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Biased Graphs. II. The Three Matroids*

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A *biased graph* Ω consists of a graph Γ and a class $\mathscr{B}$ of circles (simple, closed paths) in Γ, called *balanced* circles, such that no theta subgraph contains exactly two balanced circles. The *bias matroid* $G(\Omega)$ is a finitary matroid on the edge set E of Ω whose circuits are the balanced circles and the minimal connected edge sets of cyclomatic number two containing no balanced circle. We prove that these circuits define a matroid and we establish cryptomorphic definitions and other properties. Another finitary matroid on E, the *lift matroid* $L(\Omega)$, and its one-point extension the *complete lift matroid* $L_0(\Omega)$, are obtained from the abstract matroid lift construction applied to the graphic matroid $G(\Gamma)$ and the class $\mathscr{B}$. The circuits of $L(\Omega)$ are the balanced circles and the minimal edge sets of cyclomatic number two (not necessarily connected) containing no balanced circle. We develop cryptomorphisms and other properties of $L_0(\Omega)$ and $L(\Omega)$. There is no completely general construction rule, besides the bias and lift constructions, which assigns to each biased graph a matroid intermediate (in the sense of independent sets) between G and L and which respects subgraphs. $G(\Omega)$ has an infinitary analog for infinite graphs generalizing Klee's infinitary bicircular matroid and the Bean–Higgs infinitary graphic matroid. Whether $L(\Omega)$ has an infinitary analog is unclear.

INTRODUCTION

A *biased graph* $\Omega = (\Gamma, \mathscr{B})$ consists of an underlying graph $\Gamma = \|\Omega\|$ and a subclass $\mathscr{B} = \mathscr{B}(\Omega)$ of the class $\mathscr{C}$ of circles of Γ (edge sets of closed, simple walks) which is a *linear subclass*: that is, if $C_1, C_2 \in \mathscr{B}$ and their union is a theta graph, the third circle in $C_1 \cup C_2$ also belongs to $\mathscr{B}$.[1] A circle in $\mathscr{B}$ is called *balanced.* In this article we establish the existence and elementary properties of two finitary matroids on the edge set of a biased graph, which we call the *bias matroid* $G(\Omega)$ and the *lift matroid* $L(\Omega)$, and a

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[1] The different definition given in the introduction of [32] is erroneous.

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matroid $L_0(\Omega)$, the *complete lift matroid*, which is a canonical one-point extension of $L(\Omega)$. The circuits of the bias matroid are the balanced circles and the minimal connected edge sets with cyclomatic number two that do not contain a balanced circle. Those of the lift matroid are the same except that the circuits of the second kind are not required to be connected. The lift matroid is the result of the general matroid elementary-lift construction [4, 5, 16] applied to the graphic matroid $G(\Gamma)$; the bias matroid, on the contrary, depends on the graph itself and does not generalize to abstract matroids, and this is one reason to think it is interesting. (There are other means of generalizing it but that is outside our scope here.)

The bias matroid generalizes several previously known matroids on the edge set of a graph. The familiar graphic or polygon matroid is $G(\Gamma, \mathscr{C})$. The bicircular matroid of Γ, introduced by Simões-Pereira [22, 23] and by Klee [13] in an infinitary version, is $G(\Gamma, \varnothing)$; it has been further studied in [17, 28]. The even-cycle matroid employed by Doob [7] is $G(\Gamma, \mathscr{B}_2)$, where $\mathscr{B}_2$ is the class of even-length circles (Example I.6.3).[2] The bias matroids of graphs with signed edges appeared in [27]; biased graphs were invented as a natural generalization. The optimization matroids of networks with gains are still another example; they are the bias matroids of graphs with real (or real-positive) multipliers or "gains" on their edges; here a circle is balanced if its gain product is 1. (One of many references for networks with gains is [20]. For the bias determined by group gains see Section I.5.) Matthews' two matroids on directed graphs [18] are bias matroids of graphs with particular gains (Examples I.6.5 and I.6.6). Dowling's lattices of a group ([9], foreshadowed in [8]), which are the geometric lattices of the bias matroids of the maximal gain graphs of each number of nodes over the particular group, were indirectly the inspiration for this entire project. (They are also in a sense "universal" matroids, in company with the projective geometries, as Kahn and Kung showed in [12].)

The lift matroid has been less often discovered, as far as I am aware. $L(\Gamma, \mathscr{C})$ is the graphic matroid. The lift matroid of a sign-biased graph is a subject of Shih's thesis [21] and was the main tool employed by Lovász and Schrijver in their work on disjoint unbalanced circles in signed graphs [15].

Our principal aim is to present eleven cryptomorphic definitions of the bias matroid (in Section 2) and the lift and complete lift (Section 3), to show what operations on the biased graph correspond to matroid restriction (take a subgraph) and contraction (contract the biased graph, for the bias matroid; contract the graph and sometimes forget the bias, in the lift

[2] Citations in the style I.6.3, IV.2.4, etc., refer to [29, 30, 31], which are Parts I, III, and IV of this series.

matroid $L_0(\Omega)$, the *complete lift matroid*, which is a canonical one-point extension of $L(\Omega)$. The circuits of the bias matroid are the balanced circles and the minimal connected edge sets with cyclomatic number two that do not contain a balanced circle. Those of the lift matroid are the same except that the circuits of the second kind are not required to be connected. The lift matroid is the result of the general matroid elementary-lift construction [4, 5, 16] applied to the graphic matroid $G(\Gamma)$; the bias matroid, on the contrary, depends on the graph itself and does not generalize to abstract matroids, and this is one reason to think it is interesting. (There are other means of generalizing it but that is outside our scope here.)

