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Biased Graphs. III. Chromatic and Dichromatic Invariants*

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A *biased graph* Ω consists of a graph Γ and a class of circles in Γ (edge sets of simple, closed paths), called *balanced*, such that no theta subgraph contains exactly two balanced circles. An edge set is *balanced* if (simplifying slightly) every circle in it is balanced. Biased graphs generalize ordinary graphs, which behave like biased graphs in which every circle is balanced. We define and study the chromatic, dichromatic, and Whitney number polynomials of a biased graph, which generalize those of an ordinary graph. We employ an algebraic definition since not all biased graphs can be colored. We show that the polynomials enjoy many properties that are familiar in ordinary graph theory, such as convolutional and partition expansions, close connections with the bias and lift matroids of Ω, and deletion–contraction invariance of the dichromatic polynomial. They also have the novel feature of being reducible to more readily computable related polynomials that have no analogs in ordinary graph theory. We apply our results to evaluate Whitney numbers and other invariants of the bias and lift matroids, to characterize the biased graphs which have an unbalanced edge at every node and whose bias matroid is a series–parallel network, and to calculate the invariants of some types of biased graphs, such as those where no circle is balanced and some which are similar to Dowling lattices and classical root systems. For the latter we also characterize supersolvability, a matroid property which implies the characteristic polynomial has positive integral roots.

INTRODUCTION

In the coloring theory of ordinary graphs three polynomials naturally arise. G. D. Birkhoff's *chromatic polynomial* $\chi_\Gamma(\lambda)$ counts the proper colorings of Γ in λ colors, when the value of λ is a nonnegative integer. The *dichromatic polynomial* $Q_\Gamma(u, v)$ is a modified generating function of all colorings in λ colors by the number of improperly colored edges (both endpoints the same color): if we normalize by defining $\bar{Q}_\Gamma(uv, v) = v^n Q_\Gamma(u, v)$, where Γ

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has n nodes, then $\bar{Q}_\Gamma(\lambda, w-1) = \sum w^{\# I(f)}$, summed over all colorings f, where $I(f)$ is the set of improper edges. The *Whitney number polynomial* is the generating function of λ-colorings by connected components: $w_\Gamma(x, \lambda) = \sum x^{n-c(I(f))}$, where $c(I(f))$ is the number of components into which the improper edges connect the n nodes.

These polynomials have interesting algebraic and combinatorial properties. They are *invariant*, depending only on the isomorphism type of Γ. They have algebraic formulas expressing them as sums of monomials attached to edge subsets of Γ, and these formulas depend essentially only on the polygon matroid ("cycle" or "graphic" matroid) of Γ; thus they can be used to study the matroid. They are "multiplicative": the value on Γ is the product of the values on the connected components of Γ. The dichromatic polynomial is "additive": $Q_\Gamma = Q_{\Gamma\setminus e} + Q_{\Gamma/e}$ if e is a link (a nonloop edge), where $\Gamma\setminus e$ is Γ with e deleted and Γ/e is Γ with e contracted to a point, and it is (subject to a certain reservation) the most general additive and multiplicative invariant, as Tutte has shown [19, 20]—a result generalized to matroids by Brylawski [2]. All three polynomials satisfy a convolution identity of the form

$$Q_\Gamma(\lambda+\mu, v) = \sum_{X \subseteq N} Q_{\Gamma:X}(\lambda, v)\, Q_{\Gamma:N\setminus X}(\mu, v), \tag{$*$}$$

where N is the node set of Γ and $\Gamma : X$ denotes the subgraph induced on X (see [20, Eq. (14)]). Also, they have expansions in terms of partitions of N and falling factorials $(\lambda)_k = \lambda(\lambda-1)\cdots(\lambda-k+1)$, in terms of chains of node sets, and in terms of connected subgraphs.

All these properties generalize from ordinary to biased graphs. A *biased graph* Ω consists of an underlying graph Γ and a *linear subclass* $\mathscr{B}$ of the circles of Γ (edge sets of simple, closed paths); that is, if the union of two circles in $\mathscr{B}$ is a theta graph, then the third circle in the union belongs to $\mathscr{B}$. We call an edge set *balanced* if every circle in it belongs to $\mathscr{B}$ (and it does not contain certain special edges—see Section 1). Biased graphs were introduced in Part I of this series [33]. An ordinary graph can be treated as a balanced biased graph. A biased graph is a combinatorial abstraction of a *gain graph*, a graph whose edges are labelled by elements of a group. If the group is the sign group we have a *signed graph*. We cannot define coloring of biased graphs in general, but we can provide analogs of the algebraic definitions of the chromatic, dichromatic, and Whitney number polynomials, and they enjoy all the properties listed above. For instance, they are essentially definable through the "bias matroid" of Ω introduced in Part II [34] and are closely related to invariants of the "lift matroid," also treated in Part II; each matroid generalizes in a different way the ordinary graphic matroid. There is even an extension of coloring theory if we restrict to gain graphs with finite group.

Actually, a biased graph has not one but two versions of each polynomial. One is defined by sums of monomials over all edge subsets; this one gives us invariants of the bias and lift matroids but it is relatively hard to compute. The other is defined by sums over balanced edge sets and is easier to compute (sometimes very easy indeed, as in the case of Dowling lattices; see Example 6.7). This leads to the really novel feature of biased-graph invariant theory: the balanced expansion formulas by which one is enabled to calculate the unrestricted polynomials in terms of the easier, balanced ones. We have already exploited this phenomenon in studying examples of signed graphs in [30] and plan to generalize that work to various kinds of gain graphs and their associated geometric lattices in future articles (of which [36] is a forerunner).

In this article we develop a wide variety of formulas. In part this is to show how many properties of ordinary graph polynomials generalize to biased graphs, in line with our belief that biased graphs are a natural domain for doing much of graph theory. But we are also interested in applications to geometry. Suppose, for instance, we dissect Euclidean space by means of finitely many hyperplanes having equations of the forms $x_i = 0$ and $x_i = ax_j$. Then there is a biased graph that (through the bias matroid) represents the intersection structure of the hyperplanes and by means of whose Whitney number polynomial we can enumerate the cells of each dimension formed by the dissection. This and other examples will be treated in Part IV [35]. (For special cases see [27, 31].)

An outline: After preliminary definitions (Section 1) we examine invariants of biased graphs satisfying a key convolution property analogous to (*) above. Although our real interest is in biased graphs, indeed mainly in gain graphs, we find it clearer and more natural to develop the convolution theory for a further generalization we call a "two-ideal graph," of which a biased graph is a special type. This is done in Section 2. In Section 3 we define the polynomials in which we are chiefly interested and also certain convenient four-variable polynomials q_Ω and q_Ω^{b} of which all the polynomials of interest are evaluations. We also demonstrate that the dichromatic polynomial (and its balanced counterpart) of a biased graph is a universal additive and multiplicative invariant, as for ordinary graphs. (We actually generalize slightly to "linear" invariants, since that is more natural for the polynomials of greatest interest.) We apply linearity of the normalized dichromatic polynomial in Section 4 to prove that in a gain graph with finite gain group one has exact analogs of the coloring interpretations of the chromatic, dichromatic, and Whitney number polynomials of an ordinary graph. In Section 5 we show exactly how the biased-graph polynomials permit one to calculate invariants of the bias and lift matroids and the related complete lift matroid. These two sections are intended to justify studying biased graphs and their polynomials, especially (for Section 5) in conjunction with the matroidal geometry to be discussed in Part IV.

The heart of this paper is the first half of Section 6, in which by specializing the convolutions of Section 2 we reduce the computation of the unrestricted polynomials to that of balanced ones. In most of the remaining sections we apply convolution and balanced expansion to deduce various kinds of identities. In Sections 9 and, particularly, 10 we employ our results to obtain expressions for the Whitney numbers, and some other invariants, of the matroids, especially the bias matroid. Then we give in Section 11 a brief discussion of "full" biased graphs, in which every node carries an unbalanced edge. By evaluating the beta invariant we characterize those full graphs whose bias matroid is a series–parallel network.

We carry through most of the paper several examples (introduced in Section 3) to illustrate the theory. Simple ones are a forest, a balanced circle, an unbalanced circle, and a balanced complete graph. A more substantial example is an arbitrary *contrabalanced* graph, in which no circle is balanced. Still more substantial is a family of gain graphs (and nongain analogs) whose matroid lattices include Dowling's lattices and two kinds of generalizations; among these lattices are the intersection lattices of arrangements of hyperplanes dual to the root systems B_n and D_n. These are Examples 1 to 8 in Sections 3 through 10. (Although every example does not appear in every section, each one keeps the same number throughout.) We conclude with two last general examples in Section 12, briefly treated, and in Section 13 all seven biased graphs based on K_4.

1. PRELIMINARIES

In this article, all graphs and matroids are finite. We assume some acquaintance with matroid invariant theory as in [6, 23]. Although this is Part III of a series, we include enough definitions that the reader will not have to refer back to Parts I and II in order to understand the statements of the results (with one exception involving contraction of a biased graph; see Part I). But in the proofs we assume familiarity with Parts I and II. We define some new things: two-ideal graphs and (in Section 3) various polynomials associated with a biased graph. In referring to earlier parts we employ a prefixed Roman numeral in the style "Theorem II.2.5."

If N is a set, Π_N is the set of partitions of N and $\Pi_N^\dagger$ is that of *partial partitions* of N, that is, partitions of subsets of N (including the empty set, whose only partition is the null partition $\varnothing$). The *support* supp π of $\pi \in \Pi_N^\dagger$ is the union of its blocks. We write $n = \# N$, the cardinality of N, and Π_n for Π_N.

The falling factorial $(\lambda)_k$ is $\lambda(\lambda-1)\cdots(\lambda-k+1)$ if $k>0$ and 1 if $k=0$.

The Stirling numbers of the first and second kinds are denoted by $s(n, k)$ and $S(n, k)$. We interpret them to be zero if $k > n$ or $k < 0$.

By Γ we always mean a graph (N, E) with node set $N = N(\Gamma)$ and edge set $E = E(\Gamma)$. We always let $n = \# N$. If $X \subseteq N$ and $S \subseteq E$, we write $X^c = N \setminus X$ and $S^c = E \setminus S$. Edges are of four kinds: a *link* has two distinct endpoints, a *loop* has two coincident endpoints, a *half edge* has one endpoint, and a *loose edge* has no endpoints. We write $\omega(e)$ for the multiset of endpoints of an edge e. An *ordinary graph* has no half or loose edges. A *circle* is the edge set of a simple closed walk of positive length. (A loose or half edge cannot belong to a circle.) If $X \subseteq N$, $\Gamma{:}X$ is the subgraph induced by X; it equals $(X, E{:}X)$, where $E{:}X$ consists of all edges, except loose edges, having all their endpoints in X. We call X *stable* if $E{:}X = \varnothing$. If $\pi \in \Pi_N^\dagger$, $\Gamma{:}\pi = \bigcup \{(\Gamma{:}B) \colon B \in \pi\}$. By $\Pi(\Gamma)$ we mean the set of those $\pi \in \Pi_N$ such that $\Gamma{:}B$ is connected for each $B \in \pi$. By $c(\Gamma)$ we mean the number of connected components of Γ, excluding loose edges, which we regard as belonging to no component. (We should perhaps say "node component" to exclude loose edges, but we omit the modifier because we need no other concept of component.) For $S \subseteq E$, $c(S)$ is the number of components of the spanning subgraph (N, S). But if $S \subseteq E{:}X$, where $X \subseteq N$, then $c(S)$ means $c(X, S)$. (The context will make clear which is meant.)

A particular graph is K_n, the complete (simple) graph on n nodes. Another is the n-node circle graph C_n. If Γ is an ordinary graph, $\Gamma^\bullet$ means Γ with a half edge at every node. The complement of a simple graph Γ is Γ^c.

A *biased graph* $\Omega = (\Gamma, \mathscr{B})$ consists of a graph $\Gamma = \|\Omega\|$ and a linear subclass $\mathscr{B} = \mathscr{B}(\Omega)$ of the circles of Γ. A subgraph or edge set in Ω is called *balanced* if every circle in it belongs to $\mathscr{B}$ and it contains no half edge. It is *contrabalanced* if no circle in it belongs to $\mathscr{B}$ and it contains no loose edge. We always let Ω denote a biased graph whose underlying graph is Γ; thus Ω has n nodes, edge set E, $c(\Omega) = c(\Gamma)$ components, etc. An unbiased graph Γ becomes biased if we declare every circle balanced; this biased graph is denoted by $[\Gamma]$. If Ω is a biased graph, $\Omega^\bullet$ means Ω with a half edge or unbalanced loop added at every node not already supporting one. We call $\Omega^\bullet$ *full*.

A generalization of a biased graph is a *two-ideal graph* $\mathbf{\Omega}$. This consists of an underlying graph $\Gamma = \|\mathbf{\Omega}\|$ and two order ideals of edge sets, $\mathscr{I}_0 = \mathscr{I}_0(\mathbf{\Omega})$ and $\mathscr{I}_1 = \mathscr{I}_1(\mathbf{\Omega})$ (that is, $S \subseteq T \in \mathscr{I}_i$ implies $S \in \mathscr{I}_i$), subject to the following conditions: $\varnothing \in \mathscr{I}_1 \subseteq \mathscr{I}_0$, $\{e\} \in \mathscr{I}_0$ for each edge e and $\{l_\infty\} \in \mathscr{I}_1$ for each loose edge l_∞ (nontriviality conditions), and the multiplicative property that, if $S \subseteq E$ has all its connected components in $\mathscr{I}_i$, then $S \in \mathscr{I}_i$. We call an edge set *balanced* if it belongs to $\mathscr{I}_1$. A biased graph Ω clearly determines a two-ideal graph $\mathbf{\Omega}$ by letting $\mathscr{I}_0 = \mathscr{P}(E)$ and $\mathscr{I}_1$ consist of the balanced edge sets in Ω. Conversely, $\mathbf{\Omega}$ determines $\mathscr{B}$ and, hence, Ω. So we may regard a biased graph as a particular kind of two-ideal graph. We know from the work of Dowling and Kelly [9] that a biased graph is characterized as a two-ideal graph in which $\mathscr{I}_0 = \mathscr{P}(E)$ and $\mathscr{I}_1$ is a *modular ideal of sets*: if $S, T \in \mathscr{I}_1$ and $c(S) + c(T) = c(S \cup T) + c(S \cap T)$, then $S \cup T \in \mathscr{I}_1$.

If $\mathbf{\Omega}$ is a two-ideal graph, for instance a biased graph, we write

$$\pi_b(\mathbf{\Omega}) = \{B \subseteq N : B \text{ is the node set of a balanced component of } \mathbf{\Omega}\},$$
$$b(\mathbf{\Omega}) = \#\pi_b(\mathbf{\Omega}).$$

A subgraph (X, S), or an edge set S (regarded as a spanning subgraph), supports a two-ideal structure $\mathbf{\Psi}$ by $\mathscr{I}_i(\mathbf{\Psi}) = \{R \subseteq S : R \in \mathscr{I}_i\}$. Thus induced subgraphs $\mathbf{\Omega}{:}X$ and partition-induced subgraphs $\mathbf{\Omega}{:}\pi$ are defined on $\Gamma{:}X$ or $\Gamma{:}\pi$. By $\mathscr{I}_i{:}X$ we mean $\mathscr{I}_i(\mathbf{\Omega}{:}X)$. Also, $b(S)$ means the number of balanced components of the two-ideal subgraph on (N, S). If e is a balanced edge of $\mathbf{\Omega}$, the *contraction* $\mathbf{\Omega}/e$ equals $(\Gamma/e, \mathscr{I}_0/e, \mathscr{I}_1/e)$, where $\mathscr{I}_i/e = \{S \backslash e : S \in \mathscr{I}_i$ and $e \in S\}$. We do not define other contractions of two-ideal graphs.

Suppose that a biased graph Ω is the union of subgraphs Ω_1 and Ω_2. We call Ω their *disjoint union* (written $\Omega = \Omega_1 \cup \Omega_2$) if they are disjoint, their *one-point union* at a node v (written $\Omega = \Omega_1 \cup_v \Omega_2$) if their intersection consists only of the node v. These definitions apply equally well to unbiased graphs. For two-ideal graphs we must specify more. A two-ideal graph $\mathbf{\Omega}$ is the *disjoint union* of $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$ if $\|\mathbf{\Omega}\| = \|\mathbf{\Omega}_1\| \cup \|\mathbf{\Omega}_2\|$ and $\mathscr{I}_i(\mathbf{\Omega}) = \{S_1 \cup S_2 : S_1 \in \mathscr{I}_i(\mathbf{\Omega}_1)$ and $S_2 \in \mathscr{I}_i(\mathbf{\Omega}_2)\}$ for $i = 0, 1$. (The one-point union is similar, but we do not need it.)

An *isomorphism of graphs* Γ_1 and Γ_2 is a mapping $f: N_1 \cup E_1 \to N_2 \cup E_2$ such that $f|_{N_1}$ and $f|_{E_1}$ are bijections to N_2 and E_2, respectively, and for each edge $e \in E_1$ we have f(set of endpoints of e with multiplicity) = the set of endpoints of $f(e)$ with multiplicity. (We always assume N and E are disjoint sets.) An *isomorphism of two-ideal graphs* $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$ is an isomorphism $f: \Gamma_1 \to \Gamma_2$ under which $f(S) \in I_i(\mathbf{\Omega}_2)$ if and only if $S \in I_i(\mathbf{\Omega}_1)$.

2. Multiplicative Functions of Two-Ideal Graphs

Let $\mathcal{O}$ be a *hereditary* class of two-ideal graphs; that is, $\mathcal{O}$ is nonempty and any subgraph of a member of $\mathcal{O}$ is again in $\mathcal{O}$. A function on $\mathcal{O}$ is an *invariant* on $\mathcal{O}$ if it takes the same value on isomorphic graphs. A function F from $\mathcal{O}$ to a commutative ring with unity $\mathscr{A}$ is *multiplicative* if $F(\varnothing) \neq 0$ and

$$F(\mathbf{\Omega}) = F(\mathbf{\Omega}_1)\, F(\mathbf{\Omega}_2)$$

whenever $\mathbf{\Omega} \in \mathcal{O}$ is the disjoint union of $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$. Since we do not forbid $\mathbf{\Omega}_1$ or $\mathbf{\Omega}_2$ to be the null graph, the subring generated by the values of a multiplicative invariant F has multiplicative identity element $F(\emptyset)$.

A *function of pairs* on $\mathcal{O}$ is a function $f(\mathbf{\Omega}, S)$ defined for all pairs $(\mathbf{\Omega}, S)$, where $\mathbf{\Omega} \in \mathcal{O}$ and $S \in \mathscr{I}_0(\mathbf{\Omega})$. It is *invariant* if it takes the same value on isomorphic pairs, where $(\mathbf{\Omega}_1, S_1)$ and $(\mathbf{\Omega}_2, S_2)$ are considered *isomorphic* if there is an isomorphism of $\mathbf{\Omega}_1$ with $\mathbf{\Omega}_2$ under which S_1 corresponds with S_2. A function from pairs into a commutative ring is *multiplicative* if

$$f(\mathbf{\Omega}, S) = f(\mathbf{\Omega}_1, S \cap E_1)\, f(\mathbf{\Omega}_2, S \cap E_2)$$

whenever $\mathbf{\Omega} \in \mathcal{O}$ is the disjoint union of $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$. Observe that $f(\emptyset, \emptyset)$ is the identity in the subring generated by the image of a multiplicative invariant f of pairs. Given a function f of pairs, let

$$\hat{f}_i(\mathbf{\Omega}; \lambda) = \sum_{S \in \mathscr{I}_i} f(\mathbf{\Omega}, S)\, \lambda^{b(S)} \qquad \text{for} \quad i = 0, 1.$$

Evidently, $\hat{f}_0$ and $\hat{f}_1$ are invariant if f is invariant and multiplicative if f is multiplicative.

Let f be a multiplicative invariant of pairs on $\mathcal{O}$, a hereditary class of two-ideal graphs. If $\mathbf{\Omega} = \mathbf{\Omega}_1 \cup \mathbf{\Omega}_2$, obviously

$$\hat{f}_i(\mathbf{\Omega}; \lambda) = \hat{f}_i(\mathbf{\Omega}_1; \lambda)\, \hat{f}_i(\mathbf{\Omega}_2; \lambda) \tag{2.1}$$

for $i = 0, 1$. Let $\mathbf{l}_\infty$ denote the two-ideal graph consisting of just a loose edge l_∞. Then

$$\hat{f}_i(\mathbf{\Omega}; \lambda) = [f(\mathbf{l}_\infty, \{l_\infty\}) + 1]^{\varepsilon_\infty}\, \hat{f}_i(\mathbf{\Omega} \backslash E_\infty; \lambda) \tag{2.2}$$

for $i = 0, 1$, if $\mathbf{\Omega}$ has ε_∞ loose edges and E_∞ is the set of loose edges.

Now suppose that $\mathbf{\Omega}$ has no loose edges. Then

$$\hat{f}_i(\mathbf{\Omega}; \lambda) = \sum_{W \subseteq N} \hat{f}_1(\mathbf{\Omega}{:}W; \lambda - \mu)\, \hat{f}_i(\mathbf{\Omega}{:}W^c; \mu) \tag{2.3}$$

for $i = 0, 1$. (This formula does not require f to be invariant.) To prove (2.3) we evaluate $\hat{f}_i(\mathbf{\Omega}; \lambda)$. It equals

$$\sum_{S \in \mathscr{I}_i} \sum_{\omega \subseteq \pi_b(S)} f(\mathbf{\Omega}{:}W, S{:}W)\, f(\mathbf{\Omega}{:}W^c, S{:}W^c)(\lambda - \mu)^{\#\omega} \mu^{b(S) - \#\omega},$$

where $W = \operatorname{supp}\,\omega$,

$$= \sum_{W \subseteq N} \sum_{S_1 \in \mathscr{I}_1 : W} f(\mathbf{\Omega}\!:W, S_1)(\lambda - \mu)^{b(S_1)} \sum_{S_2 \in \mathscr{I}_i : W^c} f(\mathbf{\Omega}\!:W^c, S_2)\,\mu^{b(S_2)},$$

which is the right-hand side of (2.3).

Observe that, if $\mu = 0$ in (2.3), then W need only range over node subsets for which W^c belongs to

$$\mathscr{U}(\mathbf{\Omega}) = \{Z \subseteq N : b(\mathbf{\Omega}\!:Z) = 0\}.$$

Incidentally, $\hat{f}_0$ satisfies a convolution identity as well. To prove this, simply look at the two-ideal graph $\mathbf{\Omega}'$ on $\|\mathbf{\Omega}\|$ in which $\mathscr{I}'_0 = \mathscr{I}'_1 = \mathscr{I}_0$ and apply (2.3). (The same trick, with $\mathscr{I}'_0 = \mathscr{I}'_1 = \mathscr{I}_1$, derives (2.3) for $i = 1$ from the mixed convolution, where $i = 0$. Thus we may regard the latter as fundamental.)

Equation (2.3) is best understood in terms of the incidence algebra of the set of induced subgraphs of $\mathbf{\Omega}$ or, equivalently, of $\mathscr{P}(N)$. (For the incidence algebra of a partially ordered set see [16, Section 3]. I am indebted to Richard Stanley for suggesting its use.) The incidence algebra, call it $\mathrm{IA}(N)$, consists of all functions $\alpha : \mathscr{P}(N) \times \mathscr{P}(N) \to \mathscr{A}$ such that $\alpha(X, Y) = 0$ if $X \not\subseteq Y$. Its multiplication is by convolution:

$$(\alpha * \beta)(X, Y) = \sum_{X \subseteq W \subseteq Y} \alpha(X, W)\,\beta(W, Y).$$

The Kronecker delta $\delta(X, Y)$ is the multiplicative identity. An $\alpha \in \mathrm{IA}(N)$ also multiplies a function $F : \mathscr{P}(N) \to \mathscr{A}$ on the right or left:

$$(\alpha * F)(X) = \sum_{W \supseteq X} \alpha(X, W)\,F(W),$$

$$(F * \alpha)(Y) = \sum_{W \subseteq Y} F(W)\,\alpha(W, Y).$$

Given a fixed two-ideal graph $\mathbf{\Omega}$, we define functions $\alpha_0(\lambda) : \mathscr{P}(N) \to \mathscr{A}$ and $\alpha_1(\lambda) \in \mathrm{IA}(N)$ by

$$\alpha_0(\lambda)(W) = \hat{f}_0(\mathbf{\Omega}\!:W; \lambda),$$

$$\alpha_1(\lambda)(X, Y) = \begin{cases} \hat{f}_1(\mathbf{\Omega}\!:(Y \backslash X); \lambda) & \text{if } X \subseteq Y, \\ 0 & \text{if } X \not\subseteq Y. \end{cases}$$

Then (2.3) becomes a pair of product rules:

$$\alpha_0(\lambda) = \alpha_1(\lambda - \mu) * \alpha_0(\mu) = \alpha_0(\mu) * \alpha_1(\lambda - \mu), \tag{2.4a}$$

$$\alpha_1(\lambda) = \alpha_1(\lambda - \mu) * \alpha_1(\mu). \tag{2.4b}$$

In particular, $\alpha_1(\lambda)^k = \alpha_1(k\lambda)$ if $k = 1, 2, 3, ...$; $\alpha_1(0) = \delta = \alpha_1(\lambda)^0$; and $\alpha_1(\lambda)\,\alpha_1(-\lambda) = \alpha_1(0) = \delta$, whence $\alpha_1(\lambda)^{-1} = \alpha_1(-\lambda)$. Thus $\{\alpha_1(\lambda) : \lambda \in \mathscr{A}\}$ is a one-parameter subgroup of $\mathrm{IA}(N)^\times$ which is a homomorphic image of $\mathscr{A}^+$.

3. Invariants of Biased Graphs

The main objects of our study are certain polynomials associated with a biased graph Ω. The most general is the four-variable polynomial

$$q_\Omega(w, x, \lambda, v) = \sum_{R \subseteq S \subseteq E}\sum w^{\#R} x^{n-b(R)} \lambda^{b(S)} v^{\#(S \setminus R)},$$

which we call the *polychromatic polynomial* (briefly, *polychromial*) of Ω. The *balanced polychromatic polynomial* is

$$q^{\mathrm{b}}_\Omega(w, x, \lambda, v) = \sum_{\substack{R \subseteq S \subseteq E \\ \text{balanced}}}\sum w^{\#R} x^{n-b(R)} \lambda^{b(S)} v^{\#(S \setminus R)}.$$

The (balanced) polychromial of an unbiased graph Γ is defined to be that of $[\Gamma]$, with similar definitions for other polynomials.

