Non-Attacking Chess Pieces: The Dance of Bishops

Thomas Zaslavsky

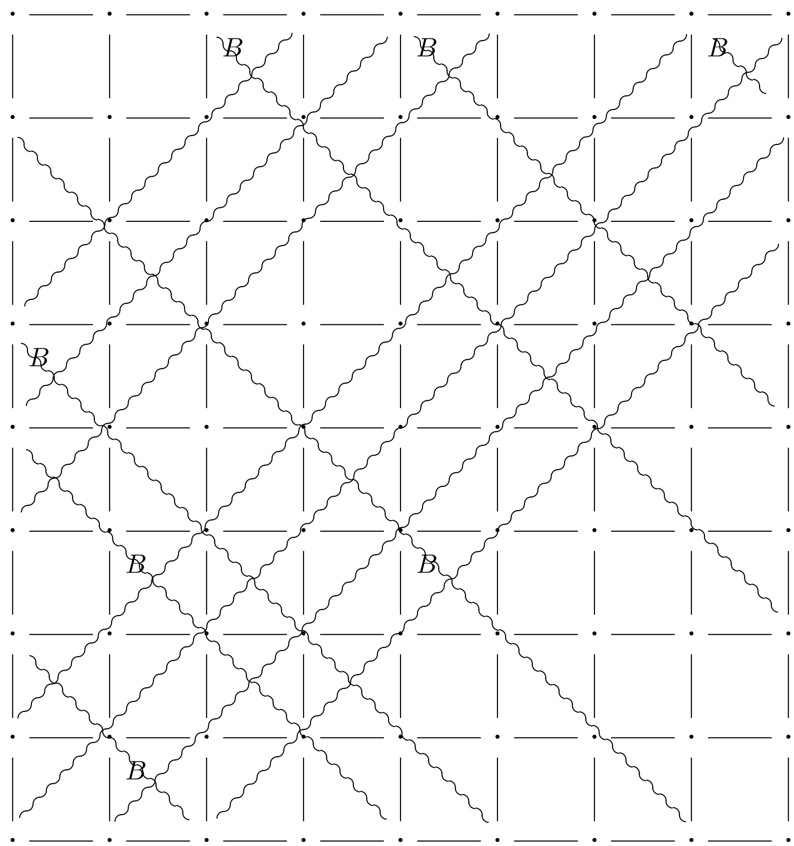
Binghamton University (State University of New York)

Joint with Seth Chaiken and Christopher R.H. Hanusa

Outline

1. Chess Problems: Non-Attacking Pieces
2. Largely Czech Numbers and Formulas
3. Riders
4. Configurations and Inside-Out Polytopes
5. The Bishop Equations
6. Signed Graphs
7. Signed Graphs to the Rescue

1. Chess Problems: Non-Attacking Pieces



7 non-attacking bishops on an 8×8 board.

Question 1: How many bishops can you get onto the board?

Question 2: Given q bishops, how many ways can you place them on the board so none attacks any other?

2. Numbers and Formulas from Our Czech Mate

Vaclàv Kotěšovec (Czech Republic) has a book of chess problems and solutions [5], called *Chess and Mathematics*. Some of the numbers:

$N_B(q;n)$:= the number of ways to place q non-attacking bishops on an $n \times n$ board.

$n =$	1	2	3	4	5	6	7	8	Period	Denom
$q = 1$	1	4	9	16	25	36	49	64	1	1
2	0	4	26	92	240	520	994	1736	1	1
3	0	0	26	232	1124	3896	10894	26192	2	2
4	0	0	8	260	2728	16428	70792	242856	2	2
5	0	0	0	112	3368	39680	282248	1444928	2	2
6	0	0	0	16	1960	53744	692320	5599888	2	2

$N_Q(q;n)$:= the number of ways to place q non-attacking queens on an $n \times n$ board.