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In Section 4 we compare the bias and lift matroids. The former is weaker than the latter (in the sense of independent sets), so one might wonder whether there are intermediate matroids associated with Ω in a systematic way. We show that there are no such matroids defined over all biased graphs and compatible with restriction to subgraphs. It remains possible, however, that systematic intermediate constructions do exist over most biased graphs. This problem is potentially interesting because one would like to axiomatize the relationship between $G(\Omega)$ and $L(\Omega)$, which so clearly depends on connectedness, and because of the interesting geometric lattices which can be derived from bias and lift matroids of particular biased graphs [35] and which one might hope to generalize.

We conclude with remarks on possible infinitary analogs of the bias and lift matroids. The latter continues to be a puzzle but the former can be defined by imitating Klee's infinitary bicircular matroid. This infinitary bias matroid $G^\infty(\Omega)$ generalizes both Klee's matroid and the Bean–Higgs infinitary graphic matroid [1, 11, 13].

This article is the sequel to Part I [29]; we assume the reader is familiar with the definitions therein. In Parts III and IV [30, 31], we continue the series with an investigation of curious reduction formulas for the chromatic and dichromatic invariants of the bias and (complete) lift matroids and with a study of geometrical and logical realizations of the matroids.

1. Preliminaries

By Ω we always mean a biased graph with underlying graph $\Gamma = (N, E) = \|\Omega\|$, whose node set is N, edge set is E, and order is $n = \#N$, not necessarily finite. The class of circles of Γ is $\mathscr{C} = \mathscr{C}(\Gamma)$ and that of balanced circles of Ω is $\mathscr{B} = \mathscr{B}(\Omega)$. The *cyclomatic number* of a graph Γ is the number of independent circles in it, which equals $\#(E \setminus T)$, where T is a maximal forest. Loose and half edges count as loops in this computation. (In the matroid theory, as will be seen, loose and half edges can be treated as balanced and unbalanced loops, respectively.) A *cutset* in Γ is a nonempty edge set consisting of all the edges with one endpoint in some node set X and the other in X^c. A *unicycle* is a connected edge set with cyclomatic number 1: that is, a tree T plus one edge (not loose) whose nodes lie in $N(T)$. The union of an edge set and a singleton edge, say $S \cup \{e\}$, we usually abbreviate by $S \cup e$.

We assume the reader is acquainted with matroid theory as presented in [6, 25, 26]. Our notation for the contraction of a matroid M by a point set A is M/A; this is Welsh's $M.(E \backslash A)$, where E is the ground set of M. A *bond* is the complement of a copoint. The closure is clos_M; the lattice of flats is $\operatorname{Lat} M$; the rank function is rk_M; the dual matroid is $M^\perp$. The *graphic* (or "circuit" or "polygon") *matroid* of an ordinary graph Γ is denoted by $G(\Gamma)$; recall that an *ordinary graph* is an unbiased graph with no loose or half edges. If Γ is not ordinary, loose edges should be treated like ordinary loops; half edges should be treated as links to an extra node v_0, not in Γ. Then $G(\Gamma)$ is defined to be the graphic matroid of this fictitious adapted Γ. We write $\operatorname{clos}_\Gamma$ and rk_Γ for the closure and rank in $G(\Gamma)$. A uniform matroid of rank r on n points is written $U_{r,n}$; if its point set is $\{e\}$ it may be denoted by $(e)_r$.

For a biased graph Ω we define

$$\operatorname{Lat}^b \Omega = \{S \subseteq E : S \text{ is balanced and balance-closed}\}.$$

If A is a balanced edge set, we define

$$(\operatorname{Lat}^b \Omega)/A = \{S \backslash A : S \in \operatorname{Lat}^b \Omega \text{ and } S \supseteq A\}.$$

A *balancing node* of Ω is a node $v \in N(\Omega)$ such that $\Omega \backslash v$ is balanced although Ω is not.

2. The Bias Matroid

The first theorem characterizes the *bias matroid* $G(\Omega)$. Recall that a *bias circuit* is (the edge set of) a balanced circle, or the union of two unbalanced figures which meet at just one node (a contrabalanced *tight handcuff*), or the union of two node-disjoint unbalanced figures and a path meeting each figure at one endpoint and nowhere else (a contrabalanced *loose handcuff*), or a contrabalanced theta graph. An alternate definition is that a bias circuit is a balanced circle or a minimal edge set that is connected and contrabalanced and has cyclomatic number two.

THEOREM 2.1. *Let Ω be a biased graph. There is a matroid $G(\Omega)$, whose points are the edges of Ω, which is determined by any of the following equivalent definitions* (a)–(k). *The matroid is finitary.*

(a) *The closure of an edge set S is*

$$\operatorname{clos}_{G(\Omega)}(S) = \operatorname{bcl}(S) \cup E{:}N_0(S).$$

nodes lie in $N(T)$. The union of an edge set and a singleton edge, say $S \cup \{e\}$, we usually abbreviate by $S \cup e$.

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(a) *The closure of an edge set S is*

$$\text{clos}_{G(\Omega)}(S) = \text{bcl}(S) \cup E{:}N_0(S).$$

(b) *An edge set A is closed $\Leftrightarrow$ it is the union of $E{:}X$, where X is some set of nodes (possibly $\varnothing$), a balanced and balance-closed subset of $E{:}X^c$, and the loose edges of Ω.*

(c) *An edge set is independent $\Leftrightarrow$ it has no loose edges and each component is a tree or an unbalanced unicycle.*

(d) *An edge set S is dependent $\Leftrightarrow$ it contains a balanced circle, two unbalanced figures connected within S, or a loose edge.*

(e) *An edge set is a matroid circuit $\Leftrightarrow$ it is a bias circuit of Ω.*

(f) *An edge set S spans $G(\Omega)$ $\Leftrightarrow$ it connects the nodes of each balanced component of Ω, and each other component of (N, S) contains an unbalanced figure.*

(g) *An edge set S is a basis $\Leftrightarrow$ it consists of a spanning tree in each balanced component of Ω, each other component of (N, S) is an unbalanced unicycle, and S contains no loose edges.*