All our polynomials occur in pairs, an unrestricted (or "unbalanced") one and a balanced one. In order to combine parallel definitions and statements about them we employ a bracket shorthand. For example, we define the *[balanced] dichromatic polynomial* to be

$$Q^{[\mathrm{b}]}_\Omega(u, v) = \sum_{\substack{S \subseteq E \\ [\text{balanced}]}} u^{b(S)} v^{\#S - n + b(S)};$$

this defines two polynomials, Q_Ω and Q^{b}_Ω, where Q_Ω equals the sum without the bracketed restriction to balanced sets S and Q^{b}_Ω equals the sum over balanced S. The dichromatic polynomials Q_Ω and Q^{b}_Ω generalize the dichromatic polynomial of an ordinary graph Γ, which equals $Q_{[\Gamma]}(u, v)$. (This was defined by Tutte [20] and studied earlier by Whitney [24, 25] in terms of the coefficients m_{ij} of the terms $u^i v^j$.) It is more convenient for us to work with the *normalized [balanced] dichromatic polynomial*

$$\bar{Q}^{[\mathrm{b}]}_\Omega(\lambda, v) = q^{[\mathrm{b}]}_\Omega(0, x, \lambda, v) = v^n Q^{[\mathrm{b}]}_\Omega(u, v) = \sum_{\substack{S \subseteq E \\ [\text{balanced}]}} \lambda^{b(S)} v^{\#S},$$

where $\lambda \equiv uv$ (a convention we preserve throughout this paper). Here the value of x is immaterial. There are other evaluations of q_Ω that reduce to $\bar{Q}_\Omega$. For example,

$$q_\Omega^{[b]}(w, 1, \lambda, v) = \bar{Q}_\Omega^{[b]}(\lambda, w+v). \tag{3.1a}$$

Thus if $x = 1$, w and v are essentially equivalent variables. This trick will be of use in many places. Other examples are

$$q_\Omega(w, x, 1, v) = x^n (v+1)^{\# E} \bar{Q}_\Omega\left(\frac{1}{x}, \frac{w}{v+1}\right), \tag{3.1b}$$

which will be of great help in Theorem 6.1 and Example 8.5, and

$$q_\Omega^{[b]}(w, x, \lambda, 0) = x^n \bar{Q}_\Omega^{[b]}\left(\frac{\lambda}{x}, w\right). \tag{3.1c}$$

Also,

$$q_\Omega^{[b]}(w, x, \lambda, v) = \sum_{\substack{R \subseteq E \\ [\text{balanced}]}} w^{\# R} x^{n-b(R)} \bar{Q}_{\Omega/R}^{[b]}(\lambda, v) \tag{3.1d}$$

and, as a special case of some interest in Example 8.5,

$$q_\Omega(w, x, 1, -1) = w^{\# E} x^{n-b(\Omega)}. \tag{3.1e}$$

Still another formula of this type is (3.9) below.

Other polynomials are $q_\Omega^{[b]}(w, x, \lambda, -1)$, which we call the [*zero-free* or *balanced*] *coloration generating polynomial* because of its role in Theorem 4.1, the [*balanced*] *chromatic polynomial*

$$\chi_\Omega^{[b]}(\lambda) = \bar{Q}_\Omega^{[b]}(\lambda, -1),$$

and the [*balanced*] *Whitney number polynomial*

$$w_\Omega^{[b]}(x, \lambda) = q_\Omega^{[b]}(1, x, \lambda, -1).$$

The [*balanced*] *Whitney numbers* of Ω (of the *first kind*) are the coefficients of the latter polynomials. The *simply indexed* number $w_j^{[b]}(\Omega)$ is the coefficient of λ^{n-j} in $\chi_\Omega^{[b]}(\lambda)$. The *doubly indexed* number $w_{ij}^{[b]}(\Omega)$ is the coefficient of $x^i \lambda^{n-j}$ in $w_\Omega^{[b]}(x, \lambda)$. It will follow from Theorem 5.1 that the simply indexed numbers are zero if Ω has a loose edge or balanced loop, and otherwise $w_j^{[b]}(\Omega) = w_{0j}^{[b]}(\Omega)$; also that $(-1)^{j-i} w_{ij}^{[b]}(\Omega) > 0$ if $0 \leqslant i \leqslant j \leqslant n - b(\Omega)$ (for w_{ij}) or $\leqslant n - c(\Omega)$ (for w_{ij}^{b}). The *simply indexed Whitney numbers of the second kind* of Ω are $W_j^{[b]}(\Omega) = w_{jj}^{[b]}(\Omega)$.

In applications the most important polynomials are the balanced and unbalanced chromatic, dichromatic, and Whitney number polynomials. Each is a specialization of the coloration generating polynomial $q_{\Omega}^{[b]}(w, x, \lambda, -1)$. This would therefore be sufficient for our needs. Furthermore, its unbalanced version is usually much easier to evaluate than the full polychromial, as we shall see in Theorem 6.1 and elsewhere. Nevertheless we do not restrict ourselves to $v = -1$. One reason is the convenience of the evaluation $\bar{Q}^{[b]}(\lambda, v) = q^{[b]}(0, x, \lambda, v)$; another is that, especially in doing balanced polynomials, there is rarely much to be gained by holding $v = -1$. Philosophically, moreover, since the -1 in the coloration generating polynomial is inescapable, we get more satisfactory (as well as stronger) results by letting it be a variable v. Whether or not the polychromial will turn out to be significant in its own right is not yet possible to say.

A *multiplicative invariant of biased graphs*, or of *pairs* (Ω, S), where Ω is a biased graph and $S \subseteq E(\Omega)$, is a multiplicative invariant F, or f, as in Section 2 on $\mathcal{C}$ = the class of biased graphs. Since edge contractions of biased graphs exist, we can define two more properties of a function F from biased graphs to a commutative ring. We call F *linear* if

$$F(\Omega) = \alpha F(\Omega \backslash e) + \beta F(\Omega / e) \tag{3.2}$$

for every edge e, and *balance-linear* if (3.2) holds for every balanced edge e, where α and β are constants connected with F. We call F *strongly balance-linear* if it is balance-linear and $F(\Omega)$ is unaffected by deleting the unbalanced edges from Ω. We want to classify all invariants of biased graphs that are linear (or, strongly balance-linear) and multiplicative.

But first we observe how by an appropriate choice of function f of pairs (Ω, S) we can make $\hat{f}_0$ and $\hat{f}_1$ be the unbalanced and balanced versions of the normalized or unnormalized dichromatic polynomial, the Whitney number polynomial, etc. We get

$$\bar{Q}_{\Omega}^{[b]}(\lambda, v) = \hat{f}_i(\Omega; \lambda) \qquad \text{if} \quad f(\Omega, S) = v^{\# S} \tag{3.3}$$

(where by $\bar{Q}^{[b]} = \hat{f}_i$ we mean to abbreviate $\bar{Q} = \hat{f}_0$ and $\bar{Q}^b = \hat{f}_1$),

$$Q_{\Omega}^{[b]}(u, v) = \hat{f}_i(\Omega; uv) \qquad \text{if} \quad f(\Omega, S) = v^{\# S - n}, \tag{3.4}$$

$$\chi_{\Omega}^{[b]}(\lambda) = \hat{f}_i(\Omega; \lambda) \qquad \text{if} \quad f(\Omega, S) = (-1)^{\# S}. \tag{3.5}$$

In these cases $f([l_\infty], \{l_\infty\}) = v$, v, and -1, respectively, where $[l_\infty]$ is the graph of a loose edge. We also get

$$q_{\Omega}^{[b]}(w, x, \lambda, v) = \hat{f}_i(\Omega; \lambda) \qquad \text{if} \quad f(\Omega; S) = \sum_{R \subsetneq S} w^{\# R} x^{n - b(R)} v^{\# (S \backslash R)}, \tag{3.6}$$

with $f([l_\infty], \{l_\infty\}) = v + w + 1$; and

$$w_\Omega^{[b]}(x, \lambda) = \hat{f}_i(\Omega; \lambda) \quad \text{if} \quad w = 1 \quad \text{and} \quad v = -1 \text{ in the preceding,} \quad (3.7)$$

with $f([l_\infty], \{l_\infty\}) = 1$.

THEOREM 3.1. *$\bar{Q}_\Omega(\lambda, v)$ is a linear, multiplicative invariant of biased graphs. $\bar{Q}^{\mathrm{b}}_\Omega(\lambda, v)$ is a strongly balance-linear, multiplicative invariant of biased graphs. In both cases $\alpha = 1$ and $\beta = v$.*

Proof. Multiplicative invariance follows from (3.3). For linearity we split the defining sum in half:

$$\bar{Q}_\Omega^{[b]}(\lambda, v) = \sum_{\substack{S \subseteq E \setminus e \\ [\text{balanced}]}} \lambda^{b(S)} v^{\# S} + v \sum_{\substack{S \subseteq E \setminus e \\ [S \cup \{e\} \text{ balanced}]}} \lambda^{b(S \cup \{e\})} v^{\# S}.$$

The former sum equals $\bar{Q}_{\Omega \setminus e}^{[b]}(\lambda, v)$. To handle the latter we observe that, by Lemma I.4.3, $b_\Omega(S \cup \{e\}) = b_{\Omega/e}(S)$. Therefore, the second sum equals

$$\sum_{\substack{S \subseteq E(\Omega/e) \\ [\text{balanced}]}} \lambda^{b(\Omega/e | S)} v^{\# S} = \bar{Q}_{\Omega/e}^{[b]}(\lambda, v). \quad ∎$$

COROLLARY 3.2. *Q_Ω and Q^{b}_Ω are multiplicative invariants of biased graphs. If Ω is a biased graph, then Q_Ω satisfies (3.2) for e a link or unbalanced edge and Q^{b}_Ω satisfies (3.2) for e a link, in both cases with $\alpha = \beta = 1$.*

COROLLARY 3.3. *For an edge e in a biased graph Ω we have*

$$\chi_\Omega(\lambda) = \chi_{\Omega \setminus e}(\lambda) - \chi_{\Omega/e}(\lambda),$$
$$|w_k(\Omega)| = |w_k(\Omega \setminus e)| + |w_{k-1}(\Omega/e)|,$$

and if e is balanced,

$$\chi^{\mathrm{b}}_\Omega(\lambda) = \chi^{\mathrm{b}}_{\Omega \setminus e}(\lambda) - \chi^{\mathrm{b}}_{\Omega/e}(\lambda),$$
$$|w^{\mathrm{b}}_k(\Omega)| = |w^{\mathrm{b}}_k(\Omega \setminus e)| + |w^{\mathrm{b}}_{k-1}(\Omega/e)|.$$

Now we come to the main results. An arbitrary balance-linear, multiplicative invariant F of biased graphs is determined by α, β, and its values on the one-node graphs A_{pq} having p half edges and q unbalanced loops, where $p, q \geqslant 0$. If F is linear, it is determined by α, β, and $F(K_1)$. If $F(\Omega)$ is strongly balance-linear it is also determined by the same three quantities.

THEOREM 3.4. *Let F be a linear, multiplicative invariant of biased graphs with values in an integral domain and with linearity constants α and β in* (3.2). *Assume that F is not identically zero. If $\alpha \neq 0$, then*

$$F(\Omega) = \alpha^{\# E} \bar{Q}_\Omega(\lambda, \beta/\alpha),$$

where $\lambda = F(K_1)$. We have $\alpha + \beta = F([l_\infty])$ and $\lambda\alpha + \beta = F(K_1^\bullet)$. If $\alpha = 0$, then $F(\Omega) = \lambda^{b(\Omega)} \beta^{\# E}$, where $\lambda = F(K_1)$. We have $\beta = F([e])$, where $[e]$ is any one-edge biased graph except for the graph of a balanced loop.

THEOREM 3.5. *Let F be a strongly balance-linear, multiplicative invariant of biased graphs with values in an integral domain and with linearity constants α and β. Assume that F is not identically zero. Define $\lambda = F(K_1)$. If $\alpha \neq 0$, then*

$$F(\Omega) = \alpha^{\# E} \bar{Q}^{\mathrm{b}}_\Omega(\lambda, \beta/\alpha).$$

We have $\alpha + \beta = F([l_\infty])$ and $\lambda(\lambda\alpha + \beta) = F([K_2])$. If $\alpha = 0$, then $\beta = 1$ and $F(\Omega) = \lambda^{c(\Omega)}$; or $\beta = 0$ and $F(\Omega) = 0$, except for $F(\varnothing) = 1$ and $F(K_n^c) = \lambda^n$; or else $\beta = F([l_\infty])$ and $F(\Omega) = 0$ if $n > 0$, but $F(\Omega) = \beta^{\# E}$ if $n = 0$.

Proofs. If $\alpha \neq 0$, then the function $G(\Omega) = \alpha^{\# E} \bar{Q}_\Omega(\lambda, \beta/\alpha)$ or $\alpha^{\# E} \bar{Q}^{\mathrm{b}}_\Omega(\lambda, \beta/\alpha)$ agrees with F on K_1, is a multiplicative invariant, and is linear or strongly balance-linear with the same linearity constants as F. Therefore, $F = G$.

If $\alpha = 0$ in Theorem 3.4, a similar argument succeeds.

If $\alpha = 0$ in Theorem 3.5, we evaluate β by considering a biased graph Ω with underlying graph $3K_2$, having parallel edges e, f, g and the one balanced digon $\{e, f\}$. Then $F(\Omega) = \beta F(\Omega/e) = \beta^2 F(\Omega/ef) = \beta^2 F(K_1)$, since Ω/ef is the graph of an unbalanced loop. On the other hand, $F(\Omega) = \beta F(\Omega/g) = \beta F(K_1)$, since Ω/g is one node supporting two unbalanced loops. If $F(K_1) \neq 0$, then $\beta = 0$ or $\beta = 1$. If $\beta = 0$, then $F(\Omega) = 0$ if Ω has an edge and $F(\Omega) = F(K_1)^n$ if Ω has no edges. If $\beta = 1$, then $F(\Omega) = F(K_1)^{c(\Omega)}$. If $F(K_1) = 0$, then $F(\Omega) = 0$ if $n > 0$ and $F(\Omega) = \beta^{\# E}$ if $n = 0$. It is clear that each of these cases yields a valid invariant. ∎

The most important case of each theorem is $\alpha = 1$, since that is the value associated with the dichromatic and chromatic polynomials. We treat $\alpha \neq 1$ as well in order not to be too arbitrarily specialized in defining linearity. If $\alpha \neq 0, 1$, we may think of β/α as calculated in the field of quotients of the integral domain in which F takes values. Then we may reduce to the case $\alpha = 1$ by scaling F to $\alpha^{\# E} F(\Omega)$, which satisfies (3.2) with constants $\alpha' = 1$, $\beta' = \beta/\alpha$. This explains in terms of the principal case $\alpha = 1$ the forms of the invariants. The case $\alpha = 0$ is included for completeness.

COROLLARY 3.6. *A function F on biased graphs is a multiplicative and linear (respectively, strongly balance-linear) invariant of biased graphs whose value on a loose edge (equivalently, provided $\alpha \neq 0$, on a balanced loop) is zero if, and only if,* $F(\Omega) = \alpha^{\# E}\chi_\Omega(F(K_1))$ *(respectively,* $F(\Omega) = \alpha^{\# E}\chi^{\mathrm{b}}_\Omega(F(K_1))$*).*

A simple case of (3.2) is

$$F(\Omega) = F(\Omega \backslash e) + F(\Omega/e). \tag{3.8}$$

We call *F additive* if it satisfies (3.8) for each link and unbalanced edge. We call *F balance-additive* if it satisfies (3.8) for each link and *strongly balance-additive* if, furthermore, $F(\Omega)$ is unaltered by deletion from Ω of the unbalanced edges. Corollary 3.2 says that Q_Ω is a multiplicative and additive invariant; also, Q^{b}_Ω is multiplicative and strongly balance-additive. Both invariants also have the property that converting a balanced loop in Ω to a loose edge does not affect their values. There are converses to these statements.

THEOREM 3.7. *Let F be a multiplicative and additive (or, strongly balance-additive) invariant of biased graphs, such that if Ω has a balanced loop e and Ω_1 is the same but with e changed to a loose edge then* $F(\Omega) = F(\Omega_1)$. *Then F is the evaluation of* $Q_\Omega(\lambda, v)$ *(or,* $Q^{\mathrm{b}}_\Omega(\lambda, v)$*) in which* $\lambda = F(K_1)$ *and* $v = F(\text{loose edge}) - 1$.

Proof. Clearly, F is determined by its values on K_1, $K_1^{\bullet}$, and $[l_\infty]$, according to the same rules as is the [balanced] dichromatic polynomial. The theorem follows. ∎

Theorem 3.7 is weaker than Theorems 3.4 and 3.5 insofar as the hypothesis of additivity is stronger than that of linearity (since it has no adjustable constants α and β). Note that additivity does not require every edge to satisfy the additive equation; on the other hand, we are forced in the theorem to assume that balanced loops and loose edges are equivalent. We state Theorem 3.7 because it is customary in ordinary graph theory (and matroid theory) to treat additive rather than linear invariants.

Although the polychromatic polynomials are not linear, they do have multiplicative properties which can be valuable in computations.

PROPOSITION 3.8. *Suppose that* $\Omega = \Omega_1 \cup \Omega_2$. *Then* $q_\Omega = q_{\Omega_1} q_{\Omega_2}$ *and* $q^b_\Omega = q^{\mathrm{b}}_{\Omega_1} q^{\mathrm{b}}_{\Omega_2}$.

Proof. These are special cases of Eq. (2.1). ∎

PROPOSITION 3.9. *Suppose that* $\Omega = \Omega_1 \cup_v \Omega_2$. *Then* $q^{\mathrm{b}}_\Omega = \lambda^{-1} q^{\mathrm{b}}_{\Omega_1} q^{\mathrm{b}}_{\Omega_2}$. *If* Ω_1 *or* Ω_2 *is balanced, then also* $q_\Omega = \lambda^{-1} q_{\Omega_1} q_{\Omega_2}$.

Proof. For a set $S \subseteq E$, let $S_1 = S \cap E_1$ and $S_2 = S \cap E_2$. The key point in proving both formulas is the observation that $b(S) = b(S_1) + b(S_2) - 1$ if S_1 or S_2 is balanced. To prove this, let $S{:}B_0$ and $S_i{:}B_{i0}$ be the components of S and of $\Omega_i | S_i$ in which v lies. The components of S are those of $\Omega_1 | S_1$ and $\Omega_2 | S_2$, in the same state of balance, except that $S{:}B_0 = S_1{:}B_{10} \cup S_2{:}B_{20}$, which is balanced if and only if both $S_i{:}B_{i0}$ are balanced. ∎

PROPOSITION 3.10. *Let Ω_1 be a biased graph and let Ω be Ω_1 with ε balanced loops and/or loose edges adjoined. We have*

$$q_{\Omega}^{[b]}(w, x, \lambda, v) = (w + v + 1)^{\varepsilon}\, q_{\Omega_1}^{[b]}(w, x, \lambda, v),$$
$$w_{\Omega}^{[b]}(x, \lambda) = w_{\Omega_1}^{[b]}(x, \lambda),$$
$$\chi_{\Omega}^{[b]}(\lambda) = 0^{\varepsilon} \chi_{\Omega_1}^{[b]}(\lambda).$$

Another useful formula expresses the polychromial in terms of the dichromatic polynomials of subgraphs:

$$q_{\Omega}^{[b]}(w, x, \lambda, v) = x^n \sum_{\substack{S \subseteq E \\ [\text{balanced}]}} \bar{Q}_{\Omega|S}^{[b]}\left(\frac{1}{x}, \frac{w}{v}\right) \lambda^{b(S)} v^{\#S}. \tag{3.9}$$

This is a stronger form of (3.1b) in that λ need not be 1, but the expression is correspondingly less simple. Specializing to the Whitney number polynomial gives the interesting formula

$$w_{\Omega}^{[b]}(x, \lambda) = \sum_{\substack{S \subseteq E \\ [\text{balanced}]}} (-1)^{\#S} \lambda^{b(S)} x^n \chi_{\Omega|S}^{[b]}(x^{-1}). \tag{3.10}$$

Equation (3.9) enables us to compute some simple examples.

EXAMPLE 3.1 (*Forests*). Let $F_{n,m}$ denote a forest of order n with m edges and, consequently, $c = n - m$ components (all balanced, of course, so we are really talking about an ordinary forest graph). Directly from the definition we have

$$\bar{Q}_{F_{n,m}}^{[b]}(\lambda, v) = \lambda^{n-m}(\lambda + v)^m.$$

Substitution in (3.9) yields

$$q_{F_{n,m}}^{[b]}(w, x, \lambda, v) = \lambda^{n-m}(\lambda + v + wx)^m.$$

This example is continued in Sections 4, 5, and 6.

EXAMPLE 3.2 (*Balanced circles*). Here we have

$$\bar{Q}^{[b]}_{[C_n]}(\lambda, v) = (\lambda + v)^n + v^n(\lambda - 1),$$

$$q^{[b]}_{[C_n]}(w, x, \lambda, v) = (\lambda + v + wx)^n + (\lambda - 1)(v + wx)^n + \lambda w^n x^{n-1}(1 - x)$$

for $n \geqslant 1$. The former is a standard result, easily proved by induction on n using deletion and contraction. The latter follows from (3.9). We also have

$$w^{[b]}_{[C_n]}(x, \lambda) = (x + \lambda - 1)^n + (\lambda - 1)(x - 1)^n - \lambda x^{n-1}(x - 1).$$

For more on this example see Sections 4, 5, and 6.

EXAMPLE 3.3 (*Unbalanced circles*). The formulas

$$\bar{Q}^{b}_{(C_n, \varnothing)}(\lambda, v) = (\lambda + v)^n - v^n,$$

$$\bar{Q}_{(C_n, \varnothing)}(\lambda, v) = (\lambda + v)^n,$$

$$q^{b}_{(C_n, \varnothing)}(w, x, \lambda, v) = (\lambda + v + wx)^n - (v + wx)^n,$$

$$q_{(C_n, \varnothing)}(w, x, \lambda, v) = (\lambda + v + wx)^n,$$

valid for $n \geqslant 1$, are established as follows: the first by deducting from $\bar{Q}_{[C_n]}$ the term λv^n corresponding to $S = E(C_n)$, the second by adding to the first the term v^n corresponding to $S = E(C_n)$ (now unbalanced), the fourth from the second by (3.9) and the observation that $\# S = n - b(S)$ for all $S \subseteq E(C_n, \varnothing)$, and the third by then subtracting the term for $S = E(C_n)$. We then obtain

$$w^{b}_{(C_n, \varnothing)}(x, \lambda) = (\lambda + x - 1)^n - (x - 1)^n = \bar{Q}^{b}_{(C_n, \varnothing)}(\lambda, x - 1),$$

$$w_{(C_n, \varnothing)}(x, \lambda) = (\lambda + x - 1)^n = \bar{Q}_{(C_n, \varnothing)}(\lambda, x - 1).$$

(The coincidence of the Whitney-number and normalized dichromatic polynomials is due to the fact that the bias matroid (Section 5) of $(C_n, \varnothing)$ is a free matroid. The same holds true in Example 3.1.)

This example continues in Sections 4–7, 9, and 10.

EXAMPLE 3.4 (Example I.6.2) (*Contrabalanced graphs*). This example is $(\Gamma, \varnothing)$. For convenience we here take Γ to be an ordinary graph. The balanced sets are the edge sets of spanning forests. Let $f_i(\Gamma)$ be the number of i-tree spanning forests and $t(\Gamma)$ the number of tree components in Γ. (Note that $f_0(\Gamma) = 0$ if $N \neq \varnothing$ and $f_0(\Gamma) = 1$ if $N = \varnothing$.) Some invariants of $(\Gamma, \varnothing)$ were calculated in [31], but there our main interest was in the lattice of flats of the bias matroid; so there is much left to be done here.

The definition of the balanced polychromial implies that

$$q^{\mathrm{b}}_{(\Gamma,\varnothing)}(w, x, \lambda, v) = \sum_{i=0}^{n} \lambda^i (wx+v)^{n-i} f_i(\Gamma),$$

because $b(R) = c(R) = n - \# R$ when R is a forest. Consequently,

$$Q^{\mathrm{b}}_{(\Gamma,\varnothing)}(u, v) = \sum_{i=0}^{n} u^i f_i(\Gamma),$$

in which v does not appear at all (because $\mathrm{Lat}^{\mathrm{b}}(\Gamma, \varnothing)$ is an ideal in a Boolean algebra; see Section 5).

This example demonstrates well that the unbalanced polychromial may be evaluated from its definition only with difficulty. Anticipating Theorem 6.1, we have the formula

$$\begin{aligned} q_{(\Gamma,\varnothing)}(w, x, \lambda, v) = \sum_{W \subseteq N} \sum_{i=0}^{\# W} & (\lambda-1)^i (wx+v)^{\# W - i} \\ & \times f_i(\Gamma{:}W)\, x^{\# W^c} (v+1)^{\# E{:}W^c} \\ \times \sum_{Z \subseteq W^c} & \left(\frac{w+v+1}{v+1}\right)^{\# E{:}(W^c \setminus Z)} \\ & \times \sum_{j=0}^{\# Z} \left(\frac{1}{x} - 1\right)^j \left(\frac{w}{v+1}\right)^{\# Z - j} f_j(\Gamma{:}Z) \end{aligned}$$

and, more simply,

$$\begin{aligned} q_{(\Gamma,\varnothing)}(w, x, \lambda, -1) = \sum_{W \subseteq N} \sum_{i=0}^{\# W} & (\lambda-1)^i (wx-1)^{\# W - i} \\ & \times f_i(\Gamma{:}W)\, w^{\# E{:}W^c} x^{\# W^c - t(\Gamma{:}W^c)}; \end{aligned}$$

hence, for instance,

$$\bar{Q}_{(\Gamma,\varnothing)}(\lambda, v) = \sum_{W \subseteq N} (v+1)^{\# E{:}W^c} \sum_{i=0}^{\# W} (\lambda-1)^i v^{\# W - i} f_i(\Gamma{:}W),$$

$$w_{(\Gamma,\varnothing)}(x, \lambda) = \sum_{W \subseteq N} x^{\# W^c - t(\Gamma{:}W^c)} \sum_{i=0}^{\# W} (x-1)^{\# W - i} (\lambda-1)^i f_i(\Gamma{:}W).$$

Furthermore,

$$\chi^{\mathrm{b}}_{(\Gamma,\varnothing)}(\lambda) = \sum_{i=0}^{n} (-1)^{n-i} f_i(\Gamma)\, \lambda^i$$

and, again anticipating Theorem 6.1,

$$\chi_{(\Gamma,\varnothing)}(\lambda)=\sum_{i=0}^{n}(\lambda-1)^{i}\sum_{\substack{W\subseteq N\\ W^{c}\text{ stable}}}(-1)^{\#W-i}f_{i}(\Gamma{:}W).$$

One of the more interesting contrabalanced examples is $(K_n, \varnothing)$, the contrabalanced complete graph. Since $\#W=m$ implies that $K_n{:}W=K_m$ and $t(K_n{:}W)=\delta_{m,n-1}$, we have, for instance,

$$\begin{aligned} q_{(K_n,\varnothing)}(w,x,\lambda,-1)=&\sum_{0\leq i\leq m\leq n}\sum\binom{n}{m}w^{\binom{n-m}{2}}x^{n-m}\\ &\times f_i(K_m)(wx-1)^{m-i}(\lambda-1)^{i}\\ &-n(x-1)\sum_{i=0}^{n-1}f_i(K_{n-1})(wx-1)^{n-1-i}(\lambda-1)^{i}. \end{aligned}$$

It is worthwhile recalling the formula of Rényi ([15]; see [14, Theorem 4.1]):

$$f_i(K_n)=\binom{n}{i}n^{n-i-1}\sum_{j=0}^{i}\left(-\frac{1}{2}\right)^{j}\binom{i}{j}(i+j)\frac{(n-i)_j}{n^{j}}.$$

Alas, this evaluation leads to no simplification of the contrabalanced formulas.

We continue Example 3.4 in Sections 4–7, 9, and 10.

EXAMPLE 3.5 (*Balanced complete graphs*). In Section 8 we shall derive formulas for polynomials of K_n (i.e., $[K_n]$) for reference in Examples 3.7 and 6.7 and Section 13.

EXAMPLE 3.6 (Example I.6.7) (*Group expansions*). The *expansion* $\mathfrak{G}\Delta$ of an ordinary graph Δ by a group $\mathfrak{G}$, or more briefly the $\mathfrak{G}$*-expansion* of Δ, is the gain graph (see Section 4) with gain group $\mathfrak{G}$ whose node set is $N(\Delta)$ and whose edge set is $\mathfrak{G}\times E(\Delta)$. An edge (g, e) has the same endpoints as e and has gain g. More precisely, one arbitrarily chooses an orientation $(e; v, w)$ of e and defines the gain of $(g, e; v, w)$ as g; thus the gain of the reversed edge $(g, e; w, v)$ will be g^{-1}. (On the other hand, the edge $(g^{-1}, e; v, w)$ is a different edge of $\mathfrak{G}\Delta$ if $g\neq g^{-1}$; its gain is g^{-1} in the direction from v to w, g from w to v.) The *full* $\mathfrak{G}$*-expansion* of Δ is $(\mathfrak{G}\Delta)^{\bullet}$. For simplicity we shall write $N=N(\Delta)$, $n=\#N(\Delta)$, ge for (g,e), and $\mathfrak{G}\Delta^{\bullet}$ for $(\mathfrak{G}\Delta)^{\bullet}$. Intermediate between $\mathfrak{G}\Delta$ and $\mathfrak{G}\Delta^{\bullet}$ are the partially filled $\mathfrak{G}$-expansions $\mathfrak{G}\Delta^{(H)}$, where $H\subseteq N$, which consist of $\mathfrak{G}\Delta$ and a half edge (or unbalanced loop) at each node $v \in H$. We shall lose no essential generality by supposing henceforth that Δ is simple.