$n =$	1	2	3	4	5	6	7	8	Period	Denom
$q = 1$	1	4	9	16	25	36	49	64	1	1
2	0	0	8	44	140	340	700	1288	1	1
3	0	0	0	24	204	1024	3628	10320	2	2
4	0	0	0	2	82	982	7002	34568	6	6
5	0	0	0	0	10	248	4618	46736	60	??
6	0	0	0	0	0	4	832	22708	840	??
7	0	0	0	0	0	0	40	3192	360360	??

$$N_B(1;n) = n^2.$$

$$N_B(2;n) = \frac{n}{6}\big(3n^3 - 4n^2 + 3n - 2\big).$$

$$N_B(3;n) = \frac{4n^6 - 16n^5 + 30n^4 - 40n^3 + 32n^2 - 16n + 3}{24} - (-1)^n\frac{1}{8}.$$

$$N_B(4;n) = \frac{(n-2)(15n^7 - 90n^6 + 260n^5 - 524n^4 + 727n^3 - 646n^2 + 393n - 90)}{360} - (-1)^n\frac{(n-2)^2}{8}.$$

$$N_B(5;n) = \frac{(n-2)\begin{pmatrix} 6n^9 - 68n^8 + 354n^7 - 1180n^6 + 2870n^5 \\ \quad - 5284n^4 + 6697n^3 - 6018n^2 + 3558n - 810 \end{pmatrix}}{720} - (-1)^n\frac{(n-2)(3n^3 - 22n^2 + 58n - 54)}{48}.$$

$$N_B(6;n) = \begin{cases} \dfrac{\begin{array}{l}(n-1)(n-3)(126n^{10} - 2016n^9 + 14868n^8 - 69244n^7 + 234017n^6 \\ \quad - 607984n^5 + 1211879n^4 - 1797328n^3 + 1953593n^2 - 1550820n + 722925)\end{array}}{90720} & \text{if } n \text{ is odd,} \\ \dfrac{\begin{array}{l}n(n-2)(126n^{10} - 2268n^9 + 18774n^8 - 97216n^7 + 361165n^6 \\ \quad - 1029454n^5 + 2283178n^4 - 3841960n^3 + 4676932n^2 - 3808152n + 1640160)\end{array}}{90720} & \text{if } n \text{ is even.} \end{cases}$$

3. Riders

Rider: Moves any distance in specified directions (forward and back).

The move is specified by an integral vector $(m_1, m_2) \in \mathbb{R}^2$ in the direction of the line.

Bishop: $(1,1),\ (1,-1)$.

Queen: $(1,0),\ (0,1),\ (1,1),\ (1,-1)$.

Nightrider: $(1,2),\ (2,1),\ (1,-2),\ (2,-1)$.

Configuration:
A point
$$z = (z_1, z_2, \dots, z_q) \in (\mathbb{R}^2)^q = \mathbb{R}^{2q}$$
where
$$z_i = (x_i, y_i),$$
which describes the locations of all the bishops (or queens, or ...).

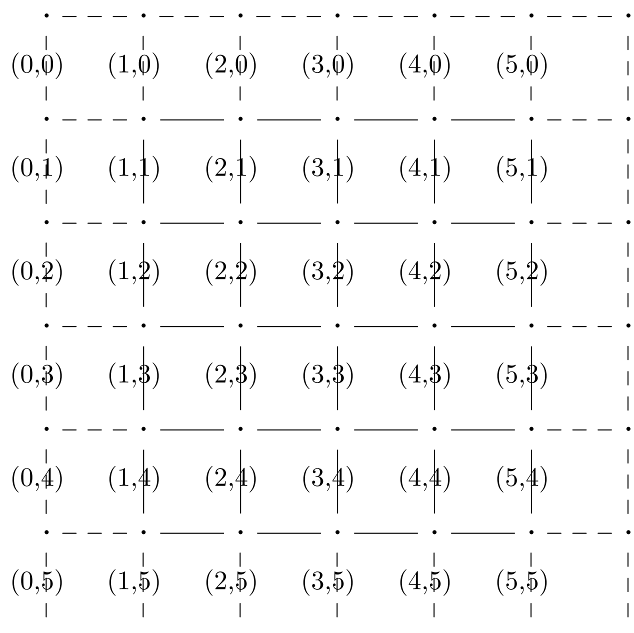
Constraints:
The equations that correspond to attacking positions:
$$z_j - z_i \in \text{a line of attack},$$
or in a formula:
$$z_j - z_i \perp m \qquad \text{for some move vector } m = (m_1, m_2),$$
or,
$$m_2(x_j - x_i) = m_1(y_j - y_i).$$