(h) *An edge set H is a copoint $\Leftrightarrow$ H has the form $A \cup E{:}X^c$ where $A \subseteq E{:}X$, $\varnothing \neq X \subseteq Y \in \pi(\Omega)$; and either $\Omega{:}Y$ is balanced, $A = E{:}X$, and both $\Omega{:}X$ and $\Omega{:}(Y \backslash X)$ are nonnull and connected, or else $\Omega{:}Y$ is unbalanced and so is every component of $\Omega{:}(Y \backslash X)$, and (X, A) is connected, balanced, and balance-closed.*

(i) *An edge set is a bond of $G(\Omega)$ $\Leftrightarrow$ it is a minimal set whose deletion increases the number of balanced components of Ω.*

(j) *The rank of an edge set S is*

$$\text{rk}_{G(\Omega)}(S) = |N_0(S)| + \sum_{B \in \pi_b(S)} (|B| - 1)$$

and if n is finite it is

$$\text{rk}_{G(\Omega)}(S) = n - b(S).$$

(k) *The corank of an edge set S is*

$$b(S{:}N_0(\Omega)) + \sum_{B \in \pi_b(\Omega)} (b(S{:}B) - 1),$$

which equals $b(S) - b(\Omega)$ if $b(\Omega)$ is finite.

This theorem (without (k)) was stated for signed graphs in [27, Theorem 5.1] with a partial proof. Unfortunately, parts (f) and (g) were stated incorrectly.

In [27] we proved that $\text{rk}_{G(\Omega)}$ is a Whitney rank function and we deduced (e) from (d), leaving the remainder to the reader. The proofs given in [27] are valid for gain-biased graphs with obvious slight changes. They are also valid for biased graphs except for the argument in the last case of case (4) in [27, p. 55], which requires Corollary I.3.8 for full generality.

A proof of (e) may be based on Simões-Pereira [22], whose Lemma 3 states, in essence, that the union of two unbalanced bias circuits, less one common edge, contains a handcuff or theta graph. It therefore contains a bias circuit. One can deduce that the bias circuits are the circuits of a matroid.

Proofs of parts (e) and (g) for real-multiplicative gain graphs have appeared in the literature of "generalized networks" or "networks with gains."

Proof. We give a complete proof based on (a) and (b). We use the observation:

LEMMA 2.2. *If Ω is balanced, then $G(\Omega)$ equals the graphic matroid $G(\|\Omega\|)$.* ∎

Proof of (a) and (b). We may ignore any loose edges. We show first that $\text{clos} = \text{clos}_{G(\Omega)}$ is a closure operator and its closed sets are the closed sets of (b). Let S be a union of balanced components S_j and unbalanced components. Evidently,

$$\text{clos}\, S = E{:}N_0(S) \cup \bigcup_j \text{bcl}\, S_j.$$

Thus any set of the form clos S is (b)-closed. It is obvious from the definitions that $S \subseteq \text{clos}\, S$ and that, for balanced sets B and B', where $B \subseteq B'$, we have $\text{bcl}\, B \subseteq \text{bcl}\, B'$. Since bcl B is balanced and balance-closed if B is balanced (Propositions I.3.1 and I.3.5), $\text{clos}^2 S = \text{clos}\, S$ and any (b)-closed set equals its own closure. If $S \subseteq T$, then $N_0(S) \subseteq N_0(T)$ and every S_j is contained either in $E{:}N_0(T)$ or in some balanced component T_k; in either case we have $\text{bcl}\, S_j \subseteq [\text{bcl}\, T_k \subseteq]\ \text{clos}\, T$. Thus (a) and (b) describe a closure operator and its closed sets.

We still have to prove the exchange property of clos. Let A be closed (in the equivalent senses of (b) or that $\text{clos}\, A = A$), $A_0 = E{:}N_0(A)$, $e \notin A$, $A' = \text{clos}(A \cup e)$, and $f \in A' \backslash A$. There are three cases.

If e connects two balanced components of A, say B_1 and B_2, then $B_1 \cup B_2 \cup e$ is still balanced and $f \in C$, where C is a balanced circle in $B_1 \cup B_2 \cup e \cup f$. Clearly f too connects B_1 and B_2 and $e \in \text{clos}(B_1 \cup B_2 \cup f)$.

If e links $N_0(A)$ to a balanced component B, then $\text{clos}(A_0 \cup B \cup e) = E{:}[N_0(A) \cup N(B)]$, call it B_0 for short, and $f \in B_0 \backslash (A_0 \cup B)$. If f also connects $N_0(A)$ to B, then $\text{clos}(A{:}N_0(A) \cup B \cup f) = B_0$. The only other possibility is that $f \in E{:}N(B)$; but since B (being a component of A) is closed and $f \notin B$, $B \cup f$ cannot be balanced. Therefore $N_0(A \cup f) = N_0(A) \cup N(B)$, which entails $\text{clos}(A_0 \cup f) = B_0 \ni e$.

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The third case is $N(e) \subseteq N(B)$, for some balanced component B of A. Since B is closed and $e \notin B$, it is impossible that $B \cup e$ could be balanced. Therefore $N_0(A \cup e) = N_0(A) \cup N(B)$, and $\mathrm{clos}(A_0 \cup B \cup e) = E{:}[N_0(A) \cup N(B)]$. There are two possibilities for f: either f links $N_0(A)$ to B, or $f \in E{:}N(B)$. These can be disposed of as under the second case of e, above.

This completes the proof that (a) and (b) describe the closure operator and the closed sets of a matroid $G = G(\Omega)$.

G is finitary because of two facts: since all balanced circles are finite, bcl is finitary; since connection is by finite paths, each edge in $E{:}N_0(S)$ is joined to an unbalanced figure F by a finite path P, hence lies in the closure of $F \cup P$.