The fundamental fact about a group expansion is that any balanced edge set S in $\mathfrak{G}\Delta$ is obtained by taking $T \subseteq E(\Delta)$ and assigning balanced gains to T. The number of ways to put balanced gains on T is $\gamma^{n-c(T)}$, where $\gamma = \#\mathfrak{G}$. Thus,

$$q^{\mathrm{b}}_{\mathfrak{G}\Delta}(w, x, \lambda, v) = \sum_{S \text{ balanced}} \lambda^{b(S)} \sum_{R \subseteq S} w^{\# R} x^{n-b(R)} v^{\#(S \setminus R)}$$

$$= \sum_{T \subseteq E(\Delta)} \lambda^{c(T)} \gamma^{n-c(T)} \sum_{R \subseteq T} w^{\# R} x^{n-c(R)} v^{\#(T \setminus R)},$$

whence the fundamental formula

$$q^{\mathrm{b}}_{\mathfrak{G}\Delta}(w, x, \lambda, v) = \gamma^n q_\Delta\left(w, x, \frac{\lambda}{\gamma}, v\right). \tag{3.11}$$

This means that the balanced polynomials of a group expansion are determined in a trivial way by those of Δ. For instance,

$$\chi^{\mathrm{b}}_{\mathfrak{G}\Delta}(\lambda) = \gamma^n \chi_\Delta(\lambda/\gamma).$$

(For a nice combinatorial proof see Example 4.6.) It is notable that the polynomials of a group expansion are independent of the structure of the group, as Dowling first discovered in connection with Dowling lattices [8].

Of course, the balanced polynomials of $\mathfrak{G}\Delta^{(H)}$ equal those of $\mathfrak{G}\Delta$.

A variety of other examples based on a group and a graph and not dissimilar to group expansions are treated in [36].

For further treatment of group expansions see Sections 4–6 and 10.

EXAMPLE 3.7 (*Dowling lattices and their relatives*). These lattices, to be defined in Example 5.7, are based on the gain graph $\Phi = \mathfrak{G}K_n^{(p)}$, that is, $\mathfrak{G}K_n^{(H)}$ where $\#H = p$. To illustrate the balanced polynomials, we have (from Example 3.6)

$$\chi^{\mathrm{b}}_{\mathfrak{G}K_n}(\lambda) = \gamma^n \left(\frac{\lambda}{\gamma}\right)_n = \lambda(\lambda - \gamma) \cdots (\lambda - [n-1]\,\gamma),$$

since $\chi_{K_n}(\lambda) = (\lambda)_n$, and

$$w^{\mathrm{b}}_{\mathfrak{G}K_n}(x, \lambda) = \gamma^n w_{K_n}\left(x, \frac{\lambda}{\gamma}\right).$$

(For w_{K_n} see Example 8.5.)

These examples will be further developed in Sections 4–6 and 10.

EXAMPLE 3.8 (*Biased expansions*). Example 3.6 has a combinatorial generalization. A *γ-fold biased expansion* of an ordinary graph Δ, which we write $\Omega = \gamma \cdot \Delta$, has node set $N = N(\Delta)$, edge set $E = [\gamma] \times E(\Delta)$ with endpoint mapping $\omega_{\gamma \cdot \Delta}(i, e) = \omega_\Delta(e)$, and balanced circle class $\mathscr{B}$ to be described in a moment. Let $p: \gamma \cdot \Delta \to \Delta$ be the natural projection. A *lift* of $S' \subseteq E(\Delta)$ is any $S \subseteq E$ such that $p|_S$ is a bijection onto S'. $\mathscr{B}$ may be any class of circles such that (i) no circle of the form $\{(i, e), (j, e)\}$ is balanced; (ii) for each circle $C' \in \mathscr{C}(\Delta)$, edge $e \in C'$, and lift P of $C' \backslash e$, there is a balanced lift C of C' which contains P; and (iii) $\mathscr{B}$ satisfies the theta-graph requirement of biased graphs.

Biased expansions include the biased graphs $[\mathfrak{G}\Delta]$ of group expansions but also nongroup expansions of some base graphs Δ. For instance, every quasigroup of order γ, or equivalently, every Latin square of order γ, gives rise to a $\gamma \cdot K_3$, which is a group expansion—and, indeed, has a gain function at all—if and only if the quasigroup is isotopic to a group—that is, the Latin square is, up to rearranging rows and columns, a group multiplication table. A similar construction applies to C_n with $n \geqslant 4$. On the other hand, Kahn and Kung have proved that a biased expansion of K_n for $n \geqslant 4$ must be a group expansion. (See the argument of [12, Section 7, pp. 490–492].) It is not known exactly which graphs have a nongroup biased expansion.

Our concern here is with the invariants of biased expansions (and in Section 6 their supersolvability). It suffices to say that all formulas and all proofs we give for group expansions, except those which explicitly involve coloring, remain valid for biased expansions. The only point that needs a separate proof is that the number of balanced lifts S of $T \subseteq E(\Delta)$ equals $\gamma^{n - c(T)}$. Any such lift S is obtained by lifting a maximal forest in T to, say, $F \subseteq E$ and taking $S = p^{-1}(T) \cap \mathrm{bcl}(F)$, which is balanced by Proposition I.3.1.

For additional remarks on biased expansions see Examples 5.8 and 6.8.

4. GAIN GRAPH COLORING

A *gain graph* Φ consists of a graph Γ, a *gain group* $\mathfrak{G}$, and a *gain mapping*, a function $\varphi: E \to \mathfrak{G}$ from the oriented edges of Γ into $\mathfrak{G}$ (but undefined on half and loose edges) such that $\varphi(e^{-1}) = \varphi(e)^{-1}$, where e^{-1} denotes e with reversed orientation. We call $\varphi(e)$ the *gain* of e. We consider only finite gain graphs and groups, letting $\gamma = \#\mathfrak{G}$. A gain graph determines a biased graph $[\Phi] = (\Gamma, \mathscr{B}(\Phi))$ by the rule: a circle is balanced when the product of its edge gains taken in cyclic order equals 1, the group identity. To keep the notation simple we write $G(\Phi)$, $\chi_\Phi(\lambda)$, etc., instead of the more precise $G([\Phi])$, $\chi_{[\Phi]}(\lambda)$, etc.

One can define coloring of gain graphs so as to generalize ordinary graph coloring (where $\gamma = 1$); then standard counting theorems generalize as well. This was shown in [29] for signed graphs (where $\gamma = 2$) and indicated therein for general gain graphs.

For a nonnegative integer k let $[k] = \{1, 2, ..., k\}$, so that $\#[k] = k$. Define

$$C_k^* = [k] \times \mathfrak{G}, \qquad C_k = C_k^* \cup \{0\}.$$

A *coloring* of Φ *in* k *colors* is a mapping $f\colon N \to C_k$; it is a *zero-free* coloring if it maps into C_k^*. An edge is *improper* or *satisfied* under f if it is a loose edge or a balanced loop, a half edge or an unbalanced loop at a node whose color is 0, or a link e whose endpoints v and w are colored so that either

$$f(v) = f(w) = 0, \qquad \text{or} \qquad f(v) = (i, \alpha) \text{ and } f(w) = (i, \alpha\varphi(e)),$$

where $\varphi(e)$ is computed with e oriented from v to w. The set of improper edges of f is denoted by $I(f)$. A coloring is *proper* if it has no improper edges. Note that $I(f)$ is closed in $G(\Phi)$ and is balanced if f is zero-free.

LEMMA 4.1. *Let f be a coloring of Φ and let $T \subseteq I(f)$. Assume that every edge in each balanced component of T has gain 1. Let f_T be the function on $\pi_b(T)$ defined by $f_T(V) = f(v)$ for $v \in V \in \pi_b(T)$. Then f_T is a well-defined coloring of Φ/T, zero-free if f is, and $I(f_T) = I(f) \backslash T$.*

Proof. In an unbalanced component of $I(f)$, every node is colored 0 by f. Hence, this holds true in any unbalanced component of T. In a balanced component of T, all nodes have the same color. Therefore f_T is well defined. If f was zero-free, so is f_T; also, $I(f)$ and, therefore, T are balanced.

Consider an edge e of Φ/T. If e is a loose edge, either it was loose in Φ or its nodes were colored 0. Hence e is improper both for f and f_T. If e is a balanced loop, in Φ it was a balanced loop or a link with gain 1 inside a balanced component of T. Thus it is improper for f and for f_T. If e is a half edge at V, it was a half edge at v in Φ or it was a link vw with $f(w) = 0$. It is improper for $f \Leftrightarrow f(v) = 0 \Leftrightarrow f(V) = 0 \Leftrightarrow$ it is improper for f_T. If e is a link VW in Φ/T, it was a link vw in Φ with $v \in V$ and $w \in W$; also, both V and $W \in \pi_b(T)$. The gain of e is the same in Φ/T as in Φ. Evidently e is improper for f_T precisely when it is improper for f. ∎

THEOREM 4.2. *Let Φ be a gain graph with no loose edges and let k be a nonnegative integer. Then the number of proper colorings of Φ in k colors equals $\chi_\Phi(\gamma k + 1)$ and the number of zero-free proper colorings equals $\chi_\Phi^b(\gamma k)$.*

Proof. We give a proof by induction on the number of links in Φ. We may assume that Φ has no balanced loops or balanced digons. Let $p(\Phi)$ be the number of proper colorings and let $p^*(\Phi)$ be the number that are zero-free.

If Φ has no links, let i be the number of isolated nodes. Then $\chi_\Phi(\lambda) = \lambda^i(\lambda-1)^{n-i}$ and $\chi_\Phi^{\mathrm{b}}(\lambda) = \lambda^n$, directly from the definitions. On the other hand, $p(\Phi) = (\gamma k + 1)^i (\gamma k)^{n-i}$ and $p^*(\Phi) = (\gamma k)^n$. So the theorem is verified.

Now let Φ have a link e with endpoints v and w. Assume that the theorem is correct for every gain graph with fewer links than Φ. We may assume by adequate switching that e has gain 1. Every proper (zero-free) coloring of Φ is obviously proper for $\Phi\backslash e$. We consider a (zero-free) coloring f of Φ which is proper on $\Phi\backslash e$. If it is proper for Φ, it is counted in $p(\Phi)$ (or, $p^*(\Phi)$). But if it is improper for Φ, then $I(f) = \{e\}$ and, by Lemma 4.1, we have a proper (zero-free) coloring f_e of Φ/e derived from f. A different coloring g, proper on $\Phi\backslash e$ and improper on e, yields $g_e \neq f_e$. Contrariwise, given h that properly colors Φ/e (and is zero-free) there is a unique (zero-free) coloring f of Φ for which $I(f) = \{e\}$ and $f_e = h$. It follows that

$$p(\Phi\backslash e) = p(\Phi) + p(\Phi/e), \qquad p^*(\Phi\backslash e) = p^*(\Phi) + p^*(\Phi/e).$$

Since these are, by Corollary 3.3, the same relationships satisfied by the unbalanced and balanced chromatic polynomials, the theorem is valid for Φ by the induction hypothesis. ∎

A proof of this theorem by Möbius inversion, for the case $\gamma = 2$, appears in [29, Section 2.4]. That method also works for general γ.

THEOREM 4.3. *Let Φ be a gain graph. For $k = 0, 1, 2, \ldots$ we have*

$$\sum_f w^{\#I(f)} x^{n-b(I(f))} = q_\Phi(w, x, \gamma k + 1, -1),$$

$$\sum_{f \text{ zero-free}} w^{\#I(f)} x^{n-b(I(f))} = q_\Phi^{\mathrm{b}}(w, x, \gamma k, -1),$$

where f ranges over colorings of Φ in k colors.

Proof. For a fixed set $T \subseteq E$, the coefficient of $w^{\#T} x^{n-b(T)}$ on the left is the number of (zero-free) colorings f for which $I(f) = T$; on the right it is $\chi_{[\Phi]/T}(\gamma k + 1)$ (or $\chi^{\mathrm{b}}_{[\Phi]/T}(\gamma k)$). (The latter assertion may be derived easily; or see Eq. (8.1b) with $v = -1$.) We may assume, switching as necessary, that all edges in balanced components of T have gain 1. We know from Theorem I.5.4 that $[\Phi]/T = [\Phi/T]$. Therefore, the coefficient on the right is the number of proper (zero-free) colorings of Φ/T. By Lemma 4.1 this number equals the number of (zero-free) colorings of Φ whose set of improper edges is T. Thus we have the theorem. ∎

COROLLARY 4.4. *The generating functions of k-colorings and zero-free k-colorings of Φ by the number of improper edges are*

$$\sum_{f} w^{\# I(f)} = \bar{Q}_{\Phi}(\gamma k + 1, w - 1),$$

$$\sum_{f \text{ zero-free}} w^{\# I(f)} = \bar{Q}^{\mathrm{b}}_{\Phi}(\gamma k, w - 1).$$

These formulas generalize Tutte's formula [19, Theorem X] for an ordinary (in effect, a balanced) graph, in which $\gamma = 1$. They can be read in reverse as giving a combinatorial interpretation of $\bar{Q}_{\Phi}(\lambda, v)$ or $\bar{Q}^{\mathrm{b}}_{\Phi}(\lambda, v)$ for certain values of λ, by setting $w = 1 + v$.

It is particularly interesting to set $k = 1$. Then we are, in effect, talking about functions from N to $\mathfrak{G} \cup \{0\}$ (assuming that $0 \notin \mathfrak{G}$), or equivalently, partial functions from N to $\mathfrak{G}$. (A *partial function* on N is a function whose domain is a subset of N.) We may think of these as *partial group-colorings* of Φ. Given a partial function $f\colon N \to \mathfrak{G}$, call an edge *improper* or *satisfied* if either no endpoint is in the domain of f, or both ends are and (if the edge is $e\colon vw$) $f(w) = f(v)\,\varphi(e)$. (Thus a loose edge is always satisfied. A half edge $e\colon v$ is satisfied only if $v \notin \operatorname{dom} f$.) Let $i(f)$ be the number of satisfied edges. Then we have the generating functions

$$\bar{Q}^{\mathrm{b}}_{\Phi}(|\mathfrak{G}|, v) = \sum_{f} (1 + v)^{i(f)}, \tag{4.1}$$

summed over functions f on N, and

$$\bar{Q}_{\Phi}(|\mathfrak{G}| + 1, v) = \sum_{f} (1 + v)^{i(f)}, \tag{4.2}$$

summed over partial functions.

COROLLARY 4.5. *The generating functions of k-colorings and zero-free k-colorings by the number of balanced components of the improper edge set are*

$$\sum_{f} x^{n - b(I(f))} = w_{\Phi}(x, \gamma k + 1),$$

$$\sum_{f \text{ zero-free}} x^{n - b(I(f))} = w^{\mathrm{b}}_{\Phi}(x, \gamma k).$$

This corollary generalizes a result that is implicit in [29, Section 2], which concerns signed graphs ($\gamma = 2$).

EXAMPLE 4.1 (*Forests*). Here any group $\mathfrak{G}$ can be the gain group and any labelling $\varphi: E \to \mathfrak{G}$ can be the gain mapping. We apply (4.1) and (4.2). Extracting the coefficient of w^i in $\bar{Q}^{\mathrm{b}}_{F_{n,m}}(\gamma, w-1)$ and $\bar{Q}_{F_{n,m}}(\gamma+1, w-1)$, we find that $F_{n,m}$ has $\binom{m}{i} \gamma^{n-m}(\gamma-1)^{m-i}$ node $\mathfrak{G}$-colorings in which exactly i edges are satisfied and $\binom{m}{i}(\gamma+1)^{n-m} \gamma^{m-i}$ such partial colorings.

EXAMPLE 4.2 (*Balanced circles*). Again any gain group is possible but the gain function is slightly constrained by the need for balance. From (4.1) and (4.2) we see that $[C_n]$ has $\binom{n}{i}$ $[(\gamma-1)^{n-i}+(-1)^{n-i}(\gamma-1)]$ group-colorings with exactly i satisfied edges and $\binom{n}{i}$ $[\gamma^{n-i}+(-1)^{n-i}\gamma]$ such partial colorings. To get the number of (partial) group-colorings whose satisfied edges form b components in the domain of the coloring, replace i by $n-b$ and, if $b=0$ or 1, respectively deduct or add γ (or, $\gamma+1$) for group-colorings (partial group-colorings). This follows from Corollary 4.5 with $k=1$ and Example 3.2.

EXAMPLE 4.3 (*Unbalanced circles*). Any nontrivial gain group is possible. $(C_n, \varnothing)$ has $\binom{n}{i}[(\gamma-1)^{n-i}-(-1)^{n-i}]$ group-colorings with just i satisfied edges and $\binom{n}{i}\gamma^{n-i}$ such partial colorings. Counted by the number b of components of satisfied edges in the domain of the coloring, there are $\binom{n}{b}[(\gamma-1)^b-(-1)^b]$ group-colorings and $\binom{n}{b}\gamma^b$ partial group-colorings.

EXAMPLE 4.4 (*Contrabalanced graphs*). A gain function that works well for contrabalance is $\varphi(e) = e$ with gain group $\mathfrak{G}$ equal to the direct sum of groups $\mathbb{Z}_q e$, each isomorphic to the q-element cyclic group, for any fixed $q \geqslant 2$. (This gain is taken from [31, p. 495], where q was chosen to be 2.) Thus $\gamma = q^{\# E}$. One could shrink the group slightly by setting $\varphi \equiv 1$ on a maximal forest and omitting its edges from the direct sum.

Given a gain group $\mathfrak{G}$ of order γ, the number of group-colorings with exactly k satisfied edges is

$$\sum_{i=0}^{n-k} (-1)^{n-k-i} \binom{n-i}{k} f_i(\Gamma)\, \gamma^i;$$

the number of similar partial colorings equals

$$\sum_{W \subseteq N} \sum_{i=0}^{\# W} (-1)^{\# W - i - k - \# E:W^c} \binom{\# W - i}{k - \# E:W^c} f_i(\Gamma:W)\, \gamma^i,$$

since this is the coefficient of w^k in $\bar{Q}_{(\Gamma,\varnothing)}(\gamma+1, w-1)$. The Whitney number polynomials show that the number of group-colorings whose satisfied edges form b balanced components is $\sum_{i=0}^{b} \binom{n-i}{n-b} f_i(\Gamma)\gamma^i$; the number of such partial colorings is

$$\sum_{W \subseteq N} \sum_{i=0}^{\# W} (-1)^{b-i-t(E:W^c)} \binom{\# W - i}{b-i-t(E:W^c)} f_i(\Gamma{:}W)\,\gamma^i.$$

EXAMPLE 4.6 (*Group expansions*). A good illustration of the use of coloring is a simple combinatorial proof of the formula

$$\chi^{\mathrm{b}}_{\mathfrak{G}\Delta}(\lambda) = \gamma^n \chi_\Delta\left(\frac{\lambda}{\gamma}\right), \tag{4.3}$$

proved algebraically in Example 3.6. To evaluate $\chi^{\mathrm{b}}_{\mathfrak{G}\Delta}(\lambda)$ we set $\lambda = k\gamma$ and count zero-free proper colorings of $\mathfrak{G}\Delta$ in k colors. No two nodes which are adjacent in Δ can be colored by the same number because they are adjacent by every possible gain: if we colored $f(v) = (i, \alpha)$ and $f(w) = (i, \beta)$, then the edge $(\alpha^{-1}\beta, e_{vw})$, whose gain is $\alpha^{-1}\beta$ in the direction from v to w, would be improper. On the other hand, if two nodes have different color numbers i and j, no edge between them can be satisfied. Therefore we obtain a proper coloring of $\mathfrak{G}\Delta$ by coloring Δ properly with the color set $[k]$, which is possible in $\chi_\Delta(k)$ ways, and assigning an arbitrary group element to each node, which can be done in γ^n ways. So $\chi^{\mathrm{b}}_{\mathfrak{G}\Delta}(k\gamma) = \gamma^n \chi_\Delta(k)$. Since this is a polynomial equation valid for all nonnegative integers k, it is an identity; consequently we have proved (4.3).

Similar reasoning suffices to prove (3.11) combinatorically in the case $v = -1$.

EXAMPLE 4.7 (*Dowling lattices and their relatives*). Let us employ coloring to obtain the chromatic polynomial of $\Phi = \mathfrak{G}K_n^{(p)}$ without relying on any formulas except Theorem 4.2. We use the color set C_k, so $\lambda = |C_k| = 1 + \gamma k$. Because each two distinct nodes are joined by edges of every possible gain, none of the numerical colors $0, 1, \ldots, k$ can be used twice. That is, no two nodes can have colors with the same numerical part (counting 0 as the numerical part of the group-labelled color 0). Once this condition is satisfied, the group part of the color of a node can be anything. There is one other restriction on proper coloring: 0 cannot color a node which supports an unbalanced edge.

Now let us color Φ. Let H be the set of nodes supporting unbalanced edges. First, let us use the color 0. It can be assigned to any node not in H, so there are $n - |H| = n - p$ ways to assign 0. Then to each remaining node we assign a distinct number from 1 to k—there are $(k)_{n-1}$ ways to do so—and an arbitrary group element—there are γ^{n-1} ways to do that. So there are $(n-p)(k)_{n-1}\gamma^{n-1}$ ways to color using 0. But if we do not use 0, we assign distinct numbers from 1 to k and arbitrary group elements to all nodes: there are $(k)_n\gamma^n$ ways to do so.

The total number of proper colorings is thus $(n-p)(k)_{n-1}\gamma^{n-1}+(k)_n\gamma^n$. Eliminating k in favor of $\lambda=k\gamma+1$, we deduce that

$$\chi_\Phi(\lambda)=\gamma^{n-1}\left(\frac{\lambda-1}{\gamma}\right)_{n-1}[\lambda-(n-1)(\gamma-1)-p]. \tag{4.4}$$

An algebraic proof of the same formula appears, with a discussion of the integrality of the roots, in Example 6.7.

5. The Matroid Connection

We can express several invariants of the bias, lift, and complete lift matroids of a biased graph in terms of polynomials of the graph. (These matroids were introduced in Part II and are characterized by their rank functions later in this section.) First we define the relevant matroid polynomials.

Let M be a matroid on the ground set E. We denote by Lat M the set of flats (closed sets) of M, by $\text{rk}=\text{rk}_M$ the rank function, and by clos the closure operator. An *ideal in* M is a nonempty subset $\mathscr{I}$ of $\mathscr{P}(E)$ such that $S\subseteq T\in\mathscr{I}$ implies that $S\in\mathscr{I}$ and clos $T\in\mathscr{I}$. If $S\in\mathscr{I}$ we set $\mathscr{I}/S=\{T\backslash S: T\in\mathscr{I} \text{ and } T\supseteq S\}$. Let $\text{Lat}\,\mathscr{I}=\mathscr{I}\cap\text{Lat}\,M$. Let $r=\text{rk}\,M$. The *rank generating polynomial* is

$$R_{M,\mathscr{I}}(u,v)=\sum_{S\in\mathscr{I}}u^{r-\text{rk}\,S}v^{\#S-\text{rk}\,S}.$$

It is convenient for us to normalize this to

$$\bar{R}_{M,\mathscr{I}}(\lambda,v)=v^rR_{M,\mathscr{I}}(u,v)=\sum_{S\in\mathscr{I}}\lambda^{r-\text{rk}\,S}v^{\#S},$$

where as usual $\lambda\equiv uv$. The (normalized) *double rank generating* polynomial is

$$\begin{aligned}\mathscr{R}_{M,\mathscr{I}}(w,x,\lambda,v)&=\sum_{R\in\mathscr{I}}w^{\#R}x^{\text{rk}\,R}\bar{R}_{M/R,\mathscr{I}/R}(\lambda,v)\\&=\sum_{R\subseteq S\in\mathscr{I}}\sum w^{\#R}x^{\text{rk}\,R}\lambda^{r-\text{rk}\,S}v^{\#(S\backslash R)}.\end{aligned}$$

The *characteristic polynomial* is

$$p_{M,\mathscr{I}}(\lambda) = \sum_{S \in \operatorname{Lat} \mathscr{I}} \mu(\varnothing, S)\, \lambda^{r - \operatorname{rk} S},$$

where μ is the Möbius function in $\operatorname{Lat} \mathscr{I}$ (or $\operatorname{Lat} M$) supplemented by taking $\mu(R, S) = 0$ if R is not closed but S is. By the Boolean expansion formula for μ (a special case of [22, Eq. (15)]; see [32, Proposition 7.1.4]), $p_{M,\mathscr{I}}(\lambda) = \bar{R}_{M,\mathscr{I}}(\lambda, -1)$. The *Whitney number polynomial* is

$$w_{M,\mathscr{I}}(x, \lambda) = \sum_{R \in \mathscr{I}} x^{\operatorname{rk} R} p_{M/R,\mathscr{I}/R}(\lambda) = \mathscr{R}_{M,\mathscr{I}}(1, x, \lambda, -1).$$

The *doubly indexed Whitney numbers* (of the *first kind*) of $\mathscr{I}$ are $w_{ij}(\mathscr{I}) =$ coefficient of $x^i \lambda^{r-j}$ in $w_{M,\mathscr{I}}(x, \lambda)$. The *simply indexed Whitney numbers* are (the first kind) $w_j(\mathscr{I}) =$ coefficient of λ^{r-j} in $p_{M,\mathscr{I}}(\lambda)$ and (the second kind) $W_j(\mathscr{I}) = w_{jj}(\mathscr{I}) =$ the number of rank-j flats in $\mathscr{I}$. We should point out that

$$w_{ij}(\mathscr{I}) = \sum_{\substack{R, S \in \operatorname{Lat} \mathscr{I} \\ \operatorname{rk} R = i, \operatorname{rk} S = j}} \mu(R, S).$$

The polynomials and Whitney numbers of M itself are those of $\mathscr{I} = \mathscr{P}(E)$; we write $\mathscr{R}_M$, $w_{ij}(M)$, etc. In particular, the *Möbius invariant* of M is

$$\mu(M) = \mu(\varnothing, E) = w_r(M).$$

We note that the *Tutte polynomial* of M, $t_M(x, y)$, equals $R_M(x-1, y-1)$ [5]. Crapo's invariant $\beta(M)$ is $(-1)^{\operatorname{rk} M - 1} (d/d\lambda)\, p_M(1)$; it is nonnegative [4].