4. Configurations and Inside-Out Polytopes

The board. A square board with squares

$$\{(x,y) : x, y \in \{1,2,\dots,n\}\} = \{1,2,\dots,n\}^2.$$

The board is $n \times n = 4 \times 4$ with a border, coordinates shown on the left side of each square. Note the border coordinates with 0 or $n+1$, not part of the main square.



The dot picture in $\mathbb{Z}^2$. The border points are hollow.

$\circ(0,0)$	$\circ(1,0)$	$\circ(2,0)$	$\circ(3,0)$	$\circ(4,0)$	$\circ(5,0)$
$\circ(0,1)$	$\bullet(1,1)$	$\bullet(2,1)$	$\bullet(3,1)$	$\bullet(4,1)$	$\circ(5,1)$
$\circ(0,2)$	$\bullet(1,2)$	$\bullet(2,2)$	$\bullet(3,2)$	$\bullet(4,2)$	$\circ(5,2)$
$\circ(0,3)$	$\bullet(1,3)$	$\bullet(2,3)$	$\bullet(3,3)$	$\bullet(4,3)$	$\circ(5,3)$
$\circ(0,4)$	$\bullet(1,4)$	$\bullet(2,4)$	$\bullet(3,4)$	$\bullet(4,4)$	$\circ(5,4)$
$\circ(0,5)$	$\circ(1,5)$	$\circ(2,5)$	$\circ(3,5)$	$\circ(4,5)$	$\circ(5,5)$

The cube.

Reduce (x, y) to $\dfrac{1}{n+1}(x, y) \in [0, 1]^2$.

The position of a piece becomes $z_i = (x_i, y_i) \in (0, 1)^2 \cap \dfrac{1}{n+1}\mathbb{Z}^2$.

The configuration becomes $z = (z_1, \ldots, z_q) \in (0, 1)^{2q} \cap \dfrac{1}{n+1}\mathbb{Z}^{2q}$,

a $\frac{1}{n+1}$-fractional point in the open cube $\big([0, 1]^{2q}\big)^\circ$.

$\circ(\frac{0}{5}, \frac{0}{5})$	$\circ(\frac{1}{5}, \frac{0}{5})$	$\circ(\frac{2}{5}, \frac{0}{5})$	$\circ(\frac{3}{5}, \frac{0}{5})$	$\circ(\frac{4}{5}, \frac{0}{5})$	$\circ(\frac{5}{5}, \frac{0}{5})$
$\circ(\frac{0}{5}, \frac{1}{5})$	$\bullet(\frac{1}{5}, \frac{1}{5})$	$\bullet(\frac{2}{5}, \frac{1}{5})$	$\bullet(\frac{3}{5}, \frac{1}{5})$	$\bullet(\frac{4}{5}, \frac{1}{5})$	$\circ(\frac{5}{5}, \frac{1}{5})$
$\circ(\frac{0}{5}, \frac{2}{5})$	$\bullet(\frac{1}{5}, \frac{2}{5})$	$\bullet(\frac{2}{5}, \frac{2}{5})$	$\bullet(\frac{3}{5}, \frac{2}{5})$	$\bullet(\frac{4}{5}, \frac{2}{5})$	$\circ(\frac{5}{5}, \frac{2}{5})$
$\circ(\frac{0}{5}, \frac{3}{5})$	$\bullet(\frac{1}{5}, \frac{3}{5})$	$\bullet(\frac{2}{5}, \frac{3}{5})$	$\bullet(\frac{3}{5}, \frac{3}{5})$	$\bullet(\frac{4}{5}, \frac{3}{5})$	$\circ(\frac{5}{5}, \frac{3}{5})$
$\circ(\frac{0}{5}, \frac{4}{5})$	$\bullet(\frac{1}{5}, \frac{4}{5})$	$\bullet(\frac{2}{5}, \frac{4}{5})$	$\bullet(\frac{3}{5}, \frac{4}{5})$	$\bullet(\frac{4}{5}, \frac{4}{5})$	$\circ(\frac{5}{5}, \frac{4}{5})$
$\circ(\frac{0}{5}, \frac{5}{5})$	$\circ(\frac{1}{5}, \frac{5}{5})$	$\circ(\frac{2}{5}, \frac{5}{5})$	$\circ(\frac{3}{5}, \frac{5}{5})$	$\circ(\frac{4}{5}, \frac{5}{5})$	$\circ(\frac{5}{5}, \frac{5}{5})$

The Bishop Equations.

Bishops must not attack.

The forbidden equations:

$$z_i \notin y_j - y_i = x_j - x_i \quad \text{and} \quad z_i \notin y_j - y_i = -(x_j - x_i).$$

Left: The move line of slope +1. Right: The move line of slope −1.

Forbidden hyperplanes in $\mathbb{R}^{2q}$ given by the 'bishop equations'.

Summary.

We have

a convex polytope $P = [0,1]^{2q}$, and

a set $\mathcal{H} = \{h_{ij}^{+}, h_{ij}^{-}\}$ of forbidden hyperplanes,

and we want the number of ways to pick

$$z \in \left[P^\circ \cap \tfrac{1}{n+1}\mathbb{Z}^{2q}\right] \setminus \left[\bigcup \mathcal{H}\right].$$

Inside-Out Polytopes

$(P, \mathcal{H})$ is an 'inside-out polytope'. We want $E_{P^\circ,\mathcal{H}}(n+1) :=$ the number of points in

$$\left[P^\circ \cap \tfrac{1}{n+1}\mathbb{Z}^{2q}\right] \setminus \left[\bigcup \mathcal{H}\right].$$

Inside-out Ehrhart theory (Beck & Zaslavsky 2005, based on Ehrhart and Macdonald) says that $E_{P^\circ,\mathcal{H}}(n+1)$ is a quasipolynomial function of $n+1$, for $n+1 \in \mathbb{Z}_{>0}$.

Vertex of $(P, \mathcal{H})$:
A point in P determined by the intersection of hyperplanes in $\mathcal{H}$ and facets of P.

Quasipolynomial:
A cyclically repeating series of polynomials,

$$c_d(n)n^d + c_{d-1}(n)n^{d-1} + \cdots + c_1(n)n + c_0(n),$$

where the c_i are periodic functions of n that depend on $n \mod p$ for some $p \in \mathbb{Z}_{>0}$. The smallest p is called the period of the quasipolynomial.

Lemma 4.1 ([1]). *If P has rational vertices and the hyperplanes in $\mathcal{H}$ are given by an integral linear equation, then:*
(a) *$E_{P^\circ,\mathcal{H}}(n+1)$ is a quasipolynomial function of n.*
(b) *Its degree is $d = \dim P$, and its leading term is $\operatorname{vol}(P)n^d$.*
(c) *Its period is a factor of the least common denominator of all coordinates of vertices of $(P, \mathcal{H})$.*

Define
$N_R(q; n) :=$ the number of ways to place q non-attacking R-pieces on an $n \times n$ board.

Theorem 4.2. *For a rider chess piece R, $N_R(q; n)$ is a quasipolynomial function of n, for each fixed $q > 0$; the leading term of each polynomial is $\frac{1}{q!}n^{2q}$.*

Agrees with Kotěšovec's formulas!

The chess problem.

What is the quasipolyomial for q bishops (or queens, or ...)?

We have $2qp$ undetermined coefficients. The actual numbers $N_B(q;n)$ for $1 \leq n \leq 2pq$ will determine the whole thing.

Aye, there's the rub. Two rubs:

(1) We don't know p.
(2) It may be impossible to compute enough values of $N_B(q;n)$.

Finding a small upper bound on the period is not so easy. Lemma 4.1(c) says:

The period is a factor of the gcd *of the denominators of the vertices of* $(P^\circ, \mathcal{H})$.

Good, if we can find the denominator.

The Bishop Solution.

Theorem 4.3. *The bishop quasipolynomial* $N_B(q;n)$ *has period at most* 2.

Theorem 4.3 is an immediate corollary of Lemma 6.1, which bounds the denominator.

5. Signed Graphs

- **Graph** (N, E):
  - Node set $N = \{v_1, v_2, \ldots, v_q\}$.
  - Edge set E.
- 1-*Forest*: each component is a tree with one more edge. (Each component contains exactly one circle.)
- **Signed graph** $\Sigma = (N, E, \sigma)$:
  - Graph (N, E).
  - Signature $\sigma : E \to \{+, -\}$.
- Circle sign $\sigma(C)$.
- *Signed circuit*: a positive circle; or a connected subgraph that contains exactly two circles, both negative.
- *Homogeneous node*: all incident edges have the same sign.
- *Incidence matrix* $\mathrm{H}(\Sigma)$:
  - $N \times E$ matrix.
  - In column of edge $e{:}v_iv_j$,
    (a) $\eta(v, e) = \pm 1$ if v is an endpoint of e and $= 0$ if not;
    (b) $\eta(v_i, e)\eta(v_j, e) = -\sigma(e)$.
    (The column of a positive edge: one $+1$ and one -1,
    the column of a negative edge: two $+1$'s or two -1's.)

  Lemma 5.1 ([8, Theorems 5.1(j) and 8B.1])**.** *For a signed graph* Σ*:*

  $\mathrm{H}(\Sigma)$ *has full column rank iff* Σ *contains no signed circuit.*
  $\mathrm{H}(\Sigma)$ *has full row rank iff every component of* Σ *contains a negative circle.*

- The usual hyperplane arrangement $\mathcal{H}[\Sigma]$:
  Edge $e{:}v_iv_j \mapsto h_e : x_j = \sigma(e)x_i$ in $\mathbb{R}^q$.
  Vector-space dual to columns of incidence matrix.
  (Linear dependencies are those of the incidence matrix columns.)

- **Clique graph** $C(\Sigma)$:
  - *Positive clique*: maximal set of nodes connected by positive edges.
  - $\mathcal{A} := \{$positive cliques$\}$.
  - *Negative clique*: maximal set of nodes connected by negative edges.
  - $\mathcal{B} := \{$negative cliques$\}$.
  - *Signed clique*: either one.

  - $\forall\ v \in$ one positive clique and one negative clique.
  - For a signed forest with q nodes, $k_A + k_B = q + c$, where

    $c =$ number of components,

    $k_A =$ number of positive cliques,

    $k_B =$ number of negative cliques.
  - *Clique graph*:

    Node set of $C(\Sigma)$ is $\mathcal{A} \cup \mathcal{B}$.

    An edge $A_k B_l$ for each $v_i \in A_k \cap B_l$.

6. Signed Graphs to the Rescue

Lemma 6.1. *A point $z = (z_1, z_2, \ldots, z_q) \in \mathbb{R}^{2q}$, determined by a total of $2q$ bishop equations and fixations, is weakly half integral. Furthermore, in each z_i, either both coordinates are integers or both are strict half integers.*

Consequently, a vertex of the bishops' inside-out polytope $([0,1]^{2q}, \mathcal{A}_B)$ has each $z_i \in \{0,1\}^2$ or $z_i = (\frac{1}{2}, \frac{1}{2})$.

Proof. A vertex z is the intersection of $2q$ hyperplanes:

- Bishop hyperplanes,
  $h^+_{ij} : x_j - y_j = x_i - y_i$ and
  $h^-_{ij} : x_j + y_j = x_i + y_i$.
- Facet hyperplanes,
  $x_i = c_i$ and
  $y_j = d_j$.

Equations:
Bishop equations (relations between coordinates).
Fixations:
Facet hyperplanes (fix some coordinates to chosen integers).

Signed graph of z:
$\Sigma_z \longleftrightarrow$ the 'equations', i.e., hyperplanes h^ε_{ij}.
Clique graph of z:
$C_z := C(\Sigma_z)$, and
$\pm C_z$.

Meaning of a clique:

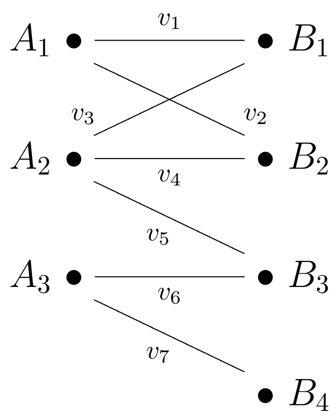
- $A_k = \{v_i, v_j, \ldots\} \implies x_i - y_i = x_j - y_j \implies$
  $$x_i - y_i = a_k \quad \forall\, v_i \in A_k.$$
- $B_l = \{v_i, v_j, \ldots\} \implies x_i + y_i = x_j + y_j \implies$
  $$x_i + y_i = b_l \quad \forall\, v_i \in B_l.$$

Method:

(1) Convert to variables a_k, b_l.
(2) Find enough fixations to determine a_k, b_l.
(3) Solve for all x_i, y_j.

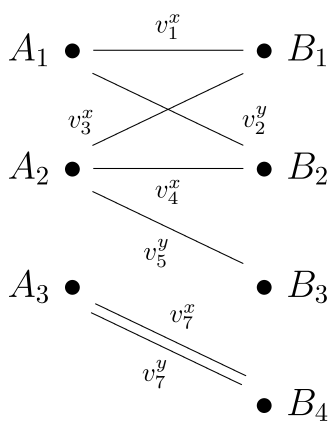
Example 6.1. $\mathcal{A} = \{A_1, A_2, A_3\}$ and $\mathcal{B} = \{B_1, B_2, B_3, B_4\}$,
$N = \{v_1, \ldots, v_8\}$:

C_z :



A suitable 1-forest $\Sigma_z \subseteq \pm C_z$
(superscript x is $+$, y is $-$):

$\Sigma_z \subseteq \pm C_z$



$\longleftrightarrow$

fixations

$x_1 = c_1$,
$y_2 = d_1$,
$x_3 = c_2$,
$x_4 = c_3$,
$y_5 = d_2$,
$x_7 = c_4$,
$y_7 = d_3$.

Incidence matrix (invertible):

$$M := \mathrm{H}(C_z) = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_3 & x_4 & x_7 & y_2 & y_5 & y_7 \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{array} .$$

In matrix form:

$$M^{\mathrm{T}} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_3 \\ x_4 \\ x_7 \\ y_2 \\ y_5 \\ y_7 \end{bmatrix} = 2 \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix},$$

where $c_i, d_j \in \mathbb{Z}$. Solution:

$$\begin{aligned}
a_1 &= x_1 - x_3 + x_4 + y_2 &&= c_1 - c_2 + c_3 + d_1, \\
a_2 &= -x_1 + x_3 + x_4 + y_2 &&= -c_1 + c_2 + c_3 + d_1, \\
a_3 &= x_7 + y_7 &&= c_4 + d_3, \\
b_1 &= -x_1 - x_3 + x_4 + y_2 &&= -c_1 - c_2 + c_3 + d_1, \\
b_2 &= -x_1 + x_3 - x_4 + y_2 &&= -c_1 + c_2 - c_3 + d_1, \\
b_3 &= x_1 - x_3 - x_4 - y_2 + 2y_5 &&= c_1 - c_2 - c_3 - d_1 + 2d_2, \\
b_4 &= -x_7 + y_7 &&= -c_4 + d_3,
\end{aligned}$$

and the unfixed variables are

$$\begin{aligned}
x_2 &= \frac{a_1 - b_2}{2} = c_1 - c_2 + c_3, \\
x_5 &= \frac{a_2 - b_3}{2} = -c_1 + c_2 + c_3 + d_1 - d_2, \\
x_6 &= \frac{a_3 - b_3}{2} = \frac{-c_1 + c_2 + c_3 + c_4 + d_1 - 2d_2 + d_3}{2}, \\
y_1 &= \frac{a_1 + b_1}{2} = -c_2 + c_3 + d_1, \\
y_3 &= \frac{a_2 + b_1}{2} = -c_1 + c_3 + d_1, \\
y_4 &= \frac{a_2 + b_2}{2} = -c_1 + c_2 + d_1, \\
y_6 &= \frac{a_3 + b_3}{2} = \frac{c_1 - c_2 - c_3 + c_4 - d_1 + 2d_2 + d_3}{2}.
\end{aligned}$$

x_6 and y_6 are the only possibly fractional coordinates; their sum is integral; therefore, either z_6 is integral or $z_6 = (\frac{1}{2}, \frac{1}{2})$.

Lemma 6.2 (Hochbaum, Megiddo, Naor, and Tamir 1993)**.** *The solution of a linear system with integral constant terms, whose coefficient matrix is the transpose of a nonsingular signed-graph incidence matrix, is weakly half-integral.*

Proof of Theorem, concluded.

$$\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix} = \mathrm{H}(\pm C_z)^{\mathrm{T}}(M^{-1})^{\mathrm{T}}\begin{bmatrix}\mathbf{c}\\ \mathbf{d}\end{bmatrix}.$$

This is half integral, because $(M^{-1})^{\mathrm{T}}\begin{bmatrix}\mathbf{c}\\ \mathbf{d}\end{bmatrix}$ is half integral by the lemma, and $\mathrm{H}(\pm C_z)^{\mathrm{T}}$ is integral. □

REFERENCES

[1] Matthias Beck and Thomas Zaslavsky, Inside-out polytopes. *Adv. Math.* **205** (2006), no. 1, 134–162. MR 2007e:52017. Zbl 1107.52009. `arXiv.org` math.CO/0309330.

[2] Seth Chaiken, Christopher R.H. Hanusa, and Thomas Zaslavsky, Mathematical analysis of a q-queens problem. In preparation.
The full paper of this talk.

[3] F. Harary, On the notion of balance of a signed graph. *Michigan Math. J.* **2** (1953–54), 143–146 and addendum preceding p. 1. MR 16, 733h. Zbl 056.42103.

[4] Dorit S. Hochbaum, Nimrod Megiddo, Joseph (Seffi) Naor, and Arie Tamir, Tight bounds and 2-approximation algorithms for integer programs with two variables per inequality. *Math. Programming Ser. B* **62** (1993), 69–83. MR 94k:90050. Zbl 802.90080.

[5] Václav Kotěšovec, *Non-attacking chess pieces (chess and mathematics)* [*Šach a matematika - počty rozmístění neohrožujících se kamenů*]. [Self-published online book], 2010; second edition 2010, URL
`http://problem64.beda.cz/silo/kotesovec_non_attacking_chess_pieces_2010.pdf`
An amazing source of formulas and conjectures; no proofs.

[6] N.J.A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, URL
`http://www.research.att.com/~njas/sequences/`
Many numbers for bishops up to 6, queens up to 7, nightriders up to 3.

[7] Thomas Zaslavsky, The geometry of root systems and signed graphs. *Amer. Math. Monthly* **88** (1981), 88–105. MR 82g:05012. Zbl 466.05058.
Hyperplanes led me to signed graphs.

[8] ——, Signed graphs. *Discrete Appl. Math.* **4** (1982), 47–74. Erratum. *Discrete Appl. Math.* **5** (1983), 248. MR 84e:05095. Zbl 503.05060.
The theory of the incidence matrix.