Parts (c) and (d) are complementary. A loose edge or a balanced circle, or two connected unbalanced figures, are obviously dependent (from (a)). A tree is equally obviously independent (cf. Lemma 2.2, for instance). A tree T with one extra edge that makes an unbalanced figure $F \subseteq T \cup e$ is independent because deleting an edge $f \in F$ leaves a balanced set, whose closure does not contain F, and deleting an edge outside F disconnects $T \cup e$ into a balanced and an unbalanced part. Thus (c) and (d) are proved.

For part (e) we have to prove every (matroid) circuit is a bias circuit. (The converse is obvious.) A balanced circuit is clearly a circle. Let S be an unbalanced matroid circuit—thus, S is connected and contrabalanced and has no univalent nodes. We may treat any half edges as unbalanced loops. Since S contains two circles, removing an edge e from one circle leaves an independent set I that consists of a tree and one more edge forming a circle. I has at most two univalent nodes, which are endpoints of e. Thus, adding back e creates a theta graph or a handcuff.

The next step is to prove (j). For a balanced set B, (j) follows from Lemma 2.2 applied to $\Omega | B$. If A is connected and unbalanced, take a spanning tree $T \subseteq A$ and let $A' = \mathrm{clos}\, T = \mathrm{bcl}\, T$. By Proposition I.3.1, A' is balanced; thus its rank is $\# N(A) - 1$. Because A is not balanced, $A' \subset A$; therefore $\mathrm{rk}\, A \geqslant \# N(A)$. On the other hand, let $e \in A \backslash A'$; then $\mathrm{clos}(A' \cup e) = A$ by part (a), so $\mathrm{rk}\, A \leqslant \# N(A)$.

Now consider a set $S \subseteq E$. Since the closure operator by its definition acts independently on components of S, the rank of S equals the sum of the ranks of its components. This is (j). (We have elided the infinite case. Here $\# N(A) = \# N(A) - 1$ so the argument simplifies slightly.)

In proving (k), we may consider each component of Ω separately. In a balanced component, (k) reduces to a well-known property of the graphic matroid. In an unbalanced component $\Omega{:}U$ of Ω, let S have b balanced components S_j and one union of unbalanced components A. Consider $\Gamma{:}U$ with each S_j and A contracted to a point. This graph is connected so it has a spanning tree T, which as an edge set in Ω connects the S_j and A and which has b edges. Adding the edges of T to $S{:}U$ one at a time and taking closures at each step gives a chain of closed sets from $S{:}U$ to $\operatorname{clos}(T \cup S{:}U)$. If $S{:}U$ is unbalanced, the latter set has corank zero; if $S{:}U$ is balanced it has corank one, as shown under the proof of (j). Thus (k) is proved. (We have neglected technicalities about infinities of different sizes—which can easily be supplied by the reader—because our main interest is when the corank is finite.)

Part (f) follows from (k). S is characterized by having corank zero. Equivalently, $\pi_b(S) = \pi_b(\Omega)$, from which (f) follows. Part (g) is immediate from (c) and (f).

Part (h) follows from (k) since H is characterized by having corank 1. One possibility is that H is obtained from E by removing a minimal cutset in a balanced component; this is the case of part (h) in which $\Omega{:}Y$ is balanced. Another possibility is that H is obtained by removing a cutset in an unbalanced component such that one side of the cutset is connected, call it $\Omega{:}X$, and taking a maximal balanced subset A of $E{:}X$. This is the case in which $\Omega{:}Y$ is unbalanced and $X \neq Y$. The third possibility is simply to take a maximal balanced subset A in an unbalanced component of Ω. This is the case in which $\Omega{:}Y$ is unbalanced and $X = Y$.

Part (i) is immediate from (k).

That concludes the proof. ▮

The argument, based on (a), that the matroid can be treated componentwise should be stated formally.

Proposition 2.3. *If Ω is a biased graph with components Ω_i, then $G(\Omega)$ is the direct sum of the $G(\Omega_i)$.* ▮

We can now extend Lemma 2.2 to cover all unbiased graphs, even those with half edges.

Proposition 2.4. *Let Γ be an unbiased graph. Then $G(\Gamma) = G([\Gamma])$.*

Proof. One can verify by inspection that $G([\Gamma])$ and $G(\Gamma)$ have the same circuits. This example was discussed in [27, Sect. 7A]. ▮

Any minor of a graphic matroid $G(\Gamma)$ is the graphic matroid of a minor of Γ. The definitions of biased restriction and contraction (Section I.4) were chosen to make this true for the bias matroid as well.

Theorem 2.5. *Let Ω be a biased graph; let A and $S \subseteq E$. Then $G(\Omega \mid S) = G(\Omega) \mid S$ and $G(\Omega/A) = G(\Omega)/A$.*

Proof. The closure of $T \subseteq S$ in $G(\Omega \mid S)$ is clearly equal to $(\operatorname{clos}_{G(\Omega)} T) \cap S$, which is the closure in $G(\Omega) \mid S$.

a spanning tree T, which as an edge set in Ω connects the S_j and A and which has b edges. Adding the edges of T to $S{:}U$ one at a time and taking closures at each step gives a chain of closed sets from $S{:}U$ to $\mathrm{clos}(T \cup S{:}U)$. If $S{:}U$ is unbalanced, the latter set has corank zero; if $S{:}U$ is balanced it has corank one, as shown under the proof of (j). Thus (k) is proved. (We have neglected technicalities about infinities of different sizes—which can easily be supplied by the reader—because our main interest is when the corank is finite.)

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Proof. The closure of $T \subseteq S$ in $G(\Omega \mid S)$ is clearly equal to $(\mathrm{clos}_{G(\Omega)}\, T) \cap S$, which is the closure in $G(\Omega) \mid S$.