The *bias matroid* $G(\Omega)$ of a biased graph Ω can be defined as follows: its points are the edges of Ω and the rank of an edge set S is $\operatorname{rk}_G S = n - b(S)$. The *lift matroid* $L(\Omega)$ has the same points; its rank function is $\operatorname{rk}_L S = n - c(S)$ if S is balanced, $n + 1 - c(S)$ if not. The *complete lift matroid* $L_0(\Omega)$ has for points the edges and an *extra point* e_0 which is not in Ω; the rank $\operatorname{rk}_{L_0} S$ agrees with that in $L(\Omega)$ if $S \subseteq E$ and equals $n + 1 - c(S)$ if $e_0 \in S$. (All this is taken from Part II.) The *polygon matroid* $G(\Gamma)$ of an unbiased graph Γ is defined to be $G([\Gamma])$; its rank function is $\operatorname{rk}_\Gamma S = n - c(S)$ if S has no half edges. If Γ has no half or loose edges, this is the usual polygon (or "cycle," or "graphic") matroid. Note that a balanced edge set has the same rank in all four matroids. An unbalanced edge set has $\operatorname{rk}_L S = 1 + \operatorname{rk}_\Gamma S$, provided S contains no half edges. Calling a set $S \subseteq E \cup \{e_0\}$ *balanced* if it is a balanced edge set (so that $e_0 \notin S$), we define

$$\begin{aligned}\mathrm{Lat}^{\mathrm{b}}\,\Omega &= \{S \in \mathrm{Lat}\,G(\Omega) \colon S \text{ is balanced}\}\\ &= \{S \in \mathrm{Lat}\,L(\Omega) \colon S \text{ is balanced}\}\\ &= \{S \in \mathrm{Lat}\,L_0(\Omega) \colon S \text{ is balanced}\}.\end{aligned}$$

THEOREM 5.1. *Let Ω be a biased graph. For the bias matroid $G(\Omega)$ we have*

$$\mathcal{R}_{G(\Omega)}(w, x, \lambda, v) = \lambda^{-b(\Omega)} q_{\Omega}(w, x, \lambda, v)$$

and similar formulas obtained by substitution or denormalizing, for instance,

$$\begin{aligned}\bar{R}_{G(\Omega)}(\lambda, v) &= \lambda^{-b(\Omega)} \bar{Q}_{\Omega}(\lambda, v),\\ R_{G(\Omega)}(u, v) &= u^{-b(\Omega)} Q_{\Omega}(u, v),\\ w_{G(\Omega)}(x, \lambda) &= \lambda^{-b(\Omega)} w_{\Omega}(x, \lambda),\\ p_{G(\Omega)}(\lambda) &= \lambda^{-b(\Omega)} \chi_{\Omega}(\lambda).\end{aligned}$$

Proof. The second formula is immediate from the definitions and the rank function in $G(\Omega)$. The first is similarly immediate, since $G(\Omega)/T = G(\Omega/T)$ by Theorem II.2.5. ∎

The normalization factor is explained by the fact that the lowest power of λ or u appearing in the matroid polynomials is the zeroth while in the graph polynomials it is the $b(\Omega)$th.

THEOREM 5.2. *Let Ω be a biased graph with no half edges, having underlying graph $\|\Omega\| = \Gamma$. For the lift matroid $L(\Omega)$ we have, if Ω is unbalanced,*

$$\begin{aligned}\lambda^{c(\Gamma)} \mathcal{R}_{L(\Omega)}(w, x, \lambda, v) = (\lambda - 1)\, q^{\mathrm{b}}_{\Omega}(w, x, \lambda, v) + x q_{\Gamma}(w, x, \lambda, v)\\ - (x - 1) \sum_{\substack{S \subseteq E\\ \text{balanced}}} w^{\# S} x^{\mathrm{rk}\, S} \bar{Q}_{\Gamma/S}(\lambda, v)\end{aligned}$$

and other formulas derived from this, such as

$$\begin{aligned}\lambda^{c(\Gamma)} \mathcal{R}_{L(\Omega)}(w, x, \lambda, -1) = (\lambda - 1)\, q^{\mathrm{b}}_{\Omega}(w, x, \lambda, -1) + x q_{\Gamma}(w, x, \lambda, -1)\\ - (x - 1) \sum_{\substack{A \in \mathrm{Lat}\,\Gamma\\ \text{balanced}}} w^{\# A} x^{\mathrm{rk}\, A} \chi_{\Gamma/A}(\lambda),\end{aligned}$$

$$\begin{aligned}\lambda^{c(\Gamma)} \bar{R}_{L(\Omega)}(\lambda, v) &= (\lambda - 1)\, \bar{Q}^{\mathrm{b}}_{\Omega}(\lambda, v) + \bar{Q}_{\Gamma}(\lambda, v),\\ p_{L(\Omega)}(\lambda) &= \lambda^{-c(\Gamma)} \{(\lambda - 1)\, \chi^{\mathrm{b}}_{\Omega}(\lambda) + \chi_{\Gamma}(\lambda)\}.\end{aligned}$$

For the complete lift matroid $L_0(\Omega)$ *we have*

$$\lambda^{c(\Gamma)}\mathcal{R}_{L_0(\Omega)}(w, x, \lambda, v) = (\lambda - 1)\, q_\Omega^{\mathrm{b}}(w, x, \lambda, v) + (w + v + 1)\, x q_\Gamma(w, x, \lambda, v) - (v+1)(x-1) \sum_{\substack{S \subseteq E \\ \text{balanced}}} w^{\# S} x^{\operatorname{rk} S} \bar{Q}_{\Gamma/S}(\lambda, v)$$

and derived formulas such as

$$\lambda^{c(\Gamma)}\mathcal{R}_{L_0(\Omega)}(w, x, \lambda, -1) = (\lambda - 1)\, q_\Omega^{\mathrm{b}}(w, x, \lambda, -1) + wxq_\Gamma(w, x, \lambda, -1),$$

$$\lambda^{c(\Gamma)}\bar{R}_{L_0(\Omega)}(\lambda, v) = (\lambda - 1)\, \bar{Q}_\Omega^{\mathrm{b}}(\lambda, v) + (v+1)\, \bar{Q}_\Gamma(\lambda, v),$$

$$p_{L_0(\Omega)}(\lambda) = \lambda^{-c(\Gamma)}(\lambda - 1)\, \chi_\Omega^{\mathrm{b}}(\lambda),$$

$$\beta(L_0(\Omega)) = (-1)^{n - c(\Gamma)}\, \chi_\Omega^{\mathrm{b}}(1).$$

Proof. We prove the first formula of each group; the others follow by substituting $w = 0$, $v = -1$, and $\lambda = 1$ in the appropriate places. In the first part, let $L = L(\Omega)$. We have

$$\begin{aligned}
\mathcal{R}_L(w, x, \lambda, v) = {} & \sum_{\substack{S \\ \text{balanced}}} w^{\# S} x^{\operatorname{rk} S} \sum_{\substack{T \supseteq S \\ \text{balanced}}} \lambda^{\operatorname{rk} L - \operatorname{rk} T} v^{\#(T \setminus S)} \\
& + \sum_{\substack{S \\ \text{balanced}}} w^{\# S} x^{\operatorname{rk} S} \sum_{\substack{T \supseteq S \\ \text{unbalanced}}} \lambda^{\operatorname{rk} L - \operatorname{rk}_L T} v^{\#(T \setminus S)} \\
& + \sum_{\substack{S \\ \text{unbalanced}}} w^{\# S} x^{\operatorname{rk}_L S} \sum_{T \supseteq S} \lambda^{\operatorname{rk} L - \operatorname{rk}_L T} v^{\#(T \setminus S)} \\
= {} & \lambda^{1 - c(\Gamma)} q_\Omega^{\mathrm{b}}(w, x, \lambda, v) \\
& + \lambda^{-c(\Gamma)} \sum_{\substack{S \\ \text{balanced}}} w^{\# S} x^{\operatorname{rk} S} \Big\{ \bar{Q}_{\Gamma/S}(\lambda, v) - \sum_{\substack{T \supseteq S \\ \text{balanced}}} \lambda^{n - \operatorname{rk} T} v^{\#(T \setminus S)} \Big\} \\
& + \lambda^{-c(\Gamma)} x \sum_{S \text{ unbalanced}} w^{\# S} x^{\operatorname{rk}_\Gamma S} \bar{Q}_{\Gamma/S}(\lambda, v),
\end{aligned}$$

which reduces to the desired expression.

To prove the first formula for the complete lift we treat the extra point e_0 as an extra unbalanced loop at some node of Ω. We let $\Omega_0 = \Omega \cup \{e_0\}$ and $\Gamma_0 = \Gamma \cup \{e_0\}$, so that $c(\Gamma_0) = c(\Gamma)$ and $L_0(\Omega) = L(\Omega_0)$, and we calculate $\lambda^{c(\Gamma_0)}\mathcal{R}_{L(\Omega_0)}$ in terms of Ω. In the first term, $q_{\Omega_0}^{\mathrm{b}} = q_\Omega^{\mathrm{b}}$. In the second term, $q_{\Gamma_0} = (w + v + 1)\, q_\Gamma$ because e_0 is a loop in Γ_0. As for the third term, e_0 is a loop in Γ_0/S; consequently, $\bar{Q}_{\Gamma_0/S} = (v+1)\, \bar{Q}_{\Gamma/S}$. This gives the result. ∎

Note that the formulas for $L(\Omega)$ do not give the correct results if Ω is balanced or has half edges. We get the right answers in the latter case by treating half edges as if they were unbalanced loops. If Ω is balanced, $L(\Omega) = G(\Gamma)$ so biased-graph formulas are superfluous.

THEOREM 5.3. *Let $\mathscr{I} = \{S \subseteq E: S$ is balanced$\}$ be the ideal of balanced sets in a biased graph Ω. Then*

$$q_\Omega^{\mathrm{b}}(w, x, \lambda, v) = \lambda^{b(\Omega)} \mathscr{R}_{G(\Omega), \mathscr{I}}(w, x, \lambda, v) = \lambda^{c(\Omega)-1} \mathscr{R}_{L_0(\Omega), \mathscr{I}}(w, x, \lambda, v).$$

This also equals $\lambda^{c(\Omega)-1} \mathscr{R}_{L(\Omega), \mathscr{I}}(w, x, \lambda, v)$ if Ω is unbalanced. In particular,

$$\chi_\Omega^{\mathrm{b}}(\lambda) = \lambda^{b(\Omega)} p_{G(\Omega), \mathscr{I}}(\lambda) = \lambda^{c(\Omega)-1} p_{L_0(\Omega), \mathscr{I}}(\lambda),$$

and this equals $\lambda^{c(\Omega)-1} p_{L(\Omega), \mathscr{I}}(\lambda)$ if Ω is unbalanced.

Proof. Similar to that of Theorem 5.1. ∎

COROLLARY 5.4. *The Whitney numbers satisfy $w_{ij}(G(\Omega)) = w_{ij}(\Omega)$ and $w_{ij}(\mathscr{I}) = w_{ij}^{\mathrm{b}}(\Omega)$, where $\mathscr{I}$ is the ideal of balanced sets.*

Theorem 5.1 implies by standard matroid theory that

$$Q_\Omega(x-1, y-1) = (x-1)^{b(\Omega)}\, t_{G(\Omega)}(x, y),$$

but that is not all we can say about Tutte polynomials. Define the *balanced Tutte polynomial* of Ω to be

$$t_\Omega^{\mathrm{b}}(x, y) = \sum_T x^{i(T)} y^{e(T)},$$

where T ranges over maximal forests of Ω and $i(T)$ and $e(T)$ are the number of edges in clos T which are, respectively, internally and externally active with respect to a fixed linear ordering of E. (See [2, Section 6; 5, Section 4; or 23, p. 271, Exercise 2] for the necessary definitions.)

THEOREM 5.5. *We have*

$$Q_\Omega^{\mathrm{b}}(x-1, y-1) = (x-1)^{c(\Omega)}\, t_\Omega^{\mathrm{b}}(x, y).$$

Proof. This is a special case of a general matroid theorem. The *Tutte polynomial* of a matroid M with a modular ideal $\mathscr{I}$ of sets is

$$t_{M, \mathscr{I}}(x, y) = \sum_T x^{i(T)} y^{e(T)},$$

where T ranges over *semibases* (maximal independent sets T whose closure lies in $\mathscr{I}$) and $i(T)$ and $e(T)$ are the internal and external activities of T in its own closure, with respect to an arbitrary linear ordering of the elements of M. Then

$$R_{M,\mathscr{I}}(x-1, y-1) = (x-1)\, t_{M,\mathscr{I}}(x, y). \tag{5.1}$$

One can prove (5.1) by induction on the number of elements, using a deletion–contraction recurrence in the usual fashion (as in the proof of Theorems 3.4 and 3.5 for $\alpha \neq 0$ or that of Theorem 4.2). Theorem 5.5 then follows from Theorem 5.3 and Eq. (5.1) with $M = L_0(\Omega)$. ∎

The definition of geometric semilattices, due to [21], implies a corresponding definition of "semimatroids," related to the former as matroids are to geometric lattices. Then $t_{M,\mathscr{I}}$ is the Tutte polynomial of the semimatroid on E corresponding to $\mathrm{Lat}^{\mathrm{b}}\, \Omega$, just as t_M is that of $G(\Omega)$.

EXAMPLE 5.1 (*Forests*). Since a forest has no circles, $G(F_{n,m}) = L(F_{n,m})$ = the free matroid $U_{m,m}$ on m points and $L_0(F_{n,m})$ = the free matroid on $m+1$ points, $U_{m+1,m+1}$.

EXAMPLE 5.2 (*Balanced circles*). $G([C_n]) = L([C_n])$ = the circuit matroid on n points, that is, the uniform matroid $U_{n,n-1}$ of size n and rank $n-1$. $L_0([C_n]) = U_{n,n-1} \oplus U_{1,1}$, in which e_0 is an isthmus.

EXAMPLE 5.3 (*Unbalanced circles*). $G(C_n, \varnothing) = L(C_n, \varnothing) = U_{n,n}$ and $L_0(C_n, \varnothing) = U_{n+1,n}$.

EXAMPLE 5.4 (*Contrabalanced graphs*). $G(\Gamma, \varnothing)$ is the *bicircular matroid* of Γ [13, 17]. In it the rank of an edge set S is $n - t(S)$. S is independent if every component is a tree or a tree with one added edge which forms an unbalanced circle. (Remember, we assume that Γ is ordinary.) It is a balanced flat if it is the edge set of a forest. Other cryptomorphic descriptions of the bias matroid can be found in [31, Theorem 1].

In $L_0(\Gamma, \varnothing)$, S has rank $\#S$ if it is a forest; otherwise its rank is $n - c(S \backslash e_0) + 1$. It is independent if it is a forest or a forest with one extra element, either e_0 or an edge forming an unbalanced circle. It is a flat in $L_0(\Gamma, \varnothing)$ if it is either a forest, or else a flat of $G(\Gamma)$ together with e_0. In $L(\Gamma, \varnothing)$, rank and independence are the same as in $L_0(\Gamma, \varnothing)$; a flat is a forest or a flat of $G(\Gamma)$.

The thing to note now about the invariants is that

$$w_{ij}^{\mathrm{b}}(\Gamma, \varnothing) = (-1)^{j-i} \binom{j}{i} f_{n-j}(\Gamma).$$

This is clear from Corollary 5.4 and the fact that $\mu(F^i, F^j) = (-1)^{j-i}$ if $F^i \subseteq F^j$, where F^k denotes a k-edge spanning forest.

We computed the chromatic polynomials in Example 3.4 (using Theorem 6.1). Thus we have

$$p_{G(\Gamma,\varnothing)}(\lambda) = \lambda^{-c(\Gamma)} \chi_{(\Gamma,\varnothing)}(\lambda),$$

$$p_{L_0(\Gamma,\varnothing)}(\lambda) = (\lambda - 1) \sum_{i=0}^{n} (-1)^{n-i} f_i(\Gamma)\, \lambda^{i-c(\Gamma)},$$

and, provided that Γ is not a forest,

$$p_{L(\Gamma,\varnothing)}(\lambda) = \lambda^{-c(\Gamma)} \left\{ (\lambda - 1) \sum_{i=0}^{n} (-1)^{n-i} f_i(\Gamma)\, \lambda^{i} + \chi_\Gamma(\lambda) \right\}.$$

EXAMPLE 5.6 (*Group expansions*). The results here and in Example 3.6 of course imply that the various polynomials of the bias, lift, and complete lift matroids of a group expansion $\mathfrak{G}\Delta^{(H)}$ are computable from the polynomials of Δ. In Example 6.6 we first use the machinery of Section 6 to calculate the unrestricted polynomials of group expansions and then the results of this section to evaluate polynomials of the three matroids.

This is the place to mention that, when $\mathfrak{G}$ is the trivial group $\{1\}$, then $G(\mathfrak{G}\Delta^{(H)}) \cong G(\Delta +_H v_0)$, where $\Delta +_H v_0$ means Δ with a new node v_0 joined by an edge to each node in H but no other node. (This is rather obvious. A proof appears in [28, Section 7A].) Furthermore, $L(\mathfrak{G}\Delta) = G(\Delta)$ and $L_0(\mathfrak{G}\Delta) = G(\Delta) \oplus e_0$, where e_0 is a point of rank 1. Thus when $\gamma = 1$ we are really dealing with ordinary graphs and their polygon matroids.

EXAMPLE 5.7 (*Dowling lattices and their relatives*). The *Dowling lattices* of a group $\mathfrak{G}$ are the lattices $Q_n^\dagger = Q_n^\dagger(\mathfrak{G}) = \text{Lat}\, G(\mathfrak{G}K_n^\bullet)$ for $n > 0$. Dowling introduced them in [8] in a nongraphic definition, abstracting and generalizing the geometrical development of [7]. (But he has told me he was aware of the graphic approach, which is clearly equivalent to his.) By analogy we call $Q_{n,p}^\dagger = \text{Lat}\, G(\mathfrak{G}K_n^{(p)})$ a *near-Dowling lattice* and $Q_n^{\S\S} = \text{Lat}\, L(\mathfrak{G}K_n)$ and $Q_n^{\S} = \text{Lat}\, L_0(\mathfrak{G}K_n) \cong \text{Lat}\, L(\mathfrak{G}K_n^\bullet)$ the *incomplete* and *complete Dowling lift lattices*. Dowling computed the characteristic polynomials and both kinds of simply indexed Whitney numbers of his lattices. Using the results of this section we can easily do the same for all related lattices. The results are in Examples 4.7, 6.7, and 10.7.

When $\#\mathfrak{G} = 2$, $Q_n^\dagger$ is the lattice of subspaces generated by the root system B_n or C_n (or the dual arrangement of hyperplanes), $Q_{n,0}^\dagger$ is that of D_n (or its dual), and the $Q_{n,p}^\dagger$ are those of intermediate systems. $\text{Lat}\, G(\mathfrak{G}\Delta^{(H)})$ is the lattice of what in [27] was called a "sign-symmetric" root-system subarrangement.

We should mention that in the case that $\gamma = 1$, $Q_n^\dagger$ is isomorphic to the partition lattice Π_{n+1}.

EXAMPLE 5.8 (*Biased expansions*). As Dowling noted [8, pp. 78–79], letting $\mathfrak{G}$ in Example 5.7 be a quasigroup gives a Dowling lattice $Q_3^\dagger(\mathfrak{G})$ which need not be the Dowling lattice of a group. (Indeed, $\mathfrak{G}K_3$ is a biased expansion $\gamma \cdot K_3$, where $\gamma = \#\mathfrak{G}$, and every biased expansion $\gamma \cdot K_3$ arises in this way.) Dowling also mentioned that when $n \geqslant 4$, $\mathfrak{G}$ must be isotopic to a group for a lattice $Q_n^\dagger(\mathfrak{G})$ to be constructible. (This is because $\mathfrak{G}K_n$ does not exist, a fact that is an immediate consequence of Kahn and Kung's calculation cited in Example 3.8.)

6. BALANCED EXPANSIONS, CHAIN SUMS, AND OTHER CONVOLUTIONAL IDENTITIES

The fundamental formula of biased-graph polynomials is the convolution identity (2.3), which takes the forms

$$q_\Omega^{[\mathrm{b}]}(w, x, \lambda+\mu, v) = \sum_{W \subseteq N} q_{\Omega:W}^{\mathrm{b}}(w, x, \lambda, v)\, q_{\Omega:W^c}^{[\mathrm{b}]}(w, x, \mu, v) \tag{6.1a}$$

and various specializations, for instance,

$$\chi_\Omega^{[\mathrm{b}]}(\lambda+\mu) = \sum_{W \subseteq N} \chi_{\Omega:W}^{\mathrm{b}}(\lambda)\, \chi_{\Omega:W^c}^{[\mathrm{b}]}(\mu). \tag{6.1b}$$

(There is also a convolution like (6.1), where all polynomials are unrestricted; see the remark on $\hat{f}_0$-convolution in Section 2.) By holding μ constant in the unbalanced version of (6.1) we obtain expressions for unrestricted polynomials in terms of their balanced counterparts. The most important such expressions are those in which μ is chosen to make $q_{\Omega:W^c}(w, x, \mu, v)$ simple. That almost always means taking $\mu = 1$.

THEOREM 6.1 (First balanced expansion). *For a biased graph Ω with no loose edges we have*

$$q_\Omega(w, x, \lambda, v) = \sum_{W \subseteq N} q_{\Omega:W}^{\mathrm{b}}(w, x, \lambda-1, v)\, q_{\Omega:W^c}(w, x, 1, v) \tag{6.2a}$$

$$= \sum_{W \subseteq N} q_{\Omega:W}^{\mathrm{b}}(w, x, \lambda-1, v) \times \sum_{T \subseteq E:W^c} w^{\#T} x^{\#W^c - b(T)} (v+1)^{\#(E:W^c) - \#T} \tag{6.2b}$$

$$= \sum_{W \subseteq N} q^{\mathrm{b}}_{\Omega:W}(w, x, \lambda-1, v)\, x^{\# W^c}(v+1)^{\#(E:W^c)}$$

$$\times \sum_{Z \subseteq W^c} \left(\frac{w+v+1}{v+1}\right)^{\#(E:W^c \setminus Z)} \bar{Q}^{\mathrm{b}}_{\Omega:Z}\left(\frac{1}{x}-1, \frac{w}{v+1}\right), \quad (6.2c)$$

$$q_\Omega(w, x, \lambda, -1) = \sum_{W \subseteq N} q^{\mathrm{b}}_{\Omega:W}(w, x, \lambda-1, -1)\, w^{\#(E:W^c)} x^{\# W^c - b(\Omega:W^c)}, \quad (6.2d)$$

and various specializations such as

$$\chi_\Omega(\lambda) = \sum_{\substack{W \subseteq N \\ W^c \text{ stable}}} \chi^{\mathrm{b}}_{\Omega:W}(\lambda-1),$$

$$\bar{Q}_\Omega(\lambda, v) = \sum_{W \subseteq N} \bar{Q}^{\mathrm{b}}_{\Omega:W}(\lambda-1, v)(v+1)^{\#(E:W^c)},$$

$$w_\Omega(x, \lambda) = \sum_{W \subseteq N} w^{\mathrm{b}}_{\Omega:W}(x, \lambda-1)\, x^{\# W^c - b(\Omega:W^c)}.$$

Proof. In (6.1a) we set $\mu = 1$. That yields the first and second forms of q_Ω. For the third we simplify $q_{\Omega:W^c}(w, x, 1, v)$ by means of (3.1b), apply (6.2a) to $\bar{Q}_{\Omega:W^c}(1/x, w/(v+1))$ to get a sum over $Z \subseteq W^c$, and simplify $\bar{Q}_{\Omega:W^c \setminus Z}(1, w/(v+1))$ by (3.1b).

The expression (6.2d) follows from (6.2a) and (3.1e). ∎

Combinatorial Proof (For chromatic polynomials of gain graphs). (This proof, for the case $\gamma = 2$, appeared in [30, Theorem 1.1].) Suppose that $\Omega = [\Phi]$, where Φ is a gain graph with finite gain group $\mathfrak{G}$. We count the ways to color Φ properly in k colors. Let $\lambda = k\gamma + 1$. First, some subset X of N receives the color 0. Clearly, X can be any stable set. Then $\Phi{:}X^c$ is properly colored in k zero-free colors; there are $\chi^{\mathrm{b}}_{\Phi:X^c}(\lambda - 1)$ ways to do that. It follows that

$$\chi_\Phi(\lambda) = \sum_{\substack{X \subseteq N \\ \text{stable}}} \chi^{\mathrm{b}}_{\Phi:X^c}(\lambda-1),$$

which is the formula of Theorem 6.1. Since it is a polynomial identity that is true for infinitely many λ (taking $\lambda = \gamma k + 1$ and $k = 0, 1, 2, \ldots$), it is true for all values of λ. ∎

The formula for $q_\Phi(w, x, \lambda, -1)$, when Φ is a gain graph whose gain group is finite, has a similar proof. We omit the details. See [30, Theorem 1.1] for the case $w = 1$ and $\gamma = 2$.

Theorem 6.1 generalizes the balanced expansion formulas for signed graphs that are proved and employed in [30]. Note that in this and the following theorem we must first eliminate loose edges by means of Proposition 3.10.

Other balanced expansions can be generated by choosing other values of μ in (2.3). An interesting one is obtained by taking $\mu = 0$.

THEOREM 6.2 (Second balanced expansion). *For a biased graph Ω with no loose edges we have*

$$q_\Omega(w, x, \lambda, v) = \sum_{\substack{W \subseteq N \\ b(\Omega : W^c) = 0}} q^{\mathrm{b}}_{\Omega : W}(w, x, \lambda, v)\, q_{\Omega : W^c}(w, x, 0, v) \tag{6.3a}$$

$$= \sum_{\substack{W \subseteq N \\ b(\Omega : W^c) = 0}} q^{\mathrm{b}}_{\Omega : W}(w, x, \lambda, v) \times \sum\sum_{\substack{S \subseteq T \subseteq E : W^c \\ b(T) = 0}} w^{\# S} x^{\# W^c - b(S)} v^{\# (T \backslash S)}, \tag{6.3b}$$

and such evaluations as

$$\chi_\Omega(\lambda) = \sum_W \chi^{\mathrm{b}}_{\Omega : W}(\lambda)\, \mu(G(\Omega : W^c)),$$

$$\bar{Q}_\Omega(\lambda, v) = \sum_W \bar{Q}^{\mathrm{b}}_\Omega(\lambda, v) \sum_{\substack{T \subseteq E : W^c \\ b(T) = 0}} v^{\# T},$$

$$w_\Omega(x, \lambda) = \sum_W w^{\mathrm{b}}_{\Omega : W}(x, \lambda) \sum_{r=0}^{\# W^c} x^r w_{r, \# W^c}(\Omega : W^c),$$

in all of which the range of W is all $W \subseteq N$ such that $b(\Omega : W^c) = 0$.

Proof. We rely for the chromatic polynomial on the conclusion from Theorem 5.1 that

$$w_n(\Omega) = \chi_\Omega(0) = \begin{cases} \mu(G(\Omega)) & \text{if } b(\Omega) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The proofs of the other formulas are routine. ∎

The trouble with the second balanced expansion is the difficulty of evaluating q_Ω at $\lambda = 0$ in most situations. A pretty expression, although complicated by comparison with the first balanced expansion, is given in Proposition 6.3. We state it and most other results in this section in the generality of a two-ideal graph Ω, *assumed to have no loose edges*, and a multiplicative function f of pairs.

PROPOSITION 6.3. *For a two-ideal graph Ω without loose edges,*

$$\hat{f}_0(\mathbf{\Omega}; 0) = \sum_{X \subseteq N} \hat{f}_0(\mathbf{\Omega}{:}X^c; 1) \sum_{r=0}^{p} (-1)^r \binom{p+1}{r+1} \hat{f}_1(\mathbf{\Omega}{:}X; r),$$

where p can be any integer $\geqslant |X|$, such as $p = n$ or $p = |X|$.

If $\hat{f}_0$ is the polychromial of a biased graph, or an evaluation of it, then $\hat{f}_0(\Omega{:}X^c; 1)$ can be simplified as in the proof of (6.2c).

We postpone the proof of Proposition 6.3 in order to prepare the ground by a series of formulas leading to an explicit evaluation of $\alpha_1(\lambda)^{-1}$, where $\alpha_1(\lambda)$ is the incidence-algebra function defined in Section 2. To begin, iterating (2.3) yields the chain-sum formula

$$\hat{f}_1(\mathbf{\Omega}; \lambda_1 + \cdots + \lambda_r) = \sum_{\varnothing = X_0 \subseteq \cdots \subseteq X_r = N} \prod_{j=1}^{r} \hat{f}_1(\mathbf{\Omega}{:}\Delta X_j; \lambda_j), \tag{6.4}$$

where ΔX_j is shorthand for $X_j \backslash X_{j-1}$ and $r \geqslant 1$. We can, therefore, by reversing (6.4) evaluate a sum over weakly ordered chains in $\mathscr{P}(N)$.

Sums over strictly ordered chains are harder. Let $[r] = \{1, 2, ..., r\}$. We have

$$\sum_{\varnothing = X_0 \subset \cdots \subset X_r = N} \prod_{j=1}^{r} \hat{f}_1(\mathbf{\Omega}{:}\Delta X_j; \lambda_j)$$

$$= \sum_{\varnothing \neq J \subseteq [r]} (-1)^{r - \# J} \hat{f}_1\Big(\mathbf{\Omega}; \sum_{j \in J} \lambda_j\Big) + (-1)^r \delta(\varnothing, N), \tag{6.5}$$

where $r \geqslant 0$ and $\delta(X, Y) = 1$ if $X = Y$ and 0 otherwise. To prove (6.5) we use the incidence algebra, in which (6.4) is the equation $\alpha_1(\lambda_1) * \alpha_1(\lambda_2) * \cdots * \alpha_1(\lambda_r) = \alpha_1(\lambda_1 + \lambda_2 + \cdots + \lambda_r)$ for $r > 0$. Since $\alpha_1(\lambda)(X, X) \equiv 1$, we have

$$[\alpha_1(\lambda_1) - \delta] * \cdots * [\alpha_1(\lambda_r) - \delta] = \sum_{J \subseteq [r]} (-1)^{r - \# J} \prod_{j \in J} \alpha_1(\lambda_j)$$

$$= \sum_{\varnothing \neq J \subseteq [r]} (-1)^{r - \# J} \alpha_1\Big(\sum_{j \in J} \lambda_j\Big) + (-1)^r \delta.$$

This is the statement of (6.5) in the incidence algebra.

Now we can invert $\alpha_1(\lambda)$. Since $\alpha_1(\lambda)(X, X) \equiv 1$, we know that $\alpha_1(\lambda)^{-1}$ exists. Equation (6.5) implies that $[\delta - \alpha_1(\lambda)]^{p+1} = 0$ in $\mathrm{IA}(N)$ if $p \geq n$. Multiplying this out and multiplying by $\alpha_1(\lambda)^{-1}$, we find that

$$\alpha_1(\lambda)^{-1} = \sum_{r=0}^{p} (-1)^r \binom{p+1}{r+1} \alpha_1(\lambda)^r.$$

Applying convolution to $\alpha_1(\lambda)^r$ and evaluating at $(\varnothing, N)$, we get

$$\hat{f}_1(\mathbf{\Omega}; \lambda)^{-1} = \sum_{r=0}^{p} (-1)^r \binom{p+1}{r+1} \hat{f}_1(\mathbf{\Omega}; r\lambda), \tag{6.6}$$

where $p \geq n$ is arbitrary. This is the desired inversion.

Proof of Proposition 6.3. More generally, we prove that

$$\hat{f}_i(\mathbf{\Omega}; \lambda) = \sum_{X \subseteq N} \hat{f}_i(\mathbf{\Omega}{:}X^c; \lambda + \lambda_0) \sum_{r=0}^{p} (-1)^r \binom{p+1}{r+1} \hat{f}_1(\mathbf{\Omega}{:}X; r\lambda_0), \tag{6.7}$$

where, again, p is any number $\geq |X|$. Substituting (6.6) into (2.3) with $\mu = \lambda + \lambda_0$ gives (6.7). Taking $i = 0$, $\lambda = 0$, and $\lambda_0 = 1$, we have Proposition 6.3. ▌

As examples of (6.7) we have, for a biased graph Ω without loose edges,

$$\chi_\Omega(\lambda) = \sum_{\substack{X \subseteq W \subseteq N \\ W \setminus X \text{ stable}}} \chi^{\mathrm{b}}_{\Omega:W^c}(\lambda) \sum_{r=0}^{|X|} (-1)^r \binom{|X|+1}{r+1} \chi^{\mathrm{b}}_{\Omega:X}(r) \tag{6.8}$$

(where we expanded $\chi_{\Omega:X^c}(\lambda + 1)$ by Theorem 6.1) and, if $b(\Omega) = 0$,

$$\mu(G(\Omega)) = \sum_{\substack{X \subseteq N \\ X^c \text{ stable}}} \sum_{r=0}^{n} (-1)^r \binom{n+1}{r+1} \chi^{\mathrm{b}}_{\Omega:X}(r). \tag{6.9}$$

EXAMPLES 6.1 AND 6.2 (*Forests and balanced circles*). Because Ω is balanced the first balanced expansion gives only more complicated formulas than those we know from Examples 3.1 and 3.2. The second balanced expansion reduces to $q_\Omega = q^{\mathrm{b}}_\Omega$, also already known.

EXAMPLE 6.3 (*Unbalanced circles*). Here also the balanced expansions are no simpler than the formulas of Example 3.3, but we develop them nonetheless as a simple illustration. To evaluate the first balanced expansion of $q_{(C_n, \varnothing)}$, we should treat separately the cases $W = N$ and $W \subset N$. The contribution of the first is $q^{\mathrm{b}}_{(C_n,\varnothing)}(w, x, \lambda-1, v)$, which was calculated in Example 3.3. That of the second is $(\lambda-1)^{b(E:W)}(\lambda-1+wx+v)^{\#E:W}\times(wx+v+1)^{\#E:W^c}$, because W and W^c both satisfy $\#X=b(E:X)+\#E:X$ (for one reason, since $G(C_n:X, \varnothing)$ is a free matroid). Combining these contributions gives

$$q_{(C_n,\varnothing)}(w, x, \lambda, v) = \sum_{W\subseteq N}(\lambda-1)^{b(E:W)}(\lambda-1+wx+v)^{\#E:W}(wx+v+1)^{\#E:W^c}-(wx+v)^n.$$

This is no great improvement on Example 3.3 but it illustrates Theorem 6.1. It specializes, for instance, to

$$\chi_{(C_n,\varnothing)}(\lambda)=\sum_{\substack{W\subseteq N\\ W^c\text{ stable}}}(\lambda-1)^{b(E:W)}(\lambda-2)^{\#E:W}-(-1)^n.$$

In the second balanced expansion there are only two terms: $W=N$ gives $q^{\mathrm{b}}_{(C_n,\varnothing)}(w, x, \lambda, v)$ and $W=\varnothing$ contributes (after simplification and since T can only equal E) $(wx+v)^n$. Their sum is precisely the form of $q_{(C_n,\varnothing)}$ in Example 3.3.

EXAMPLE 6.4 (*Contrabalanced graphs*). Here balanced expansion comes into its own. Recall that we employed Theorem 6.1 already in Example 3.4 to calculate $q_{(\Gamma,\varnothing)}$ and $\chi_{(\Gamma,\varnothing)}$. If Γ has no tree components, $p_{G(\Gamma,\varnothing)}(\lambda)=\chi_{(\Gamma,\varnothing)}(\lambda)$ by Theorem 5.1 so that

$$\mu(G(\Gamma,\varnothing))=\sum_{\substack{W\subseteq N\\ W^c\text{ stable}}}(-1)^{\#W}f_*(\Gamma:W),$$

where $f_*(\Gamma)$ denotes the total number of spanning forests in Γ, of no matter how many edges. Thus Theorem 6.2 gives the rather complicated expression

$$\chi_{(\Gamma,\varnothing)}(\lambda)=\sum_{W\subseteq X\subseteq N}\sum f_*(\Gamma:W)\sum_{i=0}^{\#X^c}(-1)^{\#X^c+\#W-i}f_i(\Gamma:X^c)\,\lambda^i,$$

in which X and W are restricted to node subsets for which $t(\Gamma:X)=0$ and $X\backslash W$ is stable. Formulas (6.8) and (6.9) yield complicated expressions for $\chi_{(\Gamma,\varnothing)}(\lambda)$ and $\mu(G(\Gamma,\varnothing))$ that, after some algebra involving the identity $S(j,j)\,j!-S(j,j-1)(j-1)!+\cdots\pm S(j,0)\,0!=1$, simplify to the same expressions just given.

EXAMPLE 6.6 (*Group expansions*). (a) *Invariants.* The first balanced expansions give nice expressions for the unrestricted polynomials of a group expansion $\Phi = \mathfrak{G}\Delta^{(H)}$. Some of the results (based upon Example 3.6) are

$$q_\Phi(w, x, \lambda, v) = \sum_{W \subseteq N} q_{\Delta:W}\left(w, x, \frac{\lambda - 1}{\gamma}, v\right) \gamma^{\# W} x^{\# W^c} (v+1)^{\gamma |E(\Delta):W^c| + |H \setminus W|}$$

$$\times \sum_{Z \subseteq W^c} \gamma^{\# Z} \bar{Q}_{\Delta:Z}\left(\frac{1-x}{x\gamma}, \frac{w}{v+1}\right)$$

$$\times \left(\frac{w+v+1}{v+1}\right)^{\gamma |E(\Delta):W^c \setminus Z| + |H \setminus (W \cup Z)|},$$

$$\chi_\Phi(\lambda) = \sum_{\substack{H \subseteq W \subseteq N \\ W^c \text{ stable in } \Delta}} \gamma^{\# W} \chi_{\Delta:W}\left(\frac{\lambda - 1}{\gamma}\right),$$

$$\bar{Q}_\Phi(\lambda, v) = \sum_{W \subseteq N} \gamma^{\# W} \bar{Q}_{\Delta:W}\left(\frac{\lambda - 1}{\gamma}, v\right) (v+1)^{\gamma |E(\Delta):W^c| + |H \setminus W|}.$$

If $\gamma \geqslant 2$, then $b(\Phi:Z) = i_H(\Delta:Z)$, the number of isolated nodes of $\Delta:Z$ which are not in H. Consequently, when $\gamma \geqslant 2$ the coloration generating polynomial is

$$q_\Phi(w, x, \lambda, -1)$$
$$= \sum_{W \subseteq N} \gamma^{\# W} q_{\Delta:W}\left(w, x, \frac{\lambda - 1}{\gamma}, -1\right) w^{\gamma |E(\Delta):W^c| + |H \setminus W|} x^{\# W^c - i_H(\Delta:W^c)}.$$

This specializes to $w_\Phi(x, \lambda)$ by setting $w = 1$.

All these formulas translate directly into expressions for polynomials of the bias matroids as explained in Theorem 5.1 or the remark following, bearing in mind that $b(\Phi) = i_H(\Delta)$ and, in particular, $b(\Phi^\bullet) = 0$.

Some formulas become substantially simpler for the full group expansion $\Phi^\bullet = \mathfrak{G}\Delta^\bullet$. Notably, we have

$$p_{G(\Phi^\bullet)}(\lambda) = \chi_{\Phi^\bullet}(\lambda) = \gamma^n \chi_\Delta\left(\frac{\lambda - 1}{\gamma}\right),$$

$$w_{G(\Phi^\bullet)}(x, \lambda) = w_{\Phi^\bullet}(x, \lambda) = \sum_{W \subseteq N} \gamma^{\# W} x^{\# W^c} w_{\Delta:W}\left(x, \frac{\lambda - 1}{\gamma}\right).$$

For the polychromials of the lift and complete lift matroids we employ Theorem 5.2. For this we must evaluate q_Γ, where $\Gamma = \|\mathfrak{G}\Delta\|$. Let $\psi\colon E(\Gamma) \to E(\Delta)$ be the projection map $ge \mapsto e$. Thus $\#\psi^{-1}(e) = \gamma$ for every edge of Δ. Consequently,

$$
\begin{aligned}
q_\Gamma &= \sum_{R' \subseteq S' \subseteq E(\Delta)} \lambda^{c(S')} x^{n-c(R')} \sum_{\substack{R \subseteq S \\ \psi(R) = R',\, \psi(S) = S'}} w^{\# R} v^{\#(S \setminus R)} \\
&= \sum_{R' \subseteq S'} \lambda^{c(S')} x^{n-c(R')} \prod_{e \in R'} \sum_{\varnothing \subset R_e \subseteq S_e \subseteq \psi^{-1}(e)} w^{\# R_e} v^{\#(S_e \setminus R_e)} \\
&\qquad \times \prod_{e \in S' \setminus R'} \sum_{\varnothing \subset S_e \subseteq \psi^{-1}(e)} v^{\# S_e} \\
&= \sum_{R' \subseteq S'} \lambda^{c(S')} x^{n-c(R')} [(w+v+1)^\gamma - (v+1)^\gamma]^{\# R'} \\
&\qquad \times [(v+1)^\gamma - 1]^{\#(S' \setminus R')} \\
&= q_\Delta((w+v+1)^\gamma - (v+1)^\gamma, x, \lambda, (v+1)^\gamma - 1).
\end{aligned}
$$

The sum in the third term in the main formulas of Theorem 5.2 equals $q_\Delta(wv^{\gamma-1}, x\gamma, \lambda, (v+1)^\gamma - 1)$. To prove this we observe first that, as usual, S is obtained by assigning balanced gains to an $S' \subseteq E(\Delta)$. There are $\gamma^{\operatorname{rk} S'}$ ways to do that. Then $c(S) = c(S')$, $\# S = \# S'$, and Γ/S is the same (up to isomorphism) for all S; it consists of $\|\mathfrak{G}(\Delta/S')\|$ with $(\gamma - 1)\# S'$ loops attached. Therefore

$$
\bar{Q}_{\Gamma/S} = (v^{\gamma-1})^{\# S'} \bar{Q}_{\|\mathfrak{G}(\Delta/S')\|} = (v^{\gamma-1})^{\# S'} \bar{Q}_{\Delta/S'}(\lambda, (v+1)^\gamma - 1)
$$

by the previous evaluation of q_Γ applied to $\|\mathfrak{G}(\Delta/S')\|$. The evaluation of the sum follows routinely.

The result of all this is the formulas

$$
\begin{aligned}
\lambda^{c(\Delta)} \mathscr{R}_{L(\mathfrak{G}\Delta)} &= (\lambda - 1)\, \gamma^n q_\Delta\left(w, x, \frac{\lambda}{\gamma}, v\right) \\
&\quad + x q_\Delta((w+v+1)^\gamma - (v+1)^\gamma, x, \lambda, (v+1)^\gamma - 1) \\
&\quad - (x-1)\, q_\Delta(wv^{\gamma-1}, x\gamma, \lambda, (v+1)^\gamma - 1)
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda^{c(\Delta)} \mathscr{R}_{L_0(\mathfrak{G}\Delta)} &= (\lambda - 1)\, \gamma^n q_\Delta\left(w, x, \frac{\lambda}{\gamma}, v\right) \\
&\quad + (w+v+1)\, x q_\Delta((w+v+1)^\gamma - (v+1)^\gamma, x, \lambda, (v+1)^\gamma - 1) \\
&\quad - (v+1)(x-1)\, q_\Delta(wv^{\gamma-1}, x\gamma, \lambda, (v+1)^\gamma - 1).
\end{aligned}
$$

When $v = -1$ these simplify well. For example, the characteristic polynomials are

$$p_{L(\mathfrak{G}\Delta)} = \lambda^{-c(\Delta)} \left\{ (\lambda - 1)\, \gamma^n \chi_\Delta\left(\frac{\lambda}{\gamma}\right) + \chi_\Delta(\lambda) \right\}$$

if $\gamma \geq 2$ and $E(\Delta) \neq \varnothing$ (so that $\mathfrak{G}\Delta$ is unbalanced) and

$$p_{L_0(\mathfrak{G}\Delta)} = \lambda^{-c(\Delta)}(\lambda - 1)\, \gamma^n \chi_\Delta\left(\frac{\lambda}{\gamma}\right);$$

and for the Whitney number polynomials we get

$$w_{L(\mathfrak{G}\Delta)}(x, \lambda) = \lambda^{-c(\Delta)} \left\{ (\lambda - 1)\, \gamma^n w_\Delta\left(x, \frac{\lambda}{\gamma}\right) + x w_\Delta(x, \lambda) - (x - 1)\, q_\Delta((-1)^{\gamma - 1}, x, \lambda, -1) \right\},$$

$$w_{L_0(\mathfrak{G}\Delta)}(x, \lambda) = \lambda^{-c(\Delta)} \left\{ (\lambda - 1)\, \gamma^n w_\Delta\left(x, \frac{\lambda}{\gamma}\right) + x w_\Delta(x, \lambda) \right\}.$$

(b) *Integral roots and supersolvability.* It follows from formulas in (a) that, if the roots of $\chi_\Delta(\lambda)$ are integers, so are those of $\chi_{\mathfrak{G}\Delta^\bullet}(\lambda) = p_{G(\mathfrak{G}\Delta^\bullet)}(\lambda)$ and $p_{L_0(\mathfrak{G}\Delta)}(\lambda)$, and if the former has nonnegative roots, so do the latter. (Even the converse is true, since a rational root of a monic integral polynomial is an integer.) One would also like to understand directly why the characteristic polynomials of these bias and complete lift matroids have integral roots and, in general, when one can expect integrality of the roots. One property that implies integrality is supersolvability of the associated matroid (or its lattice of flats). This property, introduced by Stanley [18] for a matroid (or geometric lattice) means having a complete chain of modular flats [18, Corollary 2.3]. Supersolvability implies a simple combinatorial interpretation of the roots of the characteristic polynomial [18, Theorem 4.1] which in turn entails that they are positive integers. Stanley noted that $G(\Delta)$ is supersolvable precisely when Δ is a chordal graph (also called a "triangulated" or "rigid circuit" graph) [18, Proposition 2.8]. Dowling showed that all Dowling lattices $Q^\dagger(n, n)$ are supersolvable and Whittle in [26, Theorem 3.10] showed that, at the opposite extreme, the near-Dowling lattice $Q^\dagger(n, 0)$ is never supersolvable unless $\gamma = 1$, $n \leq 2$, or both $n = 3$ and $\gamma = 2$. (Despite this, it has integral roots, as Whittle observed and as we shall see in Example 6.7.) We can easily generalize these results to a characterization of all group expansions $\Phi = \mathfrak{G}\Delta^{(H)}$ which have supersolvable bias, lift, or complete lift matroids.

A vertex of Δ is *simplicial* if its neighbors form a clique.

THEOREM 6.4. *Assume that Δ is connected and $\gamma \geqslant 2$. Then $G(\mathfrak{G}\Delta^{(H)})$ is supersolvable if and only if Δ is chordal and either $n \leqslant 2$, or H^c is a stable set of simplicial vertices, or $\gamma = 2$ and $\Delta = K_3$ and $H = \varnothing$.*

It suffices to assume that Δ is connected because $G(\Omega_1 \cup \Omega_2 \cup \cdots) = G(\Omega_1) \oplus G(\Omega_2) \oplus \cdots$ (Proposition II.2.3) and a direct sum of matroids is supersolvable just when all summands are. It is interesting that even in the generality of Theorem 6.4 the only exception is the one noted already by Whittle, since $Q^{\dagger}_{3,0} = \mathrm{Lat}(\mathfrak{G}K_3)$.

THEOREM 6.5. (a) *Assuming that $\gamma \geqslant 2$, $L_0(\mathfrak{G}\Delta)$ is supersolvable if and only if Δ is chordal.*

(b) *Assuming that $\gamma \geqslant 2$ and Δ has no isolated vertices, $L(\mathfrak{G}\Delta)$ is supersolvable if and only if either $n \leqslant 2$, or both $\gamma = 2$ and $\Delta = K_3$.*

The *proofs* depend on [3, Corollary 3.4 and Proposition 3.5] and the explicit descriptions of copoints and lines in Theorems II.2.1 and II.3.1. We omit the details, since a complete characterization of biased graphs whose matroids are supersolvable will appear in [37].

EXAMPLE 6.7 (*Dowling lattices and their relatives*). Let us begin with the near-Dowling lattices $Q^{\dagger}_{n,p}$, $p \leqslant n$. From Theorem 5.1 and either Theorem 6.1 or Example 6.6 applied to $\Phi = \mathfrak{G}K_n^{(p)}$ we obtain algebraic evaluations of the polynomials of these lattices. For instance, by Theorem 5.1 $p_{Q^{\dagger}_{n,p}}(\lambda) = \chi_{\Phi}(\lambda)$, except that it $= \chi_{\Phi}(\lambda)/\lambda$ if $p = 0$ and n or $\gamma = 1$, and we have

$$\chi_{\Phi}(\lambda) = \gamma^{n-1} \left(\frac{\lambda - 1}{\gamma}\right)_{n-1} [\lambda - 1 + (n-p) - (n-1)\gamma],$$

since $\chi_{K_n}(\lambda) = (\lambda)_n$. (This formula appears implicitly in [27, Theorem 7] for $\gamma = 2$, explicitly in [36], and independently, for $p = 0$, in [26, p. 89]. For a combinatorial proof see Example 4.7. Hanlon, also working independently, obtained a different proof for $\gamma = 2$ which raises interesting questions about possible generalizations, some of which are explored in the final version of his work [11].)

Assuming that $\gamma \geqslant 2$, the coloration generating polynomial of Φ is

$$q_\Phi(w, x, \lambda, -1) = \sum_{l=0}^{n} \gamma^l q_{K_l}\left(w, x, \frac{\lambda - 1}{\gamma}, -1\right)$$

$$\times x^{n-l} \sum_{i=0}^{l} \binom{p}{i}\binom{n-p}{l-i} w^{\gamma\binom{n-l}{2} + p - i}$$

$$-(x-1)(n-p)\,\gamma^{n-1} q_{K_{n-1}}\left(w, x, \frac{\lambda-1}{\gamma}, -1\right) w^p,$$

where q_{K_l} can be found in Example 8.5. This comes directly from Example 6.6 by summing separately over sets W of size l for each $l = 0, 1, ..., n$ and computing the number of such W with $|W \cap H| = i$ for all possible i. Here $i_H(K_n: W^c) = 0$, except that it is 1 if $W = N \backslash v$, where $v \notin H$. One obtains the normalized dichromatic polynomial by the evaluation

$$\bar{Q}_\Phi(\lambda, v) = q_\Phi(v+1, 1, \lambda, -1)$$

from (3.1a) and the Whitney number polynomial by setting $w = 1$, noting that $w_{K_l}(x, \lambda) = q_{K_l}(1, x, \lambda, -1)$ is relatively simple (Example 8.5). The polynomials of the near-Dowling lattices $Q^\dagger_{n,p}$, by Theorem 5.1, equal the corresponding ones of Φ (that is, $q_\Phi = \mathscr{R}_{Q^\dagger_{n,p}}$, $\bar{Q}_\Phi = \bar{R}_{Q^\dagger_{n,p}}$, $w_\Phi = w_{Q^\dagger_{n,p}}$, and $\chi_\Phi = \chi_{Q^\dagger_{n,p}}$) with the trivial exception of $Q^\dagger_{1,0}$.

The chromatic polynomial of $\mathfrak{G}K_n^{(p)}$ has the roots

$$1,\ 1+\gamma,\ 1+2\gamma,\ ...,\ 1+(n-2)\gamma,\ p+(n-1)(\gamma-1)$$

if $n \geqslant 2$, but just p if $n = 1$; all are nonnegative integers. Consequently the roots of the characteristic polynomial of $Q^\dagger_{n,p}$ are positive integers. This property of the roots of $Q^\dagger_n$ was of course known already, since Dowling found the roots; he even showed that $Q^\dagger_n$ is supersolvable. Whittle then showed that $Q^\dagger_{n,0}$ is not supersolvable if $n \geqslant 3$ (unless $n = 3$ and $\gamma = 2$), although, as he also noted, its roots are positive integers. A generalization, apparently first noted independently by H. Terao and by P. Hanlon, is the following corollary of Theorem 6.4.

COROLLARY 6.6. *The near-Dowling lattice $Q^\dagger_{n,p}$ of a group $\mathfrak{G}$ is supersolvable if and only if $p = n$ or $n-1$, or $\gamma = 1$, or $n \leqslant 2$, or $n = 3$ and $p = 0$ and $\gamma = 2$.*

The near-Dowling lattices are therefore a large class of geometric lattices having positive integral characteristic roots which are not accounted for by supersolvability, larger than the class of $Q^\dagger_{n,0}$ noted by Whittle. We can, however, account for them combinatorially by the coloring process of

Example 4.7, in which the factor $\gamma^{n-1}((\lambda-1)/\gamma)_{n-1} = (\lambda-1)(\lambda-1-\gamma)\cdots(\lambda-1-[n-2]\gamma)$ appears naturally.

Now let us look at the Dowling lift lattices and their matroids. The characteristic polynomials,

$$p_{Q_n^{\S}}(\lambda) = (\lambda-1)(\lambda-\gamma)(\lambda-2\gamma)\cdots(\lambda-[n-1]\gamma)$$

and, if $\gamma, n \geqslant 2$,

$$p_{Q_n^{\S\S}}(\lambda) = (\lambda-1)(\lambda-\gamma)(\lambda-2\gamma)\cdots(\lambda-[n-1]\gamma) + (\lambda-1)_{n-1},$$

follow directly from Example 6.6. Crapo's invariant of the complete Dowling lift has the simple expression

$$\beta(Q_n^{\S}) = (\gamma-1)(2\gamma-1)\cdots([n-1]\gamma-1).$$

The Whitney number polynomials are specializations of those given in Example 6.6; the only real simplifications are that $c(\Delta) = c(K_n) = 1$ and w_{K_n} has a relatively simple formula (Example 8.5).

The roots of $p_{Q_n^{\S}}(\lambda)$ are positive integers and indeed $Q_n^{\S}$ is obviously supersolvable because the sets $\varnothing$ and $A_i = E{:}\{v_1, ..., v_i\} \cup \{e_0\}$ for $i = 0, 1, 2, ..., n$, where $\{v_1, v_2, ..., v_n\} = N$, form a complete chain of modular flats (by [3, Corollary 3.4 and Proposition 3.5]). On the other hand, by Theorem 6.5, $Q_n^{\S\S}$ is supersolvable only when $n \leqslant 2$, or $\gamma = 1$, or $(n, \gamma) = (3, 2)$. In all other cases there is not even a modular copoint. What is more, it seems impossible that the roots of $p_{Q_n^{\S\S}}(\lambda)$ can all be integers or even real numbers in most cases. (They are integers if $n = 3$, but they are not all real if $n = 4$ and $\gamma \gg 0$.)

EXAMPLE 6.8 (*Biased expansions*). All the results of Examples 6.6 and 6.7 remain true of biased expansions $\gamma \cdot \Delta$, including the identities, the integrality properties of roots of chromatic and characteristic polynomials, and the supersolvability theorems, because all these depend only on the combinatorial properties of $\gamma \cdot \Delta$ and not on any group structure. Of course, in Example 6.7 there are nongroup expansions $\gamma \cdot K_n$ only for $n = 3$.

7. PARTITION IDENTITIES

By adapting chain formulas like (6.4) we get formulas involving sums over partitions. Again in this section we assume that *the two-ideal graph* $\mathbf{\Omega}$ *has no loose edges.*

THEOREM 7.1. *Let $\mathbf{\Omega}$ be a two-ideal graph without loose edges and let f be a multiplicative function of pairs, with value ring $\mathscr{A}$ a field of characteristic zero, which is defined on all subgraphs of $\mathbf{\Omega}$. Then*

$$\hat{f}_1(\mathbf{\Omega}; \mu\lambda) = \sum_{\pi \in \Pi_N} (\lambda)_{\#\pi} \hat{f}_1(\mathbf{\Omega}{:}\pi; \mu).$$

Proof. In (6.4) let $\lambda_1 = \cdots = \lambda_r = \mu$. Let $\pi = \pi(X_0, X_1, ..., X_r) = \{\Delta X_j : \Delta X_j \neq \varnothing\}$. Then π is a partition of N into at most r parts. On the other hand, given π and r, a weakly ordered chain $(X_0, X_1, ..., X_r)$ can be reconstructed by labelling the blocks of π with distinct numbers from 1 to r. There are $(r)_{\#\pi}$ ways of doing so. Thus a particular $\pi \in \Pi_N$ occurs $(r)_{\#\pi}$ times as a chain-induced partition $\pi(X_0, X_1, ..., X_r)$. This proves the proposition when $\lambda = r$. Since it is a polynomial equation valid for all positive integral λ, it is a polynomial identity. ∎

COROLLARY 7.2. *Let $\mathbf{\Omega}$ and f be as in Theorem 7.1. Then*

$$\hat{f}_1(\mathbf{\Omega}; \lambda) = \sum_{\pi \in \Pi_N} (\lambda)_{\#\pi} \hat{f}_1(\mathbf{\Omega}{:}\pi; 1),$$

$$\hat{f}_0(\mathbf{\Omega}; \lambda) = \sum_{\pi \in \Pi_N^{\dagger}} (\lambda)_{\#\pi} \hat{f}_1(\mathbf{\Omega}{:}\pi; 1) \hat{f}_0(\mathbf{\Omega}{:}(\operatorname{supp} \pi)^c; 0).$$

Proof. For the first formula take $\mu = 1$ in Theorem 7.1. For the second, substitute the first into (2.3) with $\mu = 0$. ∎

These formulas are significant because they evaluate a multiplicative function at all values of λ in terms of its values at $\lambda = 1$, with the variable λ appearing only in a way independent of the particular graph $\mathbf{\Omega}$. Thus they generalize Tutte's expression [20, Eq. (12)] for the dichromatic polynomial of a graph. In the case of a biased graph with no loose edges we have, for example,

$$\bar{Q}^{\mathrm{b}}_{\Omega}(\lambda, v) = \sum_{\pi \in \Pi_N} (\lambda)_{\#\pi} \sum_{\substack{S \subseteq E:\pi \\ \text{balanced}}} v^{\#S}, \tag{7.1}$$

which reduces to Tutte's formula if Ω is balanced, and

$$\chi^{\mathrm{b}}_{\Omega}(\lambda) = \sum_{\pi \in \Pi_N} (\lambda)_{\#\pi} \chi^{\mathrm{b}}_{\Omega:\pi}(1). \tag{7.2}$$

COROLLARY 7.3. *Let $\mathbf{\Omega}$ and f be as in Theorem 7.1, with $n > 0$. Then*

$$\hat{f}_1(\mathbf{\Omega}; \lambda) = \sum_{X \subseteq N} \hat{f}_1(\Omega{:}X; \tfrac{1}{2}\lambda) \hat{f}_1(\Omega{:}X^c; \tfrac{1}{2}\lambda).$$

Proof. In Theorem 7.1 let $\lambda = 2$. Then changing notation slightly,

$$\hat{f}_1(\mathbf{\Omega}; \lambda) = \sum_{\substack{\pi \in \Pi_N \\ \# \pi \leqslant 2}} 2\hat{f}_1(\mathbf{\Omega}{:}\pi; \tfrac{1}{2}\lambda).$$

Either $\pi = \{N\}$, corresponding to $X = N$ or $\varnothing$ in the corollary, or $\pi = \{Y, Z\}$, corresponding to $X = Y$ and $X = Z$. ∎

For a gain graph Φ (Section 4) we can set $\mu = \gamma$ in Theorem 7.1. Then we have the formula

$$\chi^{\mathrm{b}}_{\Phi}(\gamma k) = \sum_{\pi \in \Pi_N} (k)_{\# \pi} \chi^{\mathrm{b}}_{\Phi:\pi}(\gamma), \tag{7.3}$$

which has a combinatorial interpretation: to color Φ properly in k colors without using 0, we should choose a partition π, give each block a distinct color (of the k permitted) ignoring gains, and then color each block as a gain graph using one color, i.e., coloring only with group elements. One can prove (7.3) for gain graphs by means of this combinatorial interpretation.

Now let us return to $\alpha_1(\lambda)$ and its reciprocal, calculated in Section 6. From (2.4) we have

$$\alpha_i(\lambda) = \alpha_1(\lambda_0)^{-1} * \alpha_i(\lambda + \lambda_0) = \sum_{r=0}^{n} [\delta - \alpha_1(\lambda_0)]^r * \alpha_i(\lambda + \lambda_0).$$

Evaluating at N if $i = 0$ or $(\varnothing, N)$ if $i = 1$, we deduce that

$$\hat{f}_i(\mathbf{\Omega}; \lambda) = \sum_{r=0}^{n} (-1)^r \sum_{\varnothing = X_0 \subset \cdots \subset X_r} \hat{f}_i(\mathbf{\Omega}{:}X_r^c; \lambda + \lambda_0) \prod_{j=1}^{r} \hat{f}_1(\mathbf{\Omega}{:}\Delta X_j; \lambda),$$

where $\Delta X_j = X_j \backslash X_{j-1}$. We can regard this as a sum over partial partitions of N:

$$\hat{f}_i(\mathbf{\Omega}; \lambda) = \sum_{\pi \in \Pi_N^{\dagger}} (\# \pi)! \, (-1)^{\# \pi} \hat{f}_i(\mathbf{\Omega}{:}(\operatorname{supp} \pi)^c; \lambda + \lambda_0) \hat{f}_1(\mathbf{\Omega}{:}\pi; \lambda_0). \tag{7.4}$$

An example is the following identity for the chromatic polynomial of a biased graph Ω without any loose edges:

$$\chi_{\Omega}(\lambda) = \sum_{r=0}^{n} r! \, (-1)^r \sum_{\substack{\pi \in \Pi_N^{\dagger} \\ \# \pi = r}} \chi_{\Omega:(\operatorname{supp} \pi)^c}(\lambda - 1) \chi^{\mathrm{b}}_{\Omega:\pi}(-1). \tag{7.5}$$

EXAMPLE 7.3 (*Unbalanced circles*). For $(C_n, \varnothing)$, (7.1) becomes

$$\bar{Q}^{\mathrm{b}}_{(C_n,\varnothing)}(\lambda, v) = \sum_{\substack{\pi \in \Pi_N \\ \pi \neq 1}} (\lambda)_{\#\pi} (v+1)^{\#E:\pi} + \lambda[(v+1)^n - v^n]$$

and, if we let σ_i be the number of partitions of N into i stable sets, (7.2) reduces to

$$\chi^{\mathrm{b}}_{(C_n,\varnothing)}(\lambda, v) + \sum_{i=2}^{n} (\lambda)_i \sigma_i - (-1)^n \lambda.$$

EXAMPLE 7.4 (*Contrabalanced graphs*). Here (7.1) becomes

$$\bar{Q}^{\mathrm{b}}_{(\Gamma,\varnothing)}(\lambda, v) = \sum_{\pi \in \Pi_N} (\lambda)_{\#\pi} \sum_{i=0}^{n} f_i(\Gamma:\pi)\, v^{n-i};$$

upon setting $v = -1$ one has

$$\chi^{\mathrm{b}}_{(\Gamma,\varnothing)}(\lambda) = \sum_{\pi \in \Pi_N} (\lambda)_{\#\pi} \sum_{i=0}^{n} (-1)^{n-i} f_i(\Gamma:\pi),$$

which is what one gets from (7.2).

8. EXPANSIONS IN TERMS OF CHROMATIC POLYNOMIALS

We can write expressions for the polychromials in which the variable λ is isolated in chromatic polynomials. For a biased graph Ω we have

$$q^{[\mathrm{b}]}_{\Omega}(w, x, \lambda\mu, v) = \sum_{\substack{A \subseteq E \\ [\text{balanced}]}} \chi^{[\mathrm{b}]}_{\Omega/A}(\lambda)\, q_{\Omega|A}(w, x, \mu, v); \tag{8.1a}$$

hence, by setting $\mu = 1$,

$$q^{[\mathrm{b}]}_{\Omega}(w, x, \lambda, v) = \sum_{\substack{A \subseteq E \\ [\text{balanced}]}} \chi^{[\mathrm{b}]}_{\Omega/A}(\lambda) \sum_{R \subseteq A} w^{\#R} x^{n-b(R)} (v+1)^{\#(A\setminus R)}. \tag{8.1b}$$

Note that one may restrict A to be a flat of $G(\Omega)$, since otherwise $\chi^{[\mathrm{b}]}_{\Omega/A}(\lambda) \equiv 0$. The formula obtained by setting $v = -1$ in (8.1b) was employed in the proof of Theorem 4.3 on gain graph coloring.

These are really matroidal formulas. We prove them at that level of generality.

PROPOSITION 8.1. *Let M be a matroid of rank r and let $\mathscr{I}$ be an ideal in M. Then*

$$\mathscr{R}_{M,\mathscr{I}}(w, x, \lambda\mu, v) = \sum_{A \in \mathscr{I}} p_{M/A,\mathscr{I}/A}(\lambda)\, \mu^{r - \operatorname{rk} A} \mathscr{R}_{M|A}(w, x, \mu, v). \tag{8.2}$$

Proof. The right-hand side of (8.2) equals

$$\sum_{A \subseteq T \in \mathscr{I}} \sum (-1)^{\#(T \setminus A)} \lambda^{r - \operatorname{rk} T} \mu^{r - \operatorname{rk} A} \sum_{R \subseteq S \subseteq A} \sum w^{\# R} x^{\operatorname{rk} R} \mu^{\operatorname{rk} A - \operatorname{rk} S} v^{\#(S \setminus R)}$$

$$= \sum_{R \subseteq S \subseteq T \in \mathscr{I}} \sum \sum w^{\# R} x^{\operatorname{rk} R} \mu^{r - \operatorname{rk} S} \lambda^{r - \operatorname{rk} T} v^{\#(S \setminus R)} \sum_{S \subseteq A \subseteq T} (-1)^{\#(T \setminus A)},$$

which reduces to $\mathscr{R}_{M,\mathscr{I}}(w, x, \lambda\mu, v)$. ∎

In particular, for example,

$$\bar{Q}^{\mathrm{b}}_{\Omega}(\lambda, v) = \sum_{A \in \operatorname{Lat}^{\mathrm{b}} \Omega} (v+1)^{\# A} \chi^{\mathrm{b}}_{\Omega/A}(\lambda). \tag{8.3}$$

A different kind of formula employs the chromatic polynomial of the underlying graph Γ. It is

$$q^{\mathrm{b}}_{\Omega}(w, x, \lambda\mu, v) = \sum_{A \in \operatorname{Lat} \Gamma} \chi_{\Gamma/A}(\lambda)\, q^{\mathrm{b}}_{\Omega|A}(w, x, \mu, v), \tag{8.4}$$

where we assume that *Γ is ordinary* (*no half edges*). This formula can be proved for two-ideal graphs.

PROPOSITION 8.2. *Let $f(\mathbf{\Omega}, S)$ be a multiplicative function of pairs on two-ideal graphs which depends only on $\mathbf{\Omega} | S$; that is, $f(\mathbf{\Omega}, S) = f(\mathbf{\Omega} | S)$. Then*

$$\hat{f}_1(\mathbf{\Omega}; \lambda\mu) = \sum_{A \in \operatorname{Lat} \Gamma} \chi_{\Gamma/A}(\lambda)\, \hat{f}_1(\mathbf{\Omega} | A; \mu)$$

if $\Gamma = \|\Omega\|$ has no half edges.

Proof. The right-hand side equals

$$\sum_{A \subseteq E} \chi_{\Gamma/A}(\lambda)\, \hat{f}_1(\mathbf{\Omega} | A; \mu) = \sum_{\substack{S \subseteq A \subseteq T \\ S \in \mathscr{I}_1}} \sum \sum \lambda^{c(T)} (-1)^{\#(T \setminus A)} f(\mathbf{\Omega} | S)\, \mu^{b(S)}$$

$$= \sum_{\substack{S \subseteq T \\ S \in \mathscr{I}_1}} \sum f(\mathbf{\Omega} | S)\, \mu^{b(S)} \lambda^{c(T)} \sum_{S \subseteq A \subseteq T} (-1)^{\#(T \setminus A)}.$$

The inner sum vanishes unless $S=T$. Because S is balanced, then $c(T)=c(S)=b(S)$. Thus we get $\hat{f}_1(\mathbf{\Omega}; \lambda\mu)$. ∎

Formula (8.4) follows because the f which yields the polychromial, in Eq. (3.6), has the required property. As an example, we have

$$w^{\mathrm{b}}_{\Omega}(x, \lambda) = \sum_{A \in \operatorname{Lat} \Gamma} \chi_{\Gamma/A}(\lambda/\mu)\, w^{\mathrm{b}}_{\Omega|A}(x, \mu). \tag{8.5}$$

This partially generalizes [30, Theorem 2.3], in which $\mu=2$ and Ω is sign-biased. Equation (8.4) is the same as the balanced case of (8.1a), except that one sums over $A \in \operatorname{Lat} \Gamma$ and the chromatic polynomial is $\chi_{\Gamma/A}(\lambda)$, rather than $\chi^{\mathrm{b}}_{\Omega/A}(\lambda)$. Why there should be such a resemblance is a puzzle. It is no surprise, however, that (8.4), or rather its generalization in Proposition 8.2, should resemble Theorem 7.1, for when Γ is complete they are identical: a flat A of Γ is then the same as $E{:}\pi$ for $\pi \in \Pi_N$, whence $\mathbf{\Omega}|A = \mathbf{\Omega}{:}\pi$ and $\chi_{\Gamma/A}(\lambda) = (\lambda)_{\#\pi}$.

EXAMPLE 8.5 (*Balanced complete graphs*). A flat of $G(K_n)$ is precisely a set $E{:}\pi$, where $\pi \in \Pi_n$. Let us call the *type* of π the sequence $\nu=(\nu_1, \nu_2, ...)$, where ν_i is the number of i-element blocks of π. Write $\nu \vdash n$ to mean that ν is a sequence $(\nu_1, \nu_2, ...)$ of nonnegative integers with $n = 1\nu_1 + 2\nu_2 + 3\nu_3 + \cdots$. If $\nu \vdash n$, then the number of $\pi \in \Pi_n$ whose type is ν equals $n!/1!^{\nu_1}\, 2!^{\nu_2} \cdots \nu_1!\, \nu_2! \cdots$.

By appropriately specializing (8.1b) applied to $\Omega = [K_n]$, we obtain

$$\bar{Q}_{K_n}(\lambda, v) = \sum_{\nu \vdash n} \frac{n!}{1!^{\nu_1}\, 2!^{\nu_2} \cdots \nu_1!\, \nu_2! \cdots} \times (\lambda)_{\nu_1+\nu_2+\cdots} (v+1)^{\nu_1\binom{1}{2}+\nu_2\binom{2}{2}+\nu_3\binom{3}{2}+\cdots}.$$

To express the entire polychromial we note that, by (3.1b), $q_{K_i}(w, x, 1, v) = i!\, \theta_i$, where

$$\theta_i = x^i (v+1)^{\binom{i}{2}} \sum_{\iota \vdash i} \frac{1}{1!^{\iota_1}\, 2!^{\iota_2} \cdots \iota_1!\, \iota_2! \cdots} \left(\frac{1}{x}\right)_{\iota_2+\iota_2+\cdots} \times \left(\frac{w+v+1}{v+1}\right)^{\iota_1\binom{1}{2}+\iota_2\binom{2}{2}+\cdots},$$

but if $v=-1$ then (3.1e) gives

$$\theta_i = \frac{w^{\binom{i}{2}} x^{i-1}}{i!}.$$

Consequently, (8.1b) yields

$$q_{K_n}(w, x, \lambda, v) = n! \sum_{\nu \vdash n} (\lambda)_{\nu_1+\nu_2+\cdots} \prod_{i=1}^{\infty} \frac{\theta_i^{\nu_i}}{\nu_i!},$$

$$q_{K_n}(w, x, \lambda, -1) = n! \sum_{\nu \vdash n} \frac{(\lambda)_{\nu_1+\nu_2+\cdots}}{1!^{\nu_1} 2!^{\nu_2} \cdots \nu_1! \, \nu_2! \cdots} \times w^{\nu_1\binom{1}{2}+\nu_2\binom{2}{2}+\cdots} x^{n-\nu_1-\nu_2-\cdots}.$$

The Whitney number polynomial is much simpler. Because $\Omega/A = K_{b(A)}$ if one neglects multiple edges, the term of A depends only on the rank of A in Lat $G(K_n)$. The number of flats of rank $n-k$ is the Stirling number $S(n, k)$. Therefore,

$$w_{K_n}(x, \lambda) = \sum_{k=0}^{n} S(n, k)(\lambda)_k \, x^{n-k}.$$

9. Formulas Based on Connected Subgraphs

Any function $\hat{f}_0$ on two-ideal graphs, derived from a multiplicative function f of pairs, can be expressed explicitly in terms of the values of f on connected subgraphs.

PROPOSITION 9.1. *Let* $\mathbf{\Omega}$ *be a two-ideal graph with no loose edges, having underlying graph* Γ. *Then*

$$\hat{f}_0(\mathbf{\Omega}; \lambda) = \sum_{\pi \in \Pi(\Gamma)} \prod_{B \in \pi} \left\{ \sum_{\substack{T \in \mathscr{I}_0:B \\ c(T)=1}} f(\mathbf{\Omega}:B, T) + (\lambda - 1) \sum_{\substack{S \in \mathscr{I}_1:B \\ c(S)=1}} f(\mathbf{\Omega}:B, S) \right\},$$

$$\hat{f}_1(\mathbf{\Omega}; \lambda) = \sum_{\pi \in \Pi(\Gamma)} \lambda^{\#\pi} \prod_{B \in \pi} \sum_{\substack{S \in \mathscr{I}_1:B \\ c(S)=1}} f(\mathbf{\Omega}:B, S).$$

Proof. For the first formula we apply (2.3). Thus,

$$\hat{f}_0(\mathbf{\Omega}; \lambda) = \sum_{X \subseteq N} \hat{f}_0(\mathbf{\Omega}:X^c; 1)\, \hat{f}_1(\mathbf{\Omega}:X; \lambda - 1)$$

$$= \sum_{X} \left(\sum_{T \in \mathscr{I}_0:X^c} f(\mathbf{\Omega}:X^c; T) \right) \left(\sum_{S \in \mathscr{I}_1:X} f(\mathbf{\Omega}:X, S)(\lambda - 1)^{c(S)} \right)$$

$$=\sum_X \Bigg(\sum_{\tau \in \Pi(\Gamma:X^c)} \sum_{\substack{T \in \mathscr{I}_0:\tau \\ \pi(T)=\tau}} \prod_{B\in\tau} f(\mathbf{\Omega}:B, T:B) \Bigg)$$

$$\times \Bigg(\sum_{\sigma \in \Pi(\Gamma:X)} \sum_{\substack{S \in \mathscr{I}_1:\sigma \\ \pi(S)=\sigma}} \prod_{B\in\sigma} (\lambda-1) f(\mathbf{\Omega}:B, S:B) \Bigg),$$

which simplifies to the desired expression when we set $\pi = \sigma \cup \tau$.

We derive the second formula by applying the first to $\mathbf{\Omega}'$, where $\mathscr{I}'_0 = \mathscr{I}'_1 = \mathscr{I}_1$ and $\|\mathbf{\Omega}'\|$ is $\|\mathbf{\Omega}\|$ with all unbalanced edges removed. ∎

From now on let Ω be a biased graph without loose or half edges and $\Gamma = \|\Omega\|$. Choose f (as in Section 3) so that $\hat{f}_0 = q_\Omega$ and $\hat{f}_1 = q^{\mathrm{b}}_\Omega$. Let $\mathbf{C}^i_\mu p(\mu)$ denote the coefficient of μ^i in the polynomial $p(\mu)$. From Proposition 9.1 we get

$$q_\Omega(w, x, \lambda, v) = \sum_{\pi\in\Pi(\Gamma)} \prod_{B\in\pi} \Bigg\{ \sum_{\substack{R\subseteq T\subseteq E:B \\ c(T)=1}}\sum w^{\#R} x^{\#B-b(R)} v^{\#(T\setminus R)} + (\lambda-1)\, \mathbf{C}^1_\mu q^{\mathrm{b}}_{\Omega:B}(w, x, \mu, v) \Bigg\} \tag{9.1}$$

and

$$q^{\mathrm{b}}_\Omega(w, x, \lambda, v) = \sum_{\pi\in\Pi(\Gamma)} \lambda^{\#\pi} \prod_{B\in\pi} \mathbf{C}^1_\mu q^{\mathrm{b}}_{\Omega:B}(w, x, \mu, v). \tag{9.2}$$

When $x=1$, (9.1) simplifies nicely; for instance,

$$\bar{Q}_\Omega(\lambda, v) = \sum_{\pi\in\Pi(\Gamma)} \prod_{B\in\pi} \{\mathbf{C}^1_\mu \bar{Q}_{\Gamma:B}(\mu, v) + (\lambda-1)\, \mathbf{C}^1_\mu \bar{Q}^{\mathrm{b}}_{\Omega:B}(\mu, v)\},$$

which we will further simplify shortly. In particular,

$$\chi_\Omega(\lambda) = \sum_{\pi\in\Pi(\Gamma)} \prod_{B\in\pi} [w_{\#B-1}(\Gamma:B) + (\lambda-1)\, w^{\mathrm{b}}_{\#B-1}(\Omega:B)], \tag{9.3}$$

from which, by expanding the product, we obtain

$$w_k(\Omega) = \sum_{X\subseteq N} w^{\mathrm{b}}_{k-\#X}(\Omega:X^c) \times \sum_{\sigma\in\Pi(\Gamma:X)} \prod_{B\in\sigma} [w_{\#B-1}(\Gamma:B) - w^{\mathrm{b}}_{\#B-1}(\Omega:B)], \tag{9.4}$$

since for $Y = X^c$ we have

$$\sum_{\tau \in \Pi(\Gamma:Y)} \lambda^{\#\tau} \prod_{B \in \tau} w^{\mathrm{b}}_{\# B-1}(\Omega:B) = \sum_{T \in \mathrm{Lat}^{\mathrm{b}}(\Omega:Y)} \lambda^{c(T)} \mu(\varnothing, T) = \chi^{\mathrm{b}}_{\Omega:Y}(\lambda),$$

$\mu(\varnothing, T)$ being the Möbius function in $\mathrm{Lat}^{\mathrm{b}}(\Omega:Y)$.

Let $s_k(\Gamma)$ be the number of connected, spanning subgraphs of Γ having k edges. Let $s^{\mathrm{b}}_k(\Omega)$ be the number which are balanced. Note that $s_{n-1}(\Gamma) = s^{\mathrm{b}}_{n-1}(\Omega) = \tau(\Gamma)$, the number of spanning trees in Γ. If Γ is connected,

$$\mathbf{C}^1_\mu \bar{Q}_\Gamma(\mu, v) = \sum_{k \geqslant n-1} s_k(\Gamma) v^k$$

and

$$\mathbf{C}^1_\mu \bar{Q}^{\mathrm{b}}_\Omega(\mu, v) = \sum_{k \geqslant n-1} s^{\mathrm{b}}_k(\Omega) v^k.$$

Therefore,

$$\bar{Q}_\Omega(\lambda, v) = \sum_{\pi \in \Pi(\Gamma)} \prod_{B \in \pi} \Big\{ \tau(\Gamma:B)\, \lambda v^{\# B-1} + \sum_{k \geqslant \# B} [s_k(\Gamma:B) + (\lambda - 1)\, s^{\mathrm{b}}_k(\Omega:B)]\, v^k \Big\}, \quad (9.5)$$

$$\bar{Q}^{\mathrm{b}}_\Omega(\lambda, v) = \sum_{\pi \in \Pi(\Gamma)} \lambda^{\#\pi} \prod_{B \in \pi} \sum_{k \geqslant \# B-1} s^{\mathrm{b}}_k(\Omega:B) v^k. \quad (9.6)$$

The characteristic polynomial of the dual bias matroid $G^\perp(\Omega)$ satisfies

$$p_{G^\perp(\Omega)}(v) = (-1)^{\# E - n + b(\Omega)} R_{G(\Omega)}(-1, v) = (-1)^{\# E-n} v^{-n} \bar{Q}_\Omega(-v, v).$$

Consequently,

$$p_{G^\perp(\Omega)}(v) = (-1)^{\# E-n} \sum_{\pi \in \Pi(\Gamma)} \prod_{B \in \pi} \Big\{ -\tau(\Gamma:B) + \sum_{k \geqslant \# B} [s_k(\Gamma:B) - (v+1)\, s^{\mathrm{b}}_k(\Omega:B)]\, v^{k - \# B} \Big\}. \quad (9.7)$$

In particular, the dual Möbius invariant is obtained by setting $v = 0$:

$$\mu(G^\perp(\Omega)) = (-1)^{\# E-n} \sum_{\pi \in \Pi(\Gamma)} \prod_{B \in \pi} [s_{\# B}(\Gamma:B) - s^{\mathrm{b}}_{\# B}(\Omega:B) - \tau(\Gamma:B)]. \quad (9.8)$$

EXAMPLE 9.3 (*Unbalanced circles*). Let us evaluate (9.7) in this simple example. The term for $\pi = 1$ is $-\tau(C_n) + [s_n(C_n) - (v+1)\, s_n^{\mathrm{b}}(C_n, \varnothing)]\, v^0 +$ (terms equal to 0) $= -n+1$. The factor for a block $B \in \pi < 1$ is $-\tau(C_n{:}B) +$ higher terms; since $C_n{:}B$ is a path, $\tau(C_n{:}B) = 1$ and the higher terms are zero. So the term of π is $(-1)^{\#\pi}$. A nontrivial partition $\pi \in \Pi(C_n)$ is obtained by deleting any $\#\pi$ edges from C_n; hence, there are $\binom{n}{k}$ such partitions having k blocks. Thus

$$p_{G^{\perp}(C_n, \varnothing)}(v) = -n + 1 + \sum_{k=2}^{n} \binom{n}{k} (-1)^k = 0.$$

This is correct because $G^{\perp}(C_n, \varnothing)$ is the n-point matroid of rank 0.

EXAMPLE 9.4 (*Contrabalanced graphs*). Formulas (9.5) and especially (9.6) simplify markedly in this case. Since the only balanced, connected, spanning edge sets of $(\Gamma, \varnothing)$ are the spanning trees,

$$\bar{Q}_{(\Gamma, \varnothing)}(\lambda, v) = v^n \sum_{\pi \in \Pi(\Gamma)} \prod_{B \in \pi} \left\{ \tau(\Gamma{:}B) \frac{\lambda}{v} + \sum_{k \geq \#B} s_k(\Gamma{:}B)\, v^{k - \#B} \right\}$$

and

$$\bar{Q}^{\mathrm{b}}_{(\Gamma, \varnothing)}(\lambda, v) = v^n \sum_{\pi \in \Pi(\Gamma)} \left(\frac{\lambda}{v} \right)^{\#B} \prod_{B \in \pi} \tau(\Gamma{:}B).$$

In (9.3) and (9.4), $w^{\mathrm{b}}_{\#B-1}(\Omega{:}B)$ becomes $(-1)^{\#B-1}\, \tau(\Gamma{:}B)$ and $w^{\mathrm{b}}_{k-\#X}(\Omega{:}X^c)$ becomes $(-1)^{k-\#X} f_{n-k}(\Gamma{:}X^c)$, but the formulas do not simplify notably.

The dual characteristic polynomial of the bicircular matroid is

$$p_{G^{\perp}(\Gamma, \varnothing)}(v) = (-1)^{\#E-n} \sum_{\pi \in \Pi(\Gamma)} \prod_{B \in \pi} \left\{ -\tau(\Gamma{:}B) + \sum_{k \geq \#B} s_k(\Gamma{:}B)\, v^{k - \#B} \right\}.$$

Thus

$$\mu(G^{\perp}(\Gamma, \varnothing)) = (-1)^{\#E-n} \sum_{\pi \in \Pi(\Gamma)} \prod_{B \in \pi} \{ -\tau(\Gamma{:}B) + s_{\#B}(\Gamma{:}B) \}.$$

10. MATROID WHITNEY NUMBERS AND MÖBIUS INVARIANT

Many invariants of the bias, lift, and complete lift matroids can be evaluated by the results in Sections 5 to 9: the Möbius invariant, Crapo's beta invariant, the number of bases, the dual characteristic polynomial (but not the dual Whitney number polynomial), and so on. We saw some evaluations in Section 9. Here as a further illustration we show how to express the Whitney numbers of the three matroids in terms of the balanced Whitney numbers of induced subgraphs. (This supplements the bias-matroid Möbius invariant formula (6.9) and the Whitney number identity (9.4).)

COROLLARY 10.1. *The Whitney numbers of the bias matroid of a biased graph Ω are given by*

$$|w_{ij}(G(\Omega))| = \sum_{\substack{X\subseteq N,\ \#X\leqslant j\\ b(\Omega:X)\leqslant j-i}} (-1)^{b(\Omega:X)} \sum_{m=i_X}^{j} \binom{n-m}{n-j} |w^{\mathrm{b}}_{i_X-\#X,m-\#X}(\Omega:X^c)|,$$

where $i_X = i + b(\Omega:X)$. In particular,

$$|w_j(G(\Omega))| = \sum_{l=0}^{j} (-1)^l \sum_{m=l}^{j} \binom{n-m}{n-j} \sum_{\substack{X\subseteq N,\ \#X-l\\ \text{stable}}} |w^{\mathrm{b}}_{m-l}(\Omega:X^c)|,$$

except that $|w_j(G(\Omega))| = 0$ if Ω has any loose edges or balanced loops.

Proof. Note that $w_{pq} = 0$ by definition if $p<0$ or $p>q$. By Corollary 5.4 and the first balanced expansion (Theorem 6.1),

$$\begin{aligned} w_{G(\Omega)}(x,\lambda) = w_\Omega(x,\lambda) &= \sum_{X\subseteq N} w^{\mathrm{b}}_{\Omega:X^c}(x,\lambda-1)\, x^{\#X-b(\Omega:X)} \\ &= \sum_{l=0}^{n} \sum_{\substack{X\subseteq N\\ \#X=l}} \sum\sum_{0\leqslant p\leqslant q\leqslant n-l} w^{\mathrm{b}}_{pq}(\Omega:X^c)\, x^p(\lambda-1)^{n-l-q}\, x^{l-b(\Omega:X)} \\ &= \sum_{l=0}^{n} \sum_{X} \sum_{p=0}^{n-l} \sum_{q=p}^{n-l} \sum_{r=0}^{n-l-q} w^{\mathrm{b}}_{pq}(\Omega:X^c) \\ &\qquad \times x^{p+l-b(\Omega:X)} \lambda^r (-1)^{n-l-q-r} \binom{n-l-q}{r}. \end{aligned}$$

Extracting the coefficient of $x^i\lambda^{n-j}$, we have $r=n-j$, $p=i_X-l$, and

$$w_{ij}(G(\Omega)) = \sum_{l=0}^{n} \sum_{\substack{X\subseteq N\\ \#X=l}} \sum_{q=i_X-l}^{n-l} (-1)^{j-l-q}\, w^{\mathrm{b}}_{i_X-l,q}(\Omega:X^c) \binom{n-l-q}{n-j}.$$

Setting $m=q+l$ and taking account of the sign law $|w^{[\mathrm{b}]}_{pq}| = (-1)^{q-p}\, w^{[\mathrm{b}]}_{pq}$ (proved in [16]), the restriction $n-m \geqslant n-j$ from the binomial coefficient, and $i_X \leqslant m$ from the balanced Whitney number, we have the result. ∎

Corollary 10.1 incorporates as the case $i=j$ a formula for the number of rank j flats:

$$W_j(G(\Omega)) = \sum_{\substack{X \subseteq N \\ b(\Omega:X)=0}} W^{\mathrm{b}}_{j-\#X}(\Omega:X^c). \tag{10.1}$$

This can be proved directly by observing that a flat of $G(\Omega)$ consists of $E:X$, where $b(\Omega:X)=0$, and a balanced flat of $G(\Omega:X^c)$ (Theorem II.2.1(b)).

It is especially interesting to compute the Möbius invariant directly from the first balanced expansion of the chromatic polynomial. The formula is

$$\mu(G(\Omega)) = \sum_{\substack{W \subseteq N \\ W^c \text{ stable}}} \chi^{\mathrm{b}}_{\Omega:W}(-1) \tag{10.2}$$

if $b(\Omega)=0$.

From the second balanced expansion we get a different formula for the same Whitney numbers.

Corollary 10.2. *The Whitney numbers of the bias matroid satisfy*

$$|w_{ij}(G(\Omega))| = \sum_{r=0}^{i} \sum_{l=r}^{r+j-i} \sum_{\substack{X \subseteq N,\, \#X=l \\ b(\Omega:X)=0}} |w^{\mathrm{b}}_{i-r,j-l}(\Omega:X^c)| \cdot |w_{r,l}(\Omega:X)|.$$

Proof. From Theorem 6.2 we have

$$w_{G(\Omega)}(x,\lambda) = \sum_{\substack{X \subseteq N \\ b(\Omega:X)=0}} \left(\sum\sum_{0 \leqslant p \leqslant q \leqslant n} x^p \lambda^{n-\#X-q} w^{\mathrm{b}}_{pq}(\Omega:X^c) \right) \times \sum_{r=0}^{\#X} x^r w_{r,\#X}(\Omega:X)$$

$$= \sum\sum_{0 \leqslant r \leqslant l \leqslant n} \sum\sum_{0 \leqslant p \leqslant q \leqslant n-l} x^{p+r} \lambda^{n-l-q} \times \sum_{\substack{X \subseteq N,\, \#X=l \\ b(\Omega:X)=0}} w^{\mathrm{b}}_{pq}(\Omega:X^c)\, w_{rl}(\Omega:X).$$

Comparing the coefficients of $x^i\lambda^{n-j}$ gives $p=i-r$, $q=j-l$. Since a nonzero Whitney number $w^{[\mathrm{b}]}_{pq}$ has sign $(-1)^{q-p}$, the result follows. ∎

We deduce a formula for the Whitney numbers of the first kind when Ω has no loose edges. Setting $i=0$ and noting that $w_l(\Omega{:}X)=\mu(G(\Omega{:}X))$, we get

$$|w_j(G(\Omega))| = \sum_{l=0}^{j} \sum_{\substack{X \subseteq N,\ \# X = l \\ b(\Omega:X)=0}} |\mu(G(\Omega{:}X))| \cdot |w^{\mathrm{b}}_{j-l}(\Omega{:}X^c)|. \tag{10.3}$$

By the generating function method we can prove the following.

COROLLARY 10.3. *The Whitney numbers of the lift matroid of an unbalanced biased graph Ω without half edges, whose underlying graph is Γ, are given by*

$$|w_{ij}(L(\Omega))| = |w^{\mathrm{b}}_{ij}(\Omega)| + |w^{\mathrm{b}}_{i,j-1}(\Omega)| + |w_{i-1,j-1}(\Gamma)| - \sum_{\substack{S \in \operatorname{Lat} G(\Gamma) \\ \text{balanced} \\ \operatorname{rk} S = i-1 \text{ or } i}} |w_{j-1-\operatorname{rk} S}(\Gamma/S)|.$$

Those of the complete lift of any biased graph Ω without half edges are given by

$$|w_{ij}(L_0(\Omega))| = |w^{\mathrm{b}}_{ij}(\Omega)| + |w^{\mathrm{b}}_{i,j-1}(\Omega)| + |w_{i-1,j-1}(\Gamma)|.$$

Proof. The proofs are by calculations, based on Theorem 5.2, similar to that in the proof of Corollary 10.1. ∎

In particular, for any biased graph we have

$$|\mu(L_0(\Omega))| = |w^{\mathrm{b}}_{n-c(\Omega)}(\Omega)|. \tag{10.4}$$

If Ω is unbalanced, then

$$|\mu(L(\Omega))| = \begin{cases} |w^{\mathrm{b}}_{n-c(\Omega)}(\Omega)| - |\mu(G(\Gamma))| & \text{if } \Omega \text{ has no unbalanced edges,} \\ |w^{\mathrm{b}}_{n-c(\Omega)}(\Omega)| & \text{otherwise.} \end{cases} \tag{10.5}$$

More generally, for unbalanced Ω without half or loose edges or balanced loops we have

$$|w_j(L(\Omega))| = |w^{\mathrm{b}}_j(\Omega)| + |w^{\mathrm{b}}_{j-1}(\Omega)| - |w_{j-1}(\Gamma\backslash\text{loops})|, \tag{10.6}$$

and for any biased graph we have

$$|w_j(L_0(\Omega))| = |w^{\mathrm{b}}_j(\Omega)| + |w^{\mathrm{b}}_{j-1}(\Omega)|. \tag{10.7}$$

EXAMPLE 10.3 (*Unbalanced circles*). We illustrate the formulas by applying them to $(C_n, \varnothing)$. This will show how our results can generate combinatorial identities. Since $G = G(C_n, \varnothing)$ is a free matroid, $|w_j(G)| = \binom{n}{j}$. Corollary 10.1 gives

$$|w_n(G(C_n, \varnothing))| = \sum_{l=0}^{n} (-1)^l \sum_{\substack{X \text{ stable} \\ \#X = l}} \sum_{m=l}^{n} |w^{\mathrm{b}}_{m-l}(C_n{:}X^c, \varnothing)|.$$

The absolute balanced Whitney numbers equal $\binom{\#E:X^c}{m-l}$, except that $w^{\mathrm{b}}_n(C_n, \varnothing) = 0$. Thus the innermost sum totals to $2^n - 1$ if $l = 0$. When $l > 0$, since $\#E{:}X^c = n - 2l$, the absolute balanced Whitney number is $\binom{n-2l}{m-l}$ and, consequently, the innermost sum equals 2^{n-2l}. The number of different possible X, when $l > 0$, is $\binom{n-l}{l} + \binom{n-l-1}{l-1}$. Therefore,

$$|w_n(G(C_n, \varnothing))| = 2^n - 1 + \sum_{l=1}^{\lfloor n/2 \rfloor} (-1)^l 2^{n-2l} \left[\binom{n-l}{l} + \binom{n-l-1}{l-1} \right].$$

We may conclude that the right-hand side, despite appearances, equals 1 for all $n > 0$.

As for $j < n$, there the corollary yields

$$|w_j(G(C_n, \varnothing))|$$
$$= 2^j \binom{n}{j} + \sum_{1 \leqslant l \leqslant m \leqslant j} \sum (-1)^l \binom{n-m}{n-j} \binom{n-2l}{m-l} \left[\binom{n-l}{l} + \binom{n-l-1}{l-1} \right],$$

since $|w^{\mathrm{b}}_n(C_n, \varnothing)| = \binom{n}{m}$ (this is needed for $l = 0$). The right-hand side must equal $\binom{n}{j}$, since that is the value of $|w_j(U_{n,n})|$.

In Eq. (10.3) for $\Omega = (C_n, \varnothing)$, X can only be $\varnothing$ or N. The formula therefore reduces to $|w_j(G(C_n, \varnothing))| = |w^{\mathrm{b}}_j(C_n, \varnothing)| + |w_n(C_n, \varnothing)| \cdot |w^{\mathrm{b}}_{j-n}(\varnothing)|$. If $j < n$, the last factor is zero; hence $|w_j(G(C_n, \varnothing))| = |w^{\mathrm{b}}_j(C_n, \varnothing)| = \binom{n}{j}$, which is expected because all flats of rank $j < n$ are balanced. If $j = n$, the first term is zero and we obtain the value 1.

For $j < n$, Eq. (10.6) yields $|w_j(L(C_n, \varnothing))| = \binom{n}{j} + \binom{n}{j-1} - \binom{n}{j-1} = \binom{n}{j}$, as we expect from the free matroid on n points. For $j = n$, from the same formula $|w_n(L(C_n, \varnothing))| = 0 + \binom{n}{n-1} - (n-1) = 1$, also as expected.

From (10.7) we get $|w_j(L_0(C_n, \varnothing))| = \binom{n}{j} + \binom{n}{j-1} = \binom{n+1}{j}$ if $j < n$ and $\binom{n}{n-1} = n$ if $j = n$. These are the correct numbers for the uniform matroid $U_{n+1,n}$, which $L_0(C_n, \varnothing)$ is.

EXAMPLE 10.4 (*Contrabalanced graphs*). Recall Example 5.4, where $w^{\mathrm{b}}_{ij}(\Gamma, \varnothing)$ is evaluated. Corollary 10.1 gives for the bicircular matroid of Γ the formula

$$|w_{ij}(G(\Gamma, \varnothing))| = \sum_{l=0}^{j} \sum_{\substack{X \subseteq N,\, \# X = l \\ l-i \leqslant t(\Gamma:X) \leqslant j-i}} (-1)^{t(\Gamma:X)} \times \sum_{m=i+t(\Gamma:X)}^{j} \binom{n-m}{n-j} \binom{m-l}{i+t(\Gamma:X)-l} f_{\# X^c - m + l}(\Gamma:X^c)$$

and, especially,

$$|w_j(G(\Gamma, \varnothing))| = \sum_{l=0}^{j} (-1)^l \sum_{m=l}^{j} \binom{n-m}{n-j} \sum_{\substack{X \subseteq N,\, \# X = l \\ \text{stable}}} f_{\# X^c - m + l}(\Gamma:X^c).$$

From (10.1), the number of flats of rank j in the bicircular matroid is

$$W_j(G(\Gamma, \varnothing)) = \sum_{l=0}^{n} \sum_{\substack{X \subseteq N,\, \# X = l \\ t(\Gamma:X) = 0}} f_{\# X^c - j}(\Gamma:X^c).$$

Let $f(\Gamma)$ be the number of spanning forests of Γ. From (10.2) we have

$$\mu(G(\Gamma, \varnothing)) = (-1)^n \sum_{\substack{W \subseteq N \\ W^c \text{ stable}}} (-1)^{\# W^c} f(\Gamma:W)$$

if Γ has no tree components; in particular,

$$\mu(G(\Gamma^\bullet, \varnothing)) = (-1)^n f(\Gamma).$$

Corollary 10.3 tells us formulas for the lift and complete lift matroids:

$$|w_{ij}(L(\Gamma, \varnothing))| = \binom{j}{i} f_{n-j}(\Gamma) + \binom{j-1}{i} f_{n-j+1}(\Gamma) + |w_{i-1,j-1}(\Gamma)| - \sum_{(F^i)} |w_{j-i-1}(\Gamma/F^i)| - \sum_{(F^{i-1})} |w_{j-i}(\Gamma/F^{i-1})|,$$

in which $\sum_{(F^i)}$ means a sum over all i-edge forests of which each component tree is an induced subgraph, and

$$|w_{ij}(L_0(\Gamma, \varnothing))| = \binom{j}{i} f_{n-j}(\Gamma) + \binom{j-1}{i} f_{n-j+1}(\Gamma) + |w_{i-1,j-1}(\Gamma)|.$$

The Möbius invariants are (up to sign)

$$|\mu(L_0(\Gamma, \varnothing))| = f_{c(\Gamma)}(\Gamma)$$

and, provided Γ is not a forest,

$$|\mu(L(\Gamma, \varnothing))| = \begin{cases} f_{c(\Gamma)}(\Gamma) & \text{if } \Gamma \text{ has a loop,} \\ f_{c(\Gamma)}(\Gamma) - |\mu(G(\Gamma))| & \text{if not.} \end{cases}$$

EXAMPLE 10.6 (*Group expansions*). The key to these examples is a few simple facts. First, the balanced Whitney numbers of $\mathfrak{G}\Delta$ are nicely related to those of Δ by the formula

$$w_{ij}^{\mathrm{b}}(\mathfrak{G}\Delta) = \gamma^j w_{ij}(\Delta).$$

This is so because $w_{ij}^{\mathrm{b}} = \sum \{\mu(R, S)\colon R \leqslant S \text{ in } \mathrm{Lat}^{\mathrm{b}}\, \mathfrak{G}\Delta\}$. Each $S \in \mathrm{Lat}^{\mathrm{b}}\, \mathfrak{G}\Delta$ of rank j is obtained by taking a flat S' of $G(\Delta)$ and assigning balanced gains, which there are γ^j ways to do. Since the interval $[\varnothing, S]$ in $\mathrm{Lat}^{\mathrm{b}}\, \mathfrak{G}\Delta$ is isomorphic to $[\varnothing, S']$ in $\mathrm{Lat}\, G(\Delta)$, the formula for w_{ij}^{b} follows immediately. (The formula also follows easily from (3.11).)

One should recall that, for a group expansion $\Phi = \mathfrak{G}\Delta^{(H)}$ in which $\gamma \geqslant 2$, we have $b(\Phi{:}X) = i_H(\Delta{:}X)$, the number of nodes of $\Delta{:}X$ which are isolated and not in H.

In using Corollary 10.3 one needs to know that the Whitney numbers of $\Gamma = \|\mathfrak{G}\Delta\|$ equal those of Δ because $\mathrm{Lat}\, G(\Gamma) \cong \mathrm{Lat}\, G(\Delta)$ by the isomorphism induced by mapping $ge \mapsto e$ for $ge \in E(\Gamma)$. Furthermore, assuming that $\gamma \geqslant 2$ and Δ is simple (so it has no loops), the only flat $S \in \mathrm{Lat}\, G(\Gamma)$ which is balanced is $S = \varnothing$; thus

$$|w_{ij}(L(\mathfrak{G}\Delta))| = |w_{ij}(L_0(\mathfrak{G}\Delta))| - \begin{cases} 0, & \text{if } i \geqslant 2, \\ |w_{j-1}(\Delta)|, & \text{if } i = 0, 1. \end{cases}$$

EXAMPLE 10.7 (*Dowling lattices and their relatives*). Taking $\Delta = K_n$ and $\gamma \geqslant 2$ in the preceding example we get marvelously explicit results, due to the equation $w_{ij}(K_n) = S(n, n-i)\, s(n-i, n-j)$ and the fact that $i_H(K_n{:}X) = 0$, except when X is a singleton not contained in H.

Thus Corollary 10.1 gives the formula

$$\begin{aligned} &|w_{ij}(Q_{n,p}^{\dagger})| \\ &\quad = \sum_{l=0}^{j} \binom{n}{l} \sum_{m=l}^{j} \binom{n-m}{n-j} \gamma^{m-l} S(n-l, n-i)\, |s(n-i, n-m)| \\ &\quad\quad - (n-p) \sum_{m=1}^{j} \binom{n-m}{n-j} \gamma^{m-1} \{S(n-1, n-1-i)\, |s(n-1-i, n-m)| \\ &\quad\quad + S(n-1, n-i)\, |s(n-i, n-m)|\} \end{aligned}$$

when $0 \leqslant i \leqslant j$. Special cases of significance are the simply indexed Whitney numbers, which Dowling computed in [8] for the case $p = n$. For any p, they are

$$|w_j(Q^{\dagger}_{n,p})| = \sum_{m=0}^{j} \binom{n-m}{n-j} \gamma^{m-1}\{\gamma\, |s(n, n-m)| - (n-p)\, |s(n-1, n-m)|\},$$

$$W_j(Q^{\dagger}_{n,p}) = \sum_{l=0}^{j} \binom{n}{l} \gamma^{j-l} S(n-l, n-j) - (n-p)\, \gamma^{j-1}(S(n-1, n-j).$$

Since $Q^{\dagger}_{n,p}$ has rank n (if $p > 0$ or $\gamma, n \geqslant 2$) its Möbius invariant is w_n; we can also obtain it from (10.2), but it is easiest to deduce it from the chromatic polynomial in Example 6.7. It is the signed product of the roots,

$$\mu(Q^{\dagger}_{n,p}) = (-1)^n (\gamma+1)(2\gamma+1) \cdots ([n-2]\gamma+1)([n-1][\gamma-1]+p),$$

unless $p = 0$ and γ or $n = 1$.

Corollary 10.3 gives the Whitney numbers of the Dowling lift lattices. Assuming that $n \geqslant 2$ and $\gamma \geqslant 2$, we get

$$|w_{ij}(Q^{\S}_n)| = \gamma^j S(n, n-i)\, |s(n-i, n-j)| + \gamma^{j-1} S(n, n-i)\, |s(n-i, n-j+1)|$$
$$+ S(n, n-i+1)\, |s(n-i+1, n-j+1)|,$$

$$|w_{ij}(Q^{\S\S}_n)| = |w_{ij}(Q^{\S}_n)| - \begin{cases} 0, & \text{if } i \geqslant 2, \\ |s(n, n-j+1)|, & \text{if } i = 0, 1. \end{cases}$$

Consequently, the Möbius invariants are

$$|\mu(Q^{\S}_n)| = n!\, \gamma^{n-1},$$
$$|\mu(Q^{\S\S}_n)| = n!\, (\gamma^{n-1} - 1),$$

and the simply indexed Whitney numbers are

$$|w_j(Q^{\S}_n)| = \gamma^j\, |s(n, n-j)| + \gamma^{j-1}\, |s(n, n-j+1)|,$$
$$|w_j(Q^{\S\S}_n)| = \gamma^j\, |s(n, n-j)| + (\gamma^{j-1} - 1)|\, s(n, n-j+1)|,$$

and, finally,

$$W_j(Q^{\S}_n) = W_j(Q^{\S\S}_n) = \gamma^j S(n, n-j) + S(n, n-j+1),$$

except that $W_1(Q^{\S\S}_n) = \gamma\binom{n}{2} = W_1(Q^{\S}_n) - 1$.

11. Full Biased Graphs

A biased graph is *full* when every node supports an unbalanced edge. Then many of our formulas simplify. There are three reasons: The only stable node set is the null set. All induced subgraphs satisfy $b(\Omega{:}X)=0$. Provided that $n \neq 0$, $L(\Omega)$ differs only trivially from $L_0(\Omega)$, namely by deletion of the one point e_0 from the multipoint atom $\operatorname{clos}_{L_0}(e_0)$; hence, $L(\Omega)$ and $L_0(\Omega)$ have the same lattices of flats.

For instance, due to the first reason we have the very simple expression

$$\chi_\Omega(\lambda) = \chi^{\mathrm{b}}_\Omega(\lambda - 1). \tag{11.1}$$

From this we obtain the simple formulas

$$\mu(G(\Omega)) = \chi^{\mathrm{b}}_\Omega(-1), \tag{11.2}$$

$$\beta(G(\Omega)) = |w^{\mathrm{b}}_{n-1}(\Omega)|. \tag{11.3}$$

Let a *multitree* be a tree graph with multiple edges allowed. In other words, it is a connected graph in which any circle has length two. A *series-parallel matroid* is the polygon matroid of a series-parallel network (see [23, Section 14.2]). Brylawski [1] proved that a maitroid M is series-parallel if and only if $\beta(M) = 1$.

Theorem 11.1. *Let Ω be a full biased graph. Its bias matroid $G(\Omega)$ is series-parallel if and only if Ω consists of a balanced multitree, together with one or more unbalanced edges at each node.*

Proof. By (11.3) we see that $G(\Omega)$ is series-parallel if and only if $|w^{\mathrm{b}}_{n-1}(\Omega)| = 1$. Thus $G(\Omega)$ has exactly one balanced copoint. Consequently, Ω consists of a balanced graph Δ with unbalanced edges adjoined, and $|\mu(G(\Delta))| = |w^{\mathrm{b}}_{n-1}(\Omega)| = 1$. Therefore, Δ has no loose edges or balanced loops and, by the slimness theorem of Dowling and Wilson [10], $G(\Delta)$ has $n-1$ atoms. Thus Δ is a multitree. ∎

It seems a nontrivial problem to generalize this result to all biased graphs.

12. Two Examples

To illustrate once again the point of the balanced polynomials—which is, in part, that they are relatively easy to calculate and that the unrestricted polynomials can be deduced from them by balanced expansion formulas (Section 6)—we calculate invariants of two last general examples.

EXAMPLE 12.1 (Example I.6.3) (*Parity bias*). Let Γ be an ordinary graph. If all edges are signed negative we have a signed graph $-\Gamma$ whose balanced circles are those of even length. We will evaluate $q^{\mathrm{b}}_{[-\Gamma]}(w, x, \lambda, -1)$, thereby extending [30, Theorem 2.5]. Let a *cut set* of Γ be the set of links between two complementary node sets; in this definition $\varnothing$ is a cut set. From (8.1) we know that

$$q^{\mathrm{b}}_{-\Gamma}(w, x, 2, -1) = \sum_{S \in \mathrm{Lat}^{\mathrm{b}}[-\Gamma]} \chi^{\mathrm{b}}_{[-\Gamma]/S}(2)\, w^{\# S} x^{n-b(S)}.$$

The chromatic polynomial here equals $2^{c(\Gamma)}$ if $-((-\Gamma)/S)$ is balanced and 0 otherwise [30, Lemma 2.4]. In the proof of [30, Theorem 5.2] it is shown that $-((-\Gamma)/S)$ is balanced precisely when S is a cut set. Therefore,

$$q^{\mathrm{b}}_{-\Gamma}(w, x, 2, -1) = \sum_{S \text{ cut set}} 2^{c(\Gamma)} w^{\# S} x^{n-b(S)}. \qquad (*)$$

Now let us apply (8.5). We get

$$q^{\mathrm{b}}_{-\Gamma}(w, x, \lambda, -1) = \sum_{A \in \mathrm{Lat}\,\Gamma} \chi_{\Gamma/A}\left(\frac{\lambda}{2}\right) q^{\mathrm{b}}_{-\Gamma|A}(w, x, 2, -1),$$

which from (*) applied to $\Gamma | A$ is

$$= \sum_{A \in \mathrm{Lat}\,\Gamma} 2^{c(A)} \chi_{\Gamma|A}\left(\frac{\lambda}{2}\right) \sum_{\substack{S \text{ cut set} \\ \text{in } \Gamma|A}} w^{\# S} x^{n-b(S)}, \qquad (12.1)$$

where $b(S)$ here equals the number of bipartite components of S including isolated nodes.

Recalling the interpretation of q^{b}_{Φ} at $v = -1$ from Theorem 4.3, we see that (12.1) implies a formula for the number of zero-free signed colorings of $-\Gamma$ in k colors in terms of the size and rank ($= n - b(S)$) of the set S of improper edges.

EXAMPLE 12.2 (*Hamiltonian bias*). Any set $\mathscr{B}$ of Hamiltonian circles in Γ forms a linear subclass. We call $(\Gamma, \mathscr{B})$ a *Hamiltonian bias* of Γ. It is very close to the contrabalanced bias $(\Gamma, \varnothing)$ (Example 3.4), the only difference being that $H \in \mathscr{B}$ is balanced in $(\Gamma, \mathscr{B})$. Thus from the definition of the balanced polychromial we have

$$q^{\mathrm{b}}_{(\Gamma, \mathscr{B})} = q^{\mathrm{b}}_{(\Gamma, \varnothing)} + \sum_{R \subseteq H \in \mathscr{B}} \sum w^{\# R} x^{n-b(R)} \lambda^{b(H)} v^{\#(H \setminus R)},$$

which simplifies readily to

$$q^{\mathrm{b}}_{(\Gamma,\mathscr{B})}(w, x, \lambda, v) = q^{\mathrm{b}}_{(\Gamma,\varnothing)}(w, x, \lambda, v) + h\lambda\left[(wx+1)^n - (wx)^n \frac{x-1}{x}\right], \tag{12.2}$$

where $h = |\mathscr{B}|$.

We calculate the polychromial from (6.2a). The terms with $\varnothing \neq W \subset N$ are the same as in $q_{(\Gamma,\varnothing)}$, so after simplification we get

$$\begin{aligned} q_{(\Gamma,\mathscr{B})}&(w, x, \lambda, v) \\ &= q_{(\Gamma,\varnothing)}(w, x, \lambda, v) \\ &\quad + h\left\{(\lambda-1)(wx+1)^n - [\lambda - 1 + (v+1)^{m-n}](wx)^n \frac{x-1}{x}\right\}. \end{aligned} \tag{12.3}$$

Specializing the variables leads to further simplifications. For instance,

$$\begin{aligned} q_{(\Gamma,\mathscr{B})}(w, x, \lambda, -1) &= q_{(\Gamma,\varnothing)}(w, x, \lambda, -1) \\ &\quad + h(\lambda-1)\left\{(wx+1)^n - (wx)^n \frac{x-1}{x}\right\} \end{aligned}$$

if $\Gamma \neq C_n$, and in general we have

$$\begin{aligned} \bar{Q}_{(\Gamma,\mathscr{B})}(\lambda, v) &= \bar{Q}_{(\Gamma,\varnothing)}(\lambda, v) + h(\lambda - 1), \\ w_{(\Gamma,\mathscr{B})}(x, \lambda) &= w_{(\Gamma,\varnothing)}(x, \lambda) + h(\lambda-1)\{(x+1)^n - x^n + x^{n-1}\}, \\ \chi_{(\Gamma,\mathscr{B})}(\lambda) &= \chi_{(\Gamma,\varnothing)}(\lambda) + h(\lambda - 1), \\ \chi^{\mathrm{b}}_{(\Gamma,\mathscr{B})}(\lambda) &= \chi^{\mathrm{b}}_{(\Gamma,\varnothing)}(\lambda) + h\lambda, \end{aligned}$$

and so on.

The method of building on contrabalanced bias will be used again in the very last examples, the "seven dwarves."

13. Seven Dwarves: Polynomials of the Biased K_4's

In Sections I.7 and II.6 we found the seven nonisomorphic biasings of K_4 and their possible gain groups and the three matroids of each. As a final and utterly concrete illustration of our results we calculate polynomials of both the graphs and their matroids. Since it is the chromatic, dichromatic, and Whitney number polynomials (and the corresponding matroid polynomials) that are significant, all of which are easy to obtain from the balanced and unbalanced coloration generating polynomials $q^{[b]}(w, x, \lambda, -1)$, it would be sufficiently general if we were to calculate polychromials holding $v = -1$ throughout our calculation. This is, indeed, quite advantageous for the unrestricted polynomials, but for the balanced polychromial it is of no particular help, so we shall obtain the full balanced polychromials $q^{b}(w, x, \lambda, v)$ and, thence (by the first balanced expansion), the unbalanced polychromials just for $v = -1$. We shall state as well some of the polynomials of real interest (which, of course, are just specializations of the polychromials).

Even for such a small graph as K_4 there is no easy way to do the computations and the hazard of error is considerable (unless one uses computer algebra). Nontheless, in this very special situation where we have many biasings of a single underlying graph, there is a method that can reduce the labor and increase one's insight considerably. To explain our procedure is our first task (Subsection 13a). Then, after setting out certain very small examples (Subsection 13b), we calculate the polynomials we want (Subsection 13c).

13a. *Outline of the Procedure*

The central fact about our situation is that we have biased graphs, say $\Omega = (\Gamma, \mathscr{B})$ and $\Omega_i = (\Gamma, \mathscr{B}_i)$, which have not only the same base graph Γ but even comparability of bias, that is, $\mathscr{B} \subseteq \mathscr{B}_i$. (We say that Ω is more biased, or less balanced, than Ω_i.) Better yet, in our seven examples we can take Ω to be contrabalanced (Example 3.4), which makes its balanced polynomials extremely simple, but that is not essential to the general method. So in this subsection we shall assume no more than that $\|\Omega\| = \|\Omega_i\| = \Gamma$ and $\mathscr{B} \subseteq \mathscr{B}_i$.

That immediately suggests using (3.9) for the balanced polychromial, because $q^{b}_{\Omega_i}$ includes all the terms of q^{b}_{Ω}. Let us be precise. Define

$$\Delta q_i^{[b]}(w, x, \lambda, v) = q_{\Omega_i}^{[b]}(w, x, \lambda, v) - q_{\Omega}^{[b]}(w, x, \lambda, v).$$

Then

$$\Delta q_i^{b} = \sum_{S} \lambda^{b(S)} x^{n} v^{\# E} \bar{Q}_{\Gamma | S}\left(\frac{1}{x}, \frac{w}{v}\right), \tag{13.1}$$

where the range of S is all subsets of E which are balanced in Ω_i but not in Ω. If we know q^{b}_{Ω}, we need only compute Δq^{b}_i, which depends on evaluating the $Q_{\Gamma | S}$. The best way to obtain q_{Ω} is from (6.2d) (which is why we hold $v = -1$; otherwise (6.2a)–(6.2c) would be needed and the analysis would be far too complicated). Again we focus on the difference,

$$\Delta q_i(w, x, \lambda, -1) = \sum_{W \subseteq N} q^{\mathrm{b}}_{\Omega:W}(w, x, \lambda - 1, -1) \times w^{\#(E:W^c)}[x^{n-b(\Omega_i:W^c)} - x^{n-b(\Omega:W^c)}] + \sum_{W \subseteq N} \Delta q^{\mathrm{b}}_{i,W}(w, x, \lambda - 1, -1)\, w^{\#(E:W^c)} x^{\operatorname{rk} \Omega_i:W^c}, \quad (13.2)$$

where $\Delta q^{\mathrm{b}}_{i,W}$ means $q^{\mathrm{b}}_{\Omega_i:W} - q^{\mathrm{b}}_{\Omega:W}$. (The proof is immediate from (6.2d) upon replacing $q^{\mathrm{b}}_{\Omega_i:W}$ by $q^{\mathrm{b}}_{\Omega:W} + \Delta q^{\mathrm{b}}_{i,W}$.) Evidently, to use (13.2) successfully requires not only knowing $q_\Omega(w, x, \lambda, -1)$, but also knowing the $\Delta q^{\mathrm{b}}_{i,W}$ and any of the $q^{\mathrm{b}}_{\Omega:W}$ for which $b(\Omega_i:W^c) \neq b(\Omega:W^c)$. Fortunately, one can generally expect the latter two quantities to be equal (because both equal $\# W^c$) if W^c is at all large. At the same time, $\Delta q^{\mathrm{b}}_{i,W} = 0$ if $\# W \leq 2$ and frequently also if $\# W = 3$.

13b. *A Tool Kit*

Here are the normalized dichromatic polynomials of graphs we will need for the seven dwarves. Three easy examples are

$$\bar{Q}_{K_1}(\Lambda, V) = \Lambda, \qquad \bar{Q}_{K_2}(\Lambda, V) = \Lambda(\Lambda + V),$$
$$\bar{Q}_{4K_2}(\Lambda, V) = \Lambda(\Lambda - 1) + \Lambda(V + 1)^4,$$

where $4K_2$ denotes a quadruple link. From Example 3.2 we get

$$\bar{Q}_{C_3}(\Lambda, V) = (\Lambda + V)^3 + (\Lambda - 1)V^3, \qquad \bar{Q}_{C_4}(\Lambda, V) = (\Lambda + V)^4 + (\Lambda - 1)V^4,$$

and $\bar{Q}_{C_2}(\Lambda, V)$; consequently, by Propositions 2.8 and 2.9,

$$\bar{Q}_{C_3 \cup K_1}(\Lambda, V) = \bar{Q}_{K_1}\bar{Q}_{C_3} = \Lambda(\Lambda + V)^3 + \Lambda(\Lambda - 1)V^3,$$

$$\bar{Q}_{C_2 \cup_p C_2}(\Lambda, V) = \frac{1}{\Lambda}(\bar{Q}_{C_2})^2 = \frac{1}{\Lambda}[(\Lambda + V)^4 + 2(\Lambda - 1)V^2(\Lambda + V)^2 + (\Lambda - 1)^2 V^4],$$

$$\bar{Q}_P(\Lambda, V) = (\Lambda + V)^4 + (\Lambda - 1)V^3(\Lambda + V),$$

where P is the graph consisting of a triangle and one edge connecting it to a fourth node, that is, K_4 with two adjacent edges removed.

Writing Θ for $K_4\backslash$edge, let e be the edge whose endpoints are trivalent. Apply Theorem 3.1 with $\Omega = [\Theta]$:

$$\begin{aligned}\bar{Q}_\Theta(\Lambda, V) &= \bar{Q}_{C_4} + V\bar{Q}_{C_2 \cup_p C_2} \\ &= \frac{1}{\Lambda}(\Lambda + V)^5 + (\Lambda - 1)V^4 \\ &\quad + 2(1 - \Lambda^{-1})V^3(\Lambda + V)^2 + \Lambda(1 - \Lambda^{-1})^2 V^5.\end{aligned}$$

The same theorem implies that $\bar{Q}_{K_4} = \bar{Q}_\Theta + V\bar{Q}_{K_4/e}$. Let f be the edge not adjacent to e in K_4. Then K_4/e consists of $C_2 \cup_p C_2$ with the extra edge f linking the two divalent nodes. We deduce from Theorem 3.1 again (by deleting and contracting f in K_4/e) that

$$\begin{aligned}\bar{Q}_{K_4}(\Lambda, V) &= \bar{Q}_\Theta + V\bar{Q}_{C_2 \cup_p C_2} + V^2\bar{Q}_{4K_2} \\ &= \frac{1}{\Lambda}(\Lambda + V)^5 + \frac{V}{\Lambda}(\Lambda + V)^4 + (\Lambda - 1)V^4 \\ &\quad + 4(1 - \Lambda^{-1})V^3(\Lambda + V)^2 \\ &\quad + 2\Lambda(1 - \Lambda^{-1})^2 V^5 + \Lambda(\Lambda - 1)V^2 + \Lambda V^2(V + 1)^4.\end{aligned}$$

13c. *Return to the Seven Dwarves*

We call these seven biased graphs $\Omega_i = \Omega_i(K_4)$ for $i = 1, 2, \ldots, 7$. Briefly recalling their definitions from Part I: At the two extremes are the balanced $\Omega_1 = [K_4]$ and the contrabalanced $\Omega_7 = (K_4, \varnothing)$. For $j = 1, 2, 3$, Ω_{7-j} is the same as Ω_7, except for having exactly j balanced quadrilaterals. Ω_4 is the parity-biased graph $[-K_4]$. Ω_3 is like Ω_7, except for having one balanced triangle. Ω_2 has two balanced triangles and, therefore, must have just one balanced quadrilateral—the one contained in the union of the two triangles.

We write ξ_i and κ_i for the numbers of balanced triangles and quadrilaterals in Ω_i.

We compute the polynomials of the Ω_i by the method of Section 13a. The obvious choice for Ω, no matter which Ω_i we treat, is Ω_7, mainly because it is by far the easiest to solve (by Example 3.4). Then a number of nice things happen. In (13.1), each balanced triangle $S_0 \subseteq E$ gives rise to four terms of Δq_i^{b}: one for $S = S_0$ (so that $K_4 | S = (N, S) \cong C_3 \cup K_1$) and three for each $S = S_0 \cup e$ (isomorphic to P). The total of these four terms is precisely Δq_3^{b}. Thus part of Δq_i^{b} is $\xi_i \Delta q_3^{\mathrm{b}}$. The rest is terms arising from balanced subgraphs isomorphic to C_4, Θ, or K_4.

In computing Δq_i for $i \neq 1$ there are other simplifications. (In the exceptional case we have $q_{\Omega_1} = q_{\Omega_1}^{\mathrm{b}}$ so there is nothing to discuss.) First, $b(\Omega_i{:}W^c) = b(\Omega_7{:}W^c) = \# W^c$, unless $\Omega_i{:}W^c \cong [C_3]$. Consequently, the first term in (13.2) reduces to $\xi_i(\lambda-1)\,w^3(x^2-x^3)$. In the second term the only W for which $\Delta q^{\mathrm{b}}_{i,W}\neq 0$ are $W=N$ and tripletons W supporting balanced triangles. For the latter we have

$$q^{\mathrm{b}}_{i,W}(w,x,\lambda-1,v)=(\lambda-1)\,x^3v^3\bar{Q}_{C_3}\left(\frac{1}{x},\frac{w}{v}\right)$$

$$=(\lambda-1)s^3+(\lambda-1)\,w^3(x^2-x^3),$$

where $s=wx+v$.

Thus the general solutions, expressed in terms of Δq^{b}_i, are

$$q^{[\mathrm{b}]}_{\Omega_i}(w,x,\lambda,v)=q^{[\mathrm{b}]}_{\Omega_7}(w,x,\lambda,v)+q^{[\mathrm{b}]}_i(w,x,\lambda,v) \tag{13.3}$$

for all i, and

$$\Delta q_i(w,x,\lambda,-1)=\xi_i(\lambda-1)[s^3+2w^3x^2(1-x)]+\Delta q^{\mathrm{b}}_i(w,x,\lambda-1,-1) \tag{13.4}$$

if $i\neq 1$, where $s=wx-1$. Other polynomials are obtained by specializing the variables as in Section 3; for instance, the balanced and unrestricted normalized dichromatic polynomials are $q^{\mathrm{b}}_{\Omega_i}(0,1,\lambda,v)$ by definition and $q_{\Omega_i}(v+1,1,\lambda,-1)$ by (3.1a); for the chromatic polynomials the arguments are $(0,1,\lambda,-1)$.

As for the matroids, the polynomials of the bias matroid $G(\Omega_i)$ equal the corresponding polynomials of Ω_i (as shown in Theorem 5.1), except for division by λ in the case $i=1$. The lift-matroid polynomials of Ω_i equal those of the bias matroid, since $L(\Omega_i)=G(\Omega_i)$ because there are no two node-disjoint unbalanced circles. (One could also use Theorem 5.2, if $i\neq 1$.) For the complete lift matroid $L_0(\Omega_i)$ we have

$$\mathscr{R}_{L_0(\Omega_i)}(w,s,\lambda,-1)=\lambda^{-1}(\lambda-1)\,q^{\mathrm{b}}_{\Omega_i}(w,x,\lambda,-1)+\lambda^{-1}wxq_{K_4}(w,x,\lambda,-1)$$

and specializations. The quantity $q_{K_4}=q^{\mathrm{b}}_{[K_4]}$ is written out explicitly in Example 13.1. Thus, again, everything depends on Δq^{b}_i.

Now we are ready to find $\Delta q^{[\mathrm{b}]}_i$ for every i. The most natural order of presentation, odd though it may seem, is to begin with Ω_7 and proceed in reverse subscript order, for that is roughly the order of increasing balance.

EXAMPLE 13.7. Since $\Omega_7=(K_4,\varnothing)$, Example 3.4 gives

$$q^{\mathrm{b}}_{\Omega_7}(w,x,\lambda,v)=\lambda^4+6\lambda^3s+15\lambda^2s^2+16\lambda s^3,$$

where $s = wx + v$, and

$$q_{\Omega_7}(w, x, \lambda, -1) = (\lambda-1)^4 + [6s+4](\lambda-1)^3 + [15s^2+12s+6wx^2](\lambda-1)^2 + [16s^3+12s^2+6wx^2s+4w^3x^3](\lambda-1) + w^6x^4,$$

where $s = wx - 1$. The chromatic polynomials are, therefore,

$$\chi^{\mathrm{b}}_{\Omega_7}(\lambda) = \lambda^4 - 6\lambda^3 + 15\lambda^2 - 16\lambda = (\lambda)_4 + 4(\lambda)_2 - 6(\lambda)_1$$

and

$$\chi_{\Omega_7}(\lambda) = (\lambda-1)^4 - 2(\lambda-1)^3 + 3(\lambda-1)^2 - 4(\lambda-1)$$
$$= (\lambda-1)_4 + 4(\lambda-1)_3 + 20(\lambda-1)_2 + 14(\lambda-1)_1.$$

Evidently $\chi_{\Omega_7}(\lambda) > 0$ for all integral $\lambda > 1$. As for $\chi^{\mathrm{b}}_{\Omega_7}(\lambda)$, it is negative if $\lambda = 1$ or 2, but positive for integral $\lambda \geqslant 3$. (This fact has a combinatorial interpretation: Theorem 4.2 (with $k = 1$) implies that in a gain graph Φ for which $[\Phi] = (K_4, \varnothing)$ the group must have order 3 at least. This is consistent with what we know from Example I.7.7, but weaker; actually, the gain group must be a group of order at least four, but not $\mathbb{Z}_2 \times \mathbb{Z}_2$).

EXAMPLES 13.4–13.6. In Ω_i for $i = 4, 5, 6$ there are respectively exactly $\kappa_i = 3, 2, 1$ balanced quadrilaterals but no other balanced circles, so $\xi_i = 0$. Consequently, the only balanced sets S which are not forests are the balanced quadrilaterals. (So this is a case of Hamiltonian bias, Example 12.2). It follows that

$$\Delta q^{\mathrm{b}}_i(w, x, \lambda, v) = \kappa_i \lambda x^4 v^4 Q_{C_4}\left(\frac{1}{x}, \frac{w}{v}\right) = \kappa_i \lambda [s^4 + w^4x^3(1-x)],$$

where $s = wx + v$, and, by (13.4),

$$\Delta q_i\,(w, x, \lambda, -1) = \Delta q^{\mathrm{b}}_i(w, x, \lambda - 1, -1).$$

Specializing to the chromatic polynomials ($w = 0$, $s = v = -1$) yields

$$\chi^{\mathrm{b}}_{\Omega_i}(\lambda) = \lambda^4 - 6\lambda^3 + 15\lambda^2 - (16-\kappa_i)\lambda,$$

$$\chi_{\Omega_i}(\lambda) = (\lambda-1)^4 - 2(\lambda-1)^3 + 3(\lambda-1)^2 - (4-\kappa_i)(\lambda-1).$$

EXAMPLE 13.3. Since the only balanced circle is one triangle (so that $\xi_3 = 1$, $\kappa_3 = 0$), the balanced sets which are not forests are one triangle and three sets isomorphic to P. From (13.1) we see that

$$\Delta q_3^{\mathrm{b}}(w, x, \lambda, v) = x^4 v^3 \bar{Q}_{C_3 \cup K_1}\left(\frac{1}{x}, \frac{w}{v}\right) + 3x^4 v^4 \bar{Q}_P\left(\frac{1}{x}, \frac{w}{v}\right)$$
$$= (\lambda^2 + 3\lambda s)[s^3 + w^3 x^2(1-x)],$$

where $s = wx + v$. Thus (13.4) gives

$$\Delta q_3(w, x, \lambda, -1) = (\lambda - 1)^2 [s^3 + w^3 x^2 (1-x)]$$
$$+ (\lambda - 1)[3s^4 + s^3 + w^3 x^2 (1-x)(3s+2)],$$

where $s = wx - 1$. The chromatic polynomials are

$$\chi_{\Omega_3}^{\mathrm{b}}(\lambda) = \lambda^4 - 6\lambda^3 + 14\lambda^2 - 13\lambda,$$
$$\chi_{\Omega_3}(\lambda) = (\lambda-1)^4 - 2(\lambda-1)^3 + 2(\lambda-1)^2 - 2(\lambda-1).$$

EXAMPLE 13.2. Here $\xi_2 = 2$ and $\kappa_2 = 1$. The nonforest balanced subsets S are the two triangles and six associated P's, one C_4, and the Θ which is the union of the two balanced triangles. Thus from (13.1) we get

$$\Delta q_2^{\mathrm{b}}(w, x, \lambda, v) = 2\Delta q_3^{\mathrm{b}} + \lambda x^4 v^4 Q_{C_4}\left(\frac{1}{x}, \frac{w}{v}\right) + \lambda x^4 v^5 Q_\Theta\left(\frac{1}{x}, \frac{w}{v}\right)$$
$$= 2\lambda^2[s^3 + w^3 x^2(1-x)]$$
$$+ \lambda[s^5 + 7s^4 + (2s^2 + 6s + wx(v + w + 1))\, w^3 x^2 (1-x)]$$

and

$$\Delta q_2(w, x, \lambda, -1) = 2(\lambda - 1)^2 [s^3 + w^3 x^2 (1-x)]$$
$$+ (\lambda - 1)[s^5 + 7s^4 + s^3$$
$$+ (2s^2 + 6s + wx(v + w + 1) + 2)\, w^3 x^2 (1-x)].$$

For the chromatic polynomials we have

$$\chi_{\Omega_2}^{\mathrm{b}}(\lambda) = \lambda^4 - 6\lambda^3 + 13\lambda^2 - 10\lambda,$$
$$\chi_{\Omega_2}(\lambda) = (\lambda-1)^4 - 2(\lambda-1)^3 + (\lambda-1)^2 + (\lambda-1).$$

EXAMPLE 13.1. This is the balanced graph, $\Omega_1 = [K_4]$, where the balanced and unrestricted polynomials coincide. Therefore we need only find Δq_1^{b} and apply (13.3). On the other hand, we could call upon Example 8.5. Let us do both, since they yield different expressions.

In Example 8.5 let us take $v=-1$ for simplicity. Thus $\theta_i = x^{i-1} w^{\binom{i}{2}}/i!$ and we get

$$q_{\Omega_i}^{[b]}(w, x, \lambda, -1) = (\lambda)_4 + 12wx(\lambda)_3 + 4w^2(w+3)\, x^2(\lambda)_2 + w^6x^3(\lambda)_1.$$

Now we calculate Δq_1^b from (13.1). We have $\xi_1 = 4$ and $\kappa_1 = 3$. There are four triangles and 12 associated P's, three C_4's, six Θ's, and one K_4. So

$$\begin{aligned}
\Delta q_1^b(w, x, \lambda, v) &= 4\Delta q_3^6 + 3\lambda x^4 v^4 Q_{C_4}\left(\frac{1}{x}, \frac{w}{v}\right) + 6\lambda x^4 v^5 Q_\Theta\left(\frac{1}{x}, \frac{w}{v}\right) \\
&\quad + \lambda x^4 v^6 Q_{K_4}\left(\frac{1}{x}, \frac{w}{v}\right) \\
&= 4\lambda^2[s^3 + w^2x^2(1-x)] \\
&\qquad + \lambda[(6+v)s^5 + 15s^4 + w^2x^2(1-x)\{12ws^2 + 4(3+v)\, ws \\
&\qquad\quad + 3w^2x((v+1)^2 + 2w) \\
&\qquad\quad + v^4 + (w+v)^4\, x + 2vw^3x(1-x)\}\,].
\end{aligned}$$

From this and $\chi_{\Omega_7}^b(\lambda)$ we deduce that

$$\chi_{[K_4]}(\lambda) = \chi_{[K_4]}^b(\lambda) = \lambda^4 - 6\lambda^3 + 11\lambda^2 - 6\lambda = (\lambda)_4,$$

which agrees with the calculation from Example 8.5 and, of course, is correct because $\chi_{[K_4]} = \chi_{K_4}$.

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References

1. T. Brylawski, A combinatorial model for series-parallel networks, *Trans. Amer. Math. Soc.* **154** (1971), 1–22.
2. T. H. Brylawski, A decomposition for combinatorial geometries, *Trans. Amer. Math. Soc.* **171** (1972), 235–282.
3. T. Brylawski, Modular constructions for combinatorial geometries, *Trans. Amer. Math. Soc.* **203** (1975), 1–44.
4. H. H. Crapo, A higher invariant for matroids, *J. Combin. Theory* **2** (1967), 406–417.
5. H. H. Crapo, The Tutte polynomial, *Aequationes Math.* **3** (1970), 211–229.
6. H. H. Crapo and G.-C. Rota, "On the Foundations of Combinatorial Theory: Combinatorial Geometries," preliminary ed., MIT Press, Cambridge, MA, 1970.

7. T. A. Dowling, A q-analog of the partition lattice, *in* "A Survey of Combinatorial Theory" (J. N. Srivastava *et al.*, Eds.), pp. 101–115, North-Holland, Amsterdam, 1973.
8. T. A. Dowling, A class of geometric lattices based on finite groups, *J. Combin. Theory Ser. B* **14** (1973), 61–86; Erratum, *ibid.* **15** (1973), 211.
9. T. A. Dowling and D. G. Kelly, Elementary strong maps between combinatorial geometries, *in* "Colloquio Internationale sulle Teorie Combinatorie" (Proc., Rome, 1973), Vol. II, pp. 121–152. Accad. Naz. Lincei, Rome, 1976.
10. T. A. Dowling and R. M. Wilson, The slimmest geometric lattices, *Trans. Amer. Math. Soc.* **196** (1974), 203–216.
11. P. Hanlon and T. Zaslavsky, Tractable partially ordered sets derived from root systems and biased graphs, submitted.
12. J. Kahn and J. P. S. Kung, Varieties of combinatorial geometries, *Trans. Amer. Math. Soc.* **271** (1982), 485–499.
13. V. Klee, The greedy algorithm for finitary and cofinitary matroids, *in* "Combinatorics" (Proc. Symp. Pure Math., Vol. 19) (T. S. Motzkin, Ed.), pp. 137–152, Amer. Math. Soc., Providence, RI, 1971.
14. J. W. Moon, "Counting Labelled Trees" (Canad. Math. Monographs No. 1), Canadian Mathematical Congress, Montreal, 1970.
15. A. Rényi, Some remarks on the theory of trees, *Publ. Math. Inst. Hungar. Acad. Sci.* **4** (1959), 73–85.
16. G.-C. Rota, On the foundations of combinatorial theory. I. Theory of Möbius functions, *Z. Wahrsch.* **2** (1964), 340–368.
17. J. M. S. Simões-Pereira, On subgraphs as matroid cells, *Math. Z.* **127** (1972), 315–322.
18. R. P. Stanley, Supersolvable lattices, *Algebra Universalis* **2** (1972), 197–217.
19. W. T. Tutte, A ring in graph theory, *Proc. Cambridge Philos. Soc.* **43** (1947), 26–40; reprinted with commentary in "Selected Papers of W. T. Tutte" (D. McCarthy and R. G. Stanton, Eds.), Vol. I, pp. 51–69, Charles Babbage Research Centre, St. Pierre, Manitoba, Canada, 1979.
20. W. T. Tutte, On dichromatic polynomials, *J. Combin. Theory* **2** (1967), 301–320; reprinted with commentary in "Selected Papers of W. T. Tutte" (D. McCarthy and R. G. Stanton, Eds.), Vol. II, pp. 525–547, Charles Babbage Research Centre, St. Pierre, Manitoba, Canada, 1979.
21. M. L. Wachs and J. W. Walker, On geometric semilattices, *Order* **2** (1986), 367–385.
22. L. Weisner, Abstract theory of inversion of finite series, *Trans. Amer. Math. Soc.* **38** (1935), 474–492.
23. D. J. A. Welsh, "Matroid Theory," Academic Press, London, 1976.
24. H. Whitney, A logical expansion in mathematics, *Bull. Amer. Math. Soc.* **38** (1932), 572–579.
25. H. Whitney, The coloring of graphs, *Ann. of Math. (2)* **33** (1932), 688–718.
26. G. Whittle, Dowling group geometries and the critical problem, *J. Combin. Theory Ser. B* **47** (1989), 80–92.
27. T. Zaslavsky, The geometry of root systems and signed graphs, *Amer. Math. Monthly* **88**, No. 2 (1981), 88–105.
28. T. Zaslavsky, Signed graphs, *Discrete Appl. Math.* **4** (1982), 47–74; Erratum, *ibid.* **5** (1983), 248.
29. T. Zaslavsky, Signed graph coloring, *Discrete Math.* **39** (1982), 215–228.
30. T. Zaslavsky, Chromatic invariants of signed graphs, *Discrete Math.* **42** (1982), 287–312.
31. T. Zaslavsky, Bicircular geometry and the lattice of forests of a graph, *Quart. J. Math. Oxford (2)* **33** (1982), 493–511.
32. T. Zaslavsky, The Möbius function and the characteristic polynomial, Chapter 7 of "Combinatorial Geometries" (N. White, Ed.), Encycl. of Math. and its Appl., Vol. 29, pp. 114–138, Cambridge Univ. Press, Cambridge, 1987.

33. T. Zaslavsky, Biased graphs. I. Bias, balance, and gains, *J. Combin. Theory Ser. B* **47** (1989), 32–52.
34. T. Zaslavsky, Biased graphs. II. The three matroids, *J. Combin. Theory Ser. B* **51** (1991), 46–72.
35. T. Zaslavsky, Biased graphs. IV. Geometrical realizations, *J. Combin. Theory Ser. B*, to appear.
36. T. Zaslavsky, Geometric lattices of structured partitions. II. Lattices of group-valued partitions based on graphs and sets, unpublished manuscript, 1985.
37. T. Zaslavsky, Supersolvability of frame-matroid and graphic-lift lattices, in preparation.