap C_e$ is a path $P{:}xy$. Let $\delta = \mu(x) = \mu(y)$ by assumption (2). Then $\mu(C) = \mu(C')\mu(C_e)\mu(x)\mu(y) = (+)(+)\delta\delta = +$, by induction and assumption (1). □

Algorithm to detect consistency:

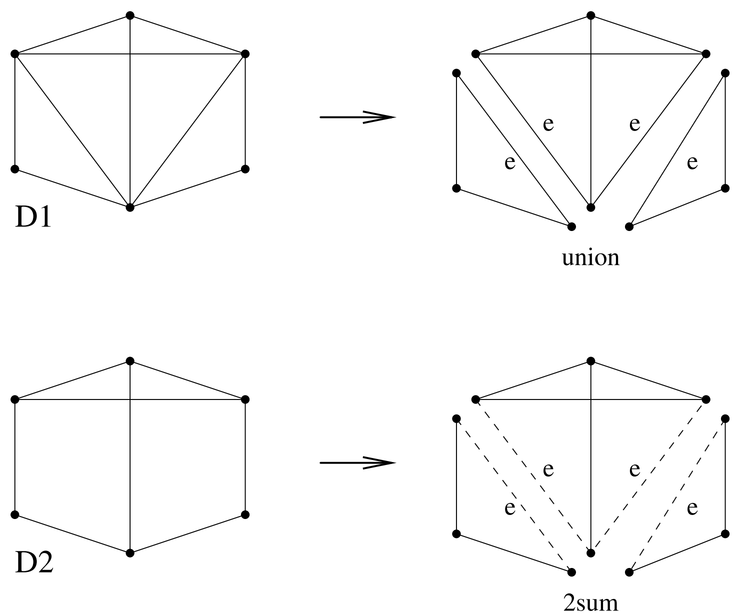
(1) Find a spanning tree T. (Time: n^2, where $n := |V|$.)
(2) For each chord of T (time n^2), find the fundamental cycle (time n^2?) and compute its sign (time n).
(3) For each pair of fundamental cycles (time n^2 if the cycles are remembered), find their common path if any (time n). Compare the endpoint signs (time n^0).
(4) If everything worked out, M is consistent.
If not, the procedure terminates early, saving time to compensate for the disappointment of inconsistency.

The total time is $O(n^2 + n^5 + n^3)$, which is n^5. (The actual time may be much less. The provable time may also be much less; this is a sloppy analysis.)

Step 5.

The night before last, T. Zaslavsky thought he had found the gold at the end of the rainbow. Is it true?

Theorem 3.5. $\mathrm{M} = (\Gamma, \mu)$ *is consistent iff, in the Tutte* 3*-decomposition of* Γ*,* μ *is constant on every* 3*-connected* 2*-block and every multiple-edge* 2*-block that has at least* 3 *edges; every* 3*-connected* 2*-block that has negative* μ *is bipartite; and every circular* 2*-block is positive.*



Proof. For Tutte's 3-decomposition, look at the lower graph in the figure.

Lemma 3.1 for a 3-connected 2-block.

The definition for a circular 2-block.

Trivial for a multiple-edge 2-block that is not a circle (not illustrated). □

Algorithm:

(1) Find the Tutte 3-decomposition. (Time: n^3 or n^4.)
(2) Compare the vertex signs in each 3-connected and multiple-edge 2-block. (Time: n.)
(3) Multiply the signs on each circular 2-block. (Time: n.)

Overall time: n^3 or n^4.

(There should be a careful analysis of the algorithmic complexity of the 3-decomposition, but I don't know of any.)

4. Vertex-Signed Digraphs

Beineke and Harary considered directed graphs with vertex signs.

(D, μ) is **consistent** if no two directed walks from v to w have opposite signs.

We can see how to approach the problem:

Give each arc the tail sign $\sigma(\vec{vw}) := \mu(v)$.
For a directed walk $W = v_0 v_1 \cdots v_l$, $\mu(W) = \mu(v_l)\sigma(W)$.
A second directed walk W' has $\mu(W') = \mu(W) \iff \sigma(W') = \sigma(W)$.
Determine directed balance in (D, σ) if you can.

Directed balance means any two directed walks from v to w in (D, σ) have the same sign.

Theorem 4.1. *If D is strongly connected, then (D, μ) is consistent iff (D, σ) is balanced.*

And then apply Harary's Balance Theorem.

5. Consistency in the Line Graph of a Signed Graph

$\Sigma = (\Gamma, \sigma)$ with $\sigma : E \to \{+, -\}$.

Then
$L(\Sigma) := (L(\Gamma), \sigma)$ has vertex signs ($\because V(L(\Gamma)) = E(\Gamma)$).

Question (Acharya, Acharya, & Sinha 2009).
For which Σ is $(L(\Gamma), \sigma)$ consistent?

Theorem 5.1 (Acharya, Acharya, & Sinha 2009)**.** *$L(\Sigma)$ is consistent $\iff$ Σ is balanced, every vertex of degree $d(v) > 3$ is totally positive, and each vertex of degree $d(v) = 3$ is either totally positive or has two negative edges which belong to all circles through the vertex.*

A 'totally positive vertex' has no negative edges.

Two Proofs. Necessity: Easy, by considering vertex triangles.
Sufficiency:

Acharya, Acharya, & Sinha 2009:
Somewhat long; based on Hoede's Theorem 3.4.

X & Zaslavsky 20xx: Shorter.
Notice that 'which belong to all circles through the vertex' means the one positive edge must be an isthmus. Thus, begin by deleting all positive isthmi (yes, it's okay); this is Σ'.

The negative subgraph of Σ has maximum degree ≤ 2, $\therefore$ is a disjoint union of paths and circles. A circle is a component of Σ'; easy. A path is connected to positive edges only at its endpoints, if at all, and only to one positive edge (due to vertex triangles). Thus, any circle in $L(\Sigma)$ that has a negative vertex (a negative edge in Σ) contains an entire negative path. By checking, the circle corresponds to a closed walk in Σ, which is positive due to balance. □

Algorithm:
Check for balance (time n^2), check vertices (time n^2). Total time: $O(n^2)$.

Construction:

(1) Start with balanced Σ_0.
(2) An edge $e \mapsto fPf'$ (optionally) where f, f' are positive and P is all negative and $\sigma(P) = \sigma(e)$.
(3) Positive loop e (if not expanded in step (2)) $\mapsto C$, an all-negative circle.
(4) Some trimming of positive edges.

Theorem 5.2 (X & Zaslavsky 20xx)**.** *This construction produces all signed graphs whose line graphs are consistent, and no other signed graphs.*

Proof. Too messy for words. See the paper. □

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