To prove $G(\Omega/A) = G(\Omega)/A$, let $S \subseteq E \backslash A$, $X = N_0(S \cup A)^c$, and $Y = N_0(S; \Omega/A)$. Then $S \cup A$ has balanced part $(S \cup A){:}X$ (the union of all balanced components). So

$$\begin{aligned}\mathrm{clos}_{G(\Omega)/A}\, S &= [\mathrm{clos}_{G(\Omega)}(S \cup A)] \backslash A \\ &= [E{:}N_0(S \cup A) \cup \mathrm{bcl}_\Omega(S \cup A){:}X] \backslash A \\ &= [A^c{:}N_0(S \cup A)] \cup [\mathrm{bcl}_\Omega((S \cup A){:}X) \backslash A] \\ &= [A^c{:}N_0(S; \Omega/A)] \cup \mathrm{bcl}_{\Omega/A}(S{:}Y)\end{aligned}$$

by Lemma I.4.4 for $A^c{:}N_0(S \cup A)$ and the same plus Proposition I.4.6 for the balance closure,

$$= \mathrm{clos}_{G(\Omega/A)}\, S,$$

as we wanted. ∎

COROLLARY 2.6. *The class of bias matroids of biased graphs is a minor-closed class.*

The contraction identity $G(\Omega/A) = G(\Omega)/A$, or in lattice form $\mathrm{Lat}\, G(\Omega/A) = (\mathrm{Lat}\, G(\Omega))/A$, has a counterpart for balanced flats.

PROPOSITION 2.7. *Let Ω be a biased graph, $A \subseteq E$, and $X = N_0(A)^c$. The natural isomorphism (via the identity map on A^c) of* $\mathrm{Lat}\, G(\Omega/A)$ *with* $\mathrm{Lat}\, G(\Omega)/A$ *is an isomorphism of* $\mathrm{Lat}^b(\Omega/A)$ *with* $(\mathrm{Lat}^b\, \Omega{:}X)/(A{:}X)$. *If in particular A is balanced,* $\mathrm{Lat}^b(\Omega/A) = (\mathrm{Lat}^b\, \Omega)/A$.

Proof. We first have to show that $\mathrm{Lat}^b(\Omega/A) = \mathrm{Lat}^b((\Omega{:}X)/(A{:}X))$. The only difference between Ω/A and $(\Omega{:}X)/(A{:}X)$ is that the former has additional half edges corresponding to the links between X and $N_0(A)$ in Ω. These do not alter the balanced sets.

Now we need only consider $A{:}X$, which is balanced. The proposition follows from Lemma I.4.3 and Theorem 2.5. ∎

Corollary 5.8 in [27] is this proposition for the case where Ω is sign-biased and A is balanced. (There is an error in its statement: I neglected to specify that A be balanced.)

Recall that, by the definition in Part I, a one-node graph is a block, and so is a loop or loose or half edge. An *inner block* of Ω is a block that is unbalanced or lies on a path between two unbalanced blocks in the block/cutpoint graph (equivalently, it has an edge that lies in an unbalanced bias circuit). Any other block having an edge is *outer*. The *core* of a component of Ω is the union of all inner blocks; it is a connected subgraph. A *plain necklace* is a biased graph of the form $\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_l$, where each Ω_i

is a block graph and is balanced and $\Omega_i \cap \Omega_j$ is a pair of nodes if $l=2$ and $i \neq j$, a single node if $l \geqslant 3$ and $i \equiv j \pm 1 \pmod{l}$, and the null graph if $i-j \not\equiv 0, \pm 1 \pmod{l}$.

THEOREM 2.8. *Let Ω be a biased graph. The irreducible separators of $G(\Omega)$ are (the edge sets of) the outer blocks and the cores of the unbalanced components, except that if a core is an unbalanced plain necklace then each block in the necklace is individually an irreducible separator.*

Proof. The proof of [27, Theorem 5.9], which is Theorem 2.8 for sign-biased graphs, is valid in general. One slight change is necessary: each time a subgraph is found to be "a circuit of $G(\Sigma)$ or a completely unbalanced [i.e., contrabalanced] theta graph" (in the case of a signed graph Σ), we should read "a circuit of $G(\Omega)$" (which may be a contrabalanced theta graph).

Another proof is based on Theorem 3.8: see the remark following that theorem. ∎

COROLLARY 2.9. *If Ω is a full biased graph, the minimal separators of $G(\Omega)$ are the loose edges and balanced loops and the connected components of the remainder of Ω.*

The existence and properties of the bias matroid raise several questions.

Problem 2.10. Which matroids are bias matroids of biased graphs? In particular, find the minor-minimal matroids not of the form $G(\Omega)$ for some biased graph Ω.

We shall call such matroids *(minor-)minimal nonbias matroids.* Two of them are the Fano and dual Fano matroids, by [32, Theorem 3] and [33, Proposition 3A]. A third is the uniform matroid of rank 3 on 7 elements.

PROPOSITION 2.11. *$U_{3,7}$ is a minor-minimal nonbias matroid.*

In the proof we employ a lemma.

LEMMA 2.12. *$G(\Omega)=U_{3,6}$ if and only if $\Omega=(2K_3, \varnothing)$. Also, $G(\Omega)=U_{3,5}$ if and only if $\Omega=(\Gamma_{5a}, \varnothing)$ or $(\Gamma_{5b}, \varnothing)$, where $\Gamma_{5a}=2K_3\backslash$edge and Γ_{5b} is K_3 with a doubled edge and a loop at the opposite node.*

Proof. Suppose $G(\Omega)=U_{3,k}$, where $k \geqslant 5$. Then Ω is unbalanced so its order is three. It is contrabalanced because it has no circuit of size three or less. It has no triple edges. If Ω contains no triangle, it must contain a path of two edges, say e_{12} and e_{23}, one of which must be doubled (say e_{12}) since there cannot be loops at adjacent nodes. Then a loop can appear only at is a block graph and is balanced and $\Omega_i \cap \Omega_j$ is a pair of nodes if $l=2$ and $i \neq j$, a single node if $l \geqslant 3$ and $i \equiv j \pm 1 \pmod{l}$, and the null graph if $i-j \not\equiv 0, \pm 1 \pmod{l}$.

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If Ω contains a triangle it can have at most one loop, so it must have a doubled edge. This leaves open the two possibilities of the lemma for $U_{3,5}$. It is easy to see that deleting any edge leaves a circuit; thus $G(\Gamma_{5a}, \varnothing)$ and $G(\Gamma_{5b}, \varnothing)$ are indeed equal to $U_{3,5}$. Only $(\Gamma_{5a}, \varnothing)$ allows a sixth edge; this gives $(2K_3, \varnothing)$. Its bias matroid is clearly $U_{3,6}$. ∎

Proof of Proposition 2.11. Obviously no edge can be added to $(2K_3, \varnothing)$ while keeping the bias matroid uniform of rank three. So $U_{3,7}$ is not a bias matroid. $U_{3,6}$ is, as we have seen. Also, $U_{2,6}=G(6K_2, \varnothing)$. ∎

Problem 2.13. Characterize biased graphs having the same bias matroid.

Problem 2.14. Characterize the biased graphs whose bias matroids are of particular types (e.g., binary, ternary, graphic, cographic).

Problem 2.13 has been solved in large part for the bicircular matroid, that is for biased graphs of the form $(\Gamma, \varnothing)$, in [24], and for full biased graphs in [34] (this case is trivial). The biased graphs whose matroids are binary have been partially characterized in [32], where the problem is reduced to the determination of all sign-biased graphs having no two node-disjoint unbalanced circles. Lovász and Schrijver have found a solution to the latter problem [15]. The biased graphs, other than sign-biased ones, whose matroids are graphic, or cographic, or in other binary classes are also implicitly characterized by [32]; the solution for sign-biased graphs was found by Lovász and Schrijver and some of it is implicit in [21]. For ternary and other bias matroids I offer

Conjecture 2.15. Let K be a field and Ω a biased graph. If $G(\Omega)$ is linearly representable over K, then either Ω is gain-biased with gain group K^*, the multiplicative group of K, or $G(\Omega)=L(\Omega)$.

It is known that, if Ω is K^*-biased, then $G(\Omega)$ is linearly representable over K; this is clear from the literature on networks with (real) gains (e.g., [20]) and is proved in Part IV. The conjecture is a proposed converse. It is true for full biased graphs (those having an unbalanced edge at every node) by Proposition IV.2.4. Exceptions, if any, should occur only when nonisomorphic biased graphs have isomorphic bias matroids. For more on the case where $G(\Omega)=L(\Omega)$ see Conjecture 3.14.

3. THE LIFT AND COMPLETE LIFT MATROIDS

The operation of (elementary) lifting constructs from a matroid M a new one of rank one greater. Each linear class $\mathscr{B}$ of circuits of M gives rise to a different lift $L(M, \mathscr{B})$. One way to define $L(M, \mathscr{B})$ is through duality. Let $M^\perp$ be the Whitney (orthogonal) dual of M and $\mathscr{B}^* = \{E \backslash C : C \in \mathscr{B}\}$, where E is the ground set of M. Since $\mathscr{B}^*$ is a linear class of copoints of $M^\perp$, it defines a strong map $M^\perp \to N$. (N is the result of adjoining a point e_0 to $M^\perp$ with respect to $\mathscr{B}^*$, then contracting by the point.) The *lift of M along $\mathscr{B}$* is $L(M, \mathscr{B}) = N^\perp$. The *complete lift of M along $\mathscr{B}$* is $(M^\perp \cup e_0)^\perp$.

The lift $L(M, \mathscr{B})$ has rank $\operatorname{rk}(M)+1$ except when $\mathscr{B} = \{$all circuits of $M\}$. This is just like the relationship which $G(\Gamma, \mathscr{B})$ bears to $G(\Gamma)$. Nevertheless the lift $L(\Gamma, \mathscr{B})$ of $G(\Gamma)$ along $\mathscr{B}$ is, in general, different from $G(\Gamma, \mathscr{B})$. The lifts instead demonstrate the essentially graphic character of bias matroids; indeed it would be fair to call the latter "connected lifts" of $G(\Gamma)$ since they are in a sense determined by lifting with the further requirement that circuits be connected subgraphs. (See Section 4.)

Recall that a *lift circuit* is a balanced circle, a contrabalanced theta graph, or a union of two unbalanced figures having at most one common node. Equivalently, it is a balanced circle or a minimal contrabalanced edge set of cyclomatic number two. We want to characterize analogously to Theorem 2.1(a)–(k) both the *lift* $L(\Omega) = L(\Gamma, \mathscr{B})$ of $G(\Gamma)$ along $\mathscr{B}$ and the *complete lift* $L_0(\Omega)$. The ground set of the former is E; that of the latter is $E \cup e_0$, where e_0, called the *extra point*, is not in Ω. Then $L(\Omega) = L_0(\Omega) \mid E$. We call a set $S \subseteq E \cup e_0$ *balanced* if it is a balanced edge set (thus $e_0 \notin S$).

THEOREM 3.1. *Let Ω be a biased graph with underlying graph $\Gamma = (N, E)$. Assume Ω has no half edges.*

(A) *The complete lift $L_0(\Omega)$ on the point set $E \cup e_0$, where e_0 is an extra point, is given by the following equivalent properties* (a)–(k). *The matroid is finitary. Let S denote any subset of $E \cup e_0$.*

(a) *The closure of S is*

$$\operatorname{clos}_{L_0(\Omega)}(S) = \begin{cases} \operatorname{bcl}_\Omega(S) & \textit{if } S \textit{ is balanced,} \\ \operatorname{clos}_{\|\Omega\|}(S \backslash e_0) \cup e_0 & \textit{otherwise.} \end{cases}$$

(b) *S is closed $\Leftrightarrow$ it is balanced and balance-closed or else it contains e_0 and $S \backslash e_0$ is closed in $G(\|\Omega\|)$.*

(c) *S is independent $\Leftrightarrow$ it contains at most e_0 or one unbalanced figure but not both, and no balanced circle.*

(d) *S is dependent $\Leftrightarrow$ it contains a balanced circle, or two unbalanced figures, or e_0 and an unbalanced figure.*

(e) *S is a matroid circuit of $L_0(\Omega)$ $\Leftrightarrow$ it is a lift circuit or the union of e_0 and an unbalanced figure.*

(f) *S spans*
of $\|\Omega\|$.
(g) *S is a b*
with one more eleme
in $F \cup e$.
(h) *S is a c*
and $S \backslash e_0$ *is a copoi*
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(a) *The closure of S is*

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(b) *S is closed $\Leftrightarrow$ it is balanced and balance-closed or else it contains e_0 and $S \backslash e_0$ is closed in $G(\|\Omega\|)$.*

(c) *S is independent $\Leftrightarrow$ it contains at most e_0 or one unbalanced figure but not both, and no balanced circle.*

(d) *S is dependent $\Leftrightarrow$ it contains a balanced circle, or two unbalanced figures, or e_0 and an unbalanced figure.*

(e) *S is a matroid circuit of $L_0(\Omega)$ $\Leftrightarrow$ it is a lift circuit or the union of e_0 and an unbalanced figure.*

(f) *S spans $L_0(\Omega)$ $\Leftrightarrow$ it is unbalanced and contains a maximal forest of $\|\Omega\|$.*

(g) *S is a basis $\Leftrightarrow$ it consists of a maximal forest F of $\|\Omega\|$ together with one more element, either e_0 or an edge e forming an unbalanced figure in $F \cup e$.*

(h) *S is a copoint $\Leftrightarrow$ it is a maximal balanced edge set or else $e_0 \in S$ and $S \backslash e_0$ is a copoint in the graphic matroid $G(\|\Omega\|)$.*

(i) *S is a bond of $L_0(\Omega)$ $\Leftrightarrow$ its complement is a copoint.*

(j) *The rank of S is*

$$\mathrm{rk}_{L_0(\Omega)}(S) = \sum_{B \in \pi(S)} (\# B - 1) + \varepsilon = \mathrm{rk}_{\|\Omega\|}(S) + \varepsilon,$$

where $\varepsilon = 0$ if S is balanced, 1 otherwise. If n is finite,

$$\mathrm{rk}_{L_0(\Omega)}(S) = \begin{cases} n - c(S) & \text{if } S \text{ is balanced,} \\ n + 1 - c(S \backslash e_0) & \text{otherwise.} \end{cases}$$

(k) *The corank of S is*

$$\sum_{B \in \pi(\Omega)} [c(S{:}B) - 1] + \delta,$$

where $\delta = 1$ if S is balanced, 0 otherwise; if $c(\Gamma)$ is finite this equals $c(S) - c(\Gamma) + \delta$.

(B) *If Ω is balanced, then $L(\Omega) = G(\|\Omega\|)$. Otherwise, the properties of $L(\Omega)$ are like those of $L_0(\Omega)$ with obvious modifications, except for:*

(b_L) *S is closed in $L(\Omega)$ $\Leftrightarrow$ it is balanced and balance-closed or it is polygon-closed.*

(h_L) *S is a copoint of $L(\Omega)$ $\Leftrightarrow$ it is a maximal balanced set in Ω or it is unbalanced and a copoint of $G(\|\Omega\|)$.*

Remark on Half Edges. In some results on the lift matroid, such as Theorems 3.1 and 3.6, half edges create a technical difficulty because they are unbalanced in $\|\Omega\|$. Perhaps the best way to handle them is to replace them by unbalanced loops, so that in $\|\Omega\|$ they are balanced loops.

Indications of Proof. Since $L(\Gamma, \mathscr{B})$ and $L_0(\Gamma, \mathscr{B})$ are special cases of general matroid constructions which have been discussed in [5, 10, 16, 4], although not from the viewpoint of balance defined by a linear class of circuits, rather than a full proof we just give the general definition and explain how one can obtain our result from more familiar parts of matroid theory.

Let M be a finitary matroid with point set E. A *linear class of circuits* of M is a class $\mathscr{B}$ of circuits (called "balanced") such that, if C_1 and C_2 are balanced circuits for which $|C_1 \cup C_2| = \text{rk}(C_1 \cup C_2) + 2$ and C is a circuit in $C_1 \cup C_2$, then C is balanced. A set of points is *balanced* if every circuit in it is balanced. The class $\mathscr{I} = \{S \subseteq E : S \text{ is balanced}\}$ is what Dowling and Kelly [10] call a *modular ideal* (*of sets*) in M. (See also [4, Proposition 7.4.15 *et seq.*].) Their Proposition 6.4 gives the rank function of the lift $L(M, \mathscr{B})$, which they call the $\mathscr{I}$*-preimage* of M (and is elsewhere called an "elementary lift" or "elementary coextension"). Their Proposition 6.6 gives the flats of the lift, but some extra argument is required to obtain our description.

The class $\mathscr{B}^*$ defined above is what Crapo and Rota [6] call a *linear subclass of copoints* of $M^\perp$. The complete lift $L_0(M, \mathscr{B})$ is then the dual of the extension $M^\perp \cup e_0$ determined by $\mathscr{B}^*$ and $L(M, \mathscr{B})$ is the dual of $(M^\perp \cup e_0)/e_0$. Thus characterizations of the lift and complete lift can be obtained by dualizing descriptions of one-point extensions. ▮

It is worth noting that the biased graphs based on a particular graph Γ correspond one-to-one with the rank-preserving one-point extensions of $G(\Gamma)^\perp$. Moreover, the isomorphism types of biased graphs correspond with the isomorphism types of extensions. (Extensions $M \cup e_1$ and $M \cup e_2$ of M are *isomorphic* when there is a matroid isomorphism $M \cup e_1 \to M \cup e_2$ which carries e_1 to e_2.)

COROLLARY 3.2. *Suppose the components of Ω are Ω_i for $i \in I$, an index set. Then $L_0(\Omega)$ is the parallel connection of the $L_0(\Omega_i)$ at the extra point.*

Proof. For the parallel connection see [3] or [4]. The corollary is immediate from the description of closed sets (Theorem 3.1(b)), for instance. ▮

COROLLARY 3.3. *If Γ is an ordinary graph, $L([\Gamma]) = G(\Gamma)$.*

COROLLARY 3.4. *$L(\Omega) = G(\Omega)$ if and only if Ω has no two node-disjoint unbalanced circles.*

Problem 3.5. Characterize the biased graphs having no two node-disjoint unbalanced circles.

This problem has been solved only for contrabalanced graphs ([14], see [2]) and sign-biased graphs [15].

THEOREM 3.6. *Let Ω be a biased graph with underlying graph Γ having no half edges. Then $L_0(\Omega) \mid S = L_0(\Omega \mid S)$ and*

$L_0(\Omega)/S =$

(*Here* $(e_0)_0$ *denotes*

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extra matroid loops

Suppose S is an

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as claimed.

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COROLLARY 3.7.

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THEOREM 3.8. *L*

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Let M be a finitary matroid with point set E. A *linear class of circuits* of M is a class $\mathscr{B}$ of circuits (called "balanced") such that, if C_1 and C_2 are balanced circuits for which $|C_1 \cup C_2| = \mathrm{rk}(C_1 \cup C_2) + 2$ and C is a circuit in $C_1 \cup C_2$, then C is balanced. A set of points is *balanced* if every circuit in it is balanced. The class $\mathscr{I} = \{S \subseteq E : S \text{ is balanced}\}$ is what Dowling and Kelly [10] call a *modular ideal* (*of sets*) in M. (See also [4, Proposition 7.4.15 *et seq.*].) Their Proposition 6.4 gives the rank function of the lift $L(M, \mathscr{B})$, which they call the *$\mathscr{I}$-preimage* of M (and is elsewhere called an "elementary lift" or "elementary coextension"). Their Proposition 6.6 gives the flats of the lift, but some extra argument is required to obtain our description.

The class $\mathscr{B}^*$ defined above is what Crapo and Rota [6] call a *linear subclass of copoints* of $M^\perp$. The complete lift $L_0(M, \mathscr{B})$ is then the dual of the extension $M^\perp \cup e_0$ determined by $\mathscr{B}^*$ and $L(M, \mathscr{B})$ is the dual of $(M^\perp \cup e_0)/e_0$. Thus characterizations of the lift and complete lift can be obtained by dualizing descriptions of one-point extensions. ∎

It is worth noting that the biased graphs based on a particular graph Γ correspond one-to-one with the rank-preserving one-point extensions of $G(\Gamma)^\perp$. Moreover, the isomorphism types of biased graphs correspond with the isomorphism types of extensions. (Extensions $M \cup e_1$ and $M \cup e_2$ of M are *isomorphic* when there is a matroid isomorphism $M \cup e_1 \to M \cup e_2$ which carries e_1 to e_2.)

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THEOREM 3.6. *Let Ω be a biased graph with underlying graph Γ having no half edges. Then $L_0(\Omega) \mid S = L_0(\Omega \mid S)$ and*

$$L_0(\Omega)/S = \begin{cases} L_0(\Omega/S) & \textit{if } S \textit{ is balanced,} \\ G(\Gamma/(S \setminus e_0)) & \textit{if } e_0 \in S, \\ G(\Gamma/S) \cup (e_0)_0 & \textit{if } e_0 \notin S \textit{ and } S \textit{ is unbalanced.} \end{cases}$$

(*Here $(e_0)_0$ denotes e_0 as a matroid loop.*)

Proof. There are three cases. We can assume S is closed since if it is not, it is clear from the definitions that we only have $L_0(\Omega)/(\mathrm{clos}\, S)$ with extra matroid loops. Let $L_0 = L_0(\Omega)$ and $G = G(\Omega)$.

Suppose S is an unbalanced flat; let $T = S \setminus e_0$. Then

$$\mathrm{clos}_{L_0/S}(A) = \mathrm{clos}_{L_0}(A \cup S) \setminus S = \mathrm{clos}_\Gamma(A \cup T) \setminus T = \mathrm{clos}_{\Gamma/T}(A),$$

as claimed.

Suppose $A \cup S$ is balanced in Ω. Then

$$\mathrm{clos}_{L_0/S}(A) = \mathrm{clos}_{L_0}(A \cup S) \setminus S = \mathrm{clos}_G(A \cup S) \setminus S$$

since both equal $\mathrm{bcl}_\Omega(A \cup S) \setminus S$,

$$= \mathrm{clos}_{G/S}(A) = \mathrm{clos}_{G(\Omega/S)}(A) = \mathrm{bcl}_{\Omega/S}(A)$$

since A is balanced in Ω/S (Lemma I.4.3)

$$= \mathrm{clos}_{L_0(\Omega/S)}(A).$$

But suppose S is balanced in Ω and $A \cup S$ is not. Then

$$\mathrm{clos}_{L_0/S}(A) = \mathrm{clos}_{L_0}(A \cup S) \setminus S = (\mathrm{clos}_\Gamma(A \cup S) \cup \{e_0\}) \setminus S$$
$$= \mathrm{clos}_{\Gamma/S}(A) \cup \{e_0\} = \mathrm{clos}_{\|\Omega/S\|}(A) \cup \{e_0\}$$

because $\|\Omega/S\| = \|\Omega\|/S$ if S is balanced,

$$= \mathrm{clos}_{L_0(\Omega/S)}(A)$$

since A is unbalanced in Ω/S (Lemma I.4.3). This concludes the proof. ∎

COROLLARY 3.7. *The class of lift matroids of biased graphs is a minor-closed class.*

THEOREM 3.8. *Let Ω be a biased graph. The irreducible separators of $L_0(\Omega)$ are the individual balanced blocks and the union of e_0 and all unbalanced blocks.*

Those of $L(\Omega)$ are the individual balanced blocks and, