

3 **CHARACTERIZATION OF LINE-CONSISTENT SIGNED**  
4 **GRAPHS**

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14 **Abstract**

15 The line graph of a graph with signed edges carries vertex signs. A  
16 vertex-signed graph is *consistent* if every circle (cycle, circuit) has positive  
17 vertex-sign product. Acharya, Acharya, and Sinha recently characterized  
18 *line-consistent* signed graphs, i.e., edge-signed graphs whose line graphs,  
19 with the naturally induced vertex signature, are consistent. Their proof ap-  
20 plies Hoede’s relatively difficult characterization of consistent vertex-signed  
21 graphs. We give a simple proof that does not depend on Hoede’s theorem  
22 as well as a structural description of line-consistent signed graphs.

23 **Keywords:** line-consistent signed graph, line graph, consistent vertex-signed  
24 graph, consistent marked graph..

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26 05C76.

27 In the first article on signed graphs—graphs whose edges are labelled positive  
28 or negative—Harary [8] gave a simple characterization of those in which the  
29 product of signs around every circle (i.e., circuit, cycle) is positive. (Such graphs  
30 are called *balanced*.) Later, Beineke and Harary [7] introduced signed vertices  
31 and asked the analogous question of characterizing the vertex-signed graphs (also  
32 called *marked graphs*) in which the product of vertex signs around every circle  
33 is positive. (These vertex-signed graphs are called *consistent*.) This was more

34 difficult. After preliminary results by Acharya [1, 2] and Rao [11], Hoede found  
 35 a definitive answer [9], which was developed more deeply in [12]. (An even more  
 36 definitive answer was found subsequently; see [10].)

37 An obvious question was never answered until recently. If a signed graph  $\Sigma$   
 38 has underlying graph  $\Gamma$  and edge signature  $\sigma$ , then the line graph  $L(\Gamma)$  has  $\sigma$   
 39 as a vertex signature. Under what conditions is this vertex signature consistent?  
 40 We call such a graph *line consistent*, as the defining property is consistency of  
 41 the line graph. Acharya, Acharya, and Sinha [4] found a simple necessary and  
 42 sufficient condition for line consistency. The necessity of their condition is easy  
 43 to determine. Sufficiency is not so easy; it depends on Hoede’s relatively complex  
 44 consistency criterion. Here we give an elementary, short proof of sufficiency as  
 45 well as a direct structural description of the signed graphs whose line graphs are  
 46 consistent. (The related paper [14] gives constructions for line-consistent signed  
 47 graphs that illuminate more about their structure.)

48 A graph may have multiple edges but not loops. A *simple* graph has neither  
 49 loops nor multiple edges. The *degree*  $d(v)$  of a vertex is the number of edges  
 50 incident with  $v$  (and also the number of neighbors of  $v$  if the graph is simple). The  
 51 length of a path  $P$  is denoted by  $l(P)$ ; if it is zero the path is *trivial*. *Suppressing*  
 52 a divalent vertex means replacing it and its two incident edges by a single edge.  
 53 Two graphs are *homeomorphic* if they are isomorphic or they can both be changed  
 54 into the same (unlabelled) graph by suppressing divalent vertices in one or both  
 55 of them. When  $e, e'$  are parallel edges in  $\Gamma$ , the line graph  $L(\Gamma)$  has a double edge  
 56 between its vertices  $e$  and  $e'$ .

57 In a signed graph  $\Sigma = (\Gamma, \sigma)$ , the sign of a circle  $C$ , written  $\sigma(C)$ , is the  
 58 product of its edge signs. The sign  $\sigma(P)$  of a path  $P$  is similar. (Formally, the  
 59 sign of any edge set  $S$  is  $\sigma(S) := \prod_{e \in S} \sigma(e)$ .) A vertex is *totally positive* (or,  
 60 *totally negative*) if all incident edges are positive (or, negative). The *negative*  
 61 *subgraph* of  $\Sigma$  is the spanning subgraph  $\Sigma^-$  whose edges are the negative edges  
 62 of  $\Sigma$ .

63 The *line graph of a signed graph*  $\Sigma$ , written  $L_\sigma(\Sigma)$ , is defined as the vertex-  
 64 signed graph  $(L(\Gamma), \sigma)$  whose underlying graph is  $L(\Gamma)$ , the line graph of  $\Gamma$ , and  
 65 whose vertices are marked by the sign function  $\sigma$  of  $\Sigma$ . (Other notions of line  
 66 graph of a signed graph exist, in which edges are signed instead of vertices, but  
 67 they are not related to this work.)

68 **Definition.** A signed graph  $\Sigma$  is called *line consistent* if  $L_\sigma(\Sigma)$  is consistent.  
 69 (This is the same as “ $S$ -consistency” in [4], where  $\Sigma$  is called  $S$ .)

70 The main result of [4], Theorem 2.1 (when corrected by revising the first  
 71 line of part (2) to read “for every vertex  $v_i \dots$  in  $S$  such that  $d(v_i) \geq 3$ ,” as the  
 72 authors obviously intended), applies to simple graphs with edge signs. It states:

73 **Theorem 1** [4, Theorem 2.1]. *A signed simple graph  $\Sigma$  is line consistent if and*  
 74 *only if it is balanced, every vertex of degree  $d(v) > 3$  is totally positive, and each*  
 75 *vertex of degree  $d(v) = 3$  is either totally positive or has two negative edges which*  
 76 *belong to all circles through the vertex.*

77 We reformulate Theorem 1 in a simpler way that leads to a short proof,  
 78 we add a second and third criterion for line consistency, and we generalize by  
 79 allowing the underlying graph to have multiple edges. An *isthmus* (called by some  
 80 a “bridge”) is an edge whose deletion raises the number of connected components;  
 81 equivalently, it is an edge that does not belong to any circle.

82 **Theorem 2.** *Each of the following conditions on a signed graph  $\Sigma$  without loops*  
 83 *is necessary and sufficient for it to be line consistent:*

84 (i)  *$\Sigma$  is balanced, each vertex of degree  $d(v) > 3$  is totally positive, and each*  
 85 *vertex of degree  $d(v) = 3$  is totally positive or has exactly one positive edge,*  
 86 *which is an isthmus.*

87 (ii)  *$\Sigma$  is balanced, its negative subgraph is a vertex-disjoint union of paths and*  
 88 *circles, and each endpoint  $v$  of a negative edge is incident with at most one*  
 89 *positive edge, which is an isthmus if  $d_{\Sigma^-}(v) = 2$ .*

90 (iii) *Each vertex  $v$  of degree  $d(v) > 3$  is totally positive, each vertex of degree*  
 91  *$d(v) = 3$  is totally positive or has exactly one positive edge, and after deleting*  
 92 *all positive isthmi, the signed graph is balanced and the endpoints of every*  
 93 *negative edge have degree at most 2.*

94 **Proof.** The equivalence of (i) and (ii) is obvious, so we give short proofs of the  
 95 necessity and sufficiency of (i) and the equivalence of (ii) and (iii).

96 To prove necessity of (i), consider the possibilities. If a vertex  $v$  has either  
 97 three negative edges, or two positive edges and one negative edge, these edges  
 98 form a negative triangle in the line graph. This implies most of (i). We have to  
 99 prove that, if  $v$  is incident with negative edges  $e$  and  $e'$  and a positive edge  $f$   
 100 (and no other edges), then  $f$  is an isthmus.

101 If not, there is a circle  $C$  on  $f$  which must contain  $e$  or  $e'$ ; let us say  $e$ . In  
 102 the line graph  $C$  generates a circle  $L(C)$ , which is positive because its vertex sign  
 103 product equals the edge sign product of  $C$ . However, in the line graph there is  
 104 another circle that interposes  $e'$  between  $e$  and  $f$ . This circle is negative. Hence,  
 105  $f$  must be an isthmus. Thus, (i) is necessary for line consistency.

106 Now we prove sufficiency. A *digon* is a circle of length 2. A *vertex triangle* is  
 107 a circle of length 3 in  $L(\Gamma)$  whose vertices are three edges that are incident with  
 108 a single vertex in  $\Gamma$ .

109 A circle  $C$  in  $L_\sigma(\Sigma)$  has the form  $e_l e_1 \cdots e_{l-1} e_l$  where  $l \geq 2$ . A digon or  
 110 vertex triangle in  $L_\sigma(\Sigma)$  is obviously positive. Any other triangle comes from a

111 triangle in  $\Sigma$ , so is also positive. Thus, we may assume  $l \geq 4$  and that any shorter  
 112 circle is positive.

113 Note that an isthmus  $e = uv$  of  $\Gamma$  that is not a pendant edge is a cutpoint  
 114 of  $L(\Gamma)$ , separating the other edges incident with  $u$  from those incident with  $v$ .  
 115 The only way it can belong to a circle in  $L(\Gamma)$  is for the preceding and following  
 116 edges of the circle to be incident with the same endpoint of  $e$ .

117 Suppose consecutive edges  $e_{i-1}e_ie_{i+1}$  are incident with  $v$ . If  $e_i$  is positive,  $C$   
 118 can be shortened by omitting it without changing the sign of the circle. If  $e_i$  is  
 119 negative, since  $d(v) = 3$  one of the edges is a positive isthmus, say  $e_{i+1}$ . Because  
 120  $e_{i+1}$  is an isthmus,  $e_{i+2}$  must be incident with  $v$  as well, which is impossible  
 121 because  $d(v) = 3$  and (since  $l \geq 4$ )  $e_{i+2} \neq e_{i-1}$ . Therefore, we may assume no  
 122 three consecutive edges in  $C$  are incident with the same vertex. That means  
 123  $C = L(W)$  for a closed walk  $W$  in  $\Sigma$ . But every closed walk is positive because  
 124 of balance. Thus,  $C$  is positive.

125 That (ii) implies (iii) is obvious. Assume (iii); we deduce (ii).  $\Sigma$  is balanced  
 126 because isthmi do not affect balance. Also,  $\Sigma^-$  has maximum degree at most 2,  
 127 so it is a disjoint union of paths and circles. Let  $\Sigma'$  be  $\Sigma$  without its positive  
 128 isthmi. If  $d_{\Sigma}(v) = 3$  and  $v$  has one positive edge  $f$  and two negative edges  $e_1, e_2$ ,  
 129 then  $d_{\Sigma'}(v) \leq 2$  so  $f$  must be an isthmus. ■

130 It is interesting to see what the theorems say about relatively well connected  
 131 graphs.

132 **Corollary 3.** *Let  $\Sigma$  of order at least 4 be 3-connected, or just edge 2-connected*  
 133 *without divalent vertices. Then  $\Sigma$  is line consistent if and only if it is all positive.*  
 134 □

135 Although (ii) and (iii) in the theorem are simple restatements of (i), their  
 136 different points of view are suggestive. Parts (ii) and (iii) suggest constructions  
 137 for line-consistent signed graphs, for which see [14].

138 Part (ii) can be interpreted as a structural description of a line-consistent  
 139 signed graph  $\Sigma$ . (Recall that we forbid loops.) A *block* is a graph that is connected  
 140 and has no cutpoint. A block is *nontrivial* if it contains a circle; thus, the trivial  
 141 blocks are the isthmi and isolated vertices. A *block of  $\Gamma$*  is a maximal block  
 142 subgraph of  $\Gamma$ ; equivalently, it is a maximal connected subgraph that is not  
 143 separated by any cutpoint of  $\Gamma$ . We apply the same terminology to  $\Sigma$  as to its  
 144 underlying graph.

145 **Theorem 4.** *A signed graph  $\Sigma$  is line consistent if and only if it has the following*  
 146 *form:*

- 147 (1) *Each component of  $\Sigma^-$  is a circle, a nontrivial path, or a single vertex.*

- 148 (2) *A circle component of  $\Sigma^-$  is a block of  $\Sigma$  and each of its vertices is incident*  
 149 *with at most one other edge, which must be a positive isthmus.*
- 150 (3) *A nontrivial path component  $P$  of  $\Sigma^-$  either is an induced subgraph of  $\Sigma$ , or*  
 151 *is all but one edge of a circle that is a block of  $\Sigma$  whose remaining edge is*  
 152 *positive. The endpoints of  $P$  are at most divalent. Each internal vertex of*  
 153  *$P$  is incident with at most one other edge, which must be a positive isthmus.*  
 154 *Furthermore,  $P$  either*
- 155 (a) *is part of a nontrivial block of  $\Sigma$  (then its endpoints are necessarily*  
 156 *divalent); or*
- 157 (b) *is composed entirely of isthmi and its endpoints are not incident with*  
 158 *any nontrivial block of  $\Sigma$  (then the second edge, if any, incident with an*  
 159 *endpoint is necessarily a positive isthmus).*

160 **Proof.** The form is stronger than the characterization in (ii), so it implies line  
 161 consistency. We verify the converse in stages.

162 The characterization of an all-negative circle follows directly from (ii).

163 Consider a nontrivial path component  $P$  of the negative subgraph.

164 If  $P$  is contained in a nontrivial block  $B$ , each endpoint must be incident with  
 165 a second edge, which is positive. Suppose the second edge at both endpoints is  
 166 the same edge  $e$ ; then  $P \cup \{e\}$  is a circle. No other edge can be incident with the  
 167 endpoints of  $P$ , but an internal vertex can be incident with one more edge, which  
 168 can only be an isthmus. If the positive edges at the endpoints are distinct, then  
 169  $P$  is induced because, again, any third edge at an internal vertex is an isthmus,  
 170 therefore has its other endpoint off  $P$ .

171 If  $P$  is not contained in a nontrivial block, it is composed entirely of isthmi.  
 172 Suppose it were not; then it would have a vertex  $v$  that is in a nontrivial block  $B$   
 173 and is incident with an edge of  $P$  that is not in  $B$ . The degree of  $v$  in  $B$  is at least 2;  
 174 therefore,  $d(v) = 3$  and by (ii) the third edge at  $v$  is a positive isthmus. However,  
 175 that leaves only one edge at  $v$  that can be in  $B$ , an impossibility. Therefore, if  $P$   
 176 is not entirely within a nontrivial block, its every edge is an isthmus. By similar  
 177 reasoning a second edge at an endpoint of  $P$  must be an isthmus. ■

178 Acharya, Acharya, and Sinha [3] examined a similar problem, where the line  
 179 graph has edge as well as vertex signs derived from  $\Sigma$ . The question is whether the  
 180 product of edge signs and vertex signs on each circle is the same—this property  
 181 is called *harmony*. In [3] the edge signs are those of the Behzad–Chartrand line  
 182 graph [6]. Other definitions of a signed line graph (such as that in [13, Sect. 5.2])  
 183 could be considered, though if the edge signature is balanced (as in the  $\times$ -line  
 184 signed graph of M. Acharya [5]) the answer is the same as in Theorem 1 since  
 185 switching (defined in, e.g., [13]) the line graph’s edge signature does not alter the  
 186 characterization of harmony.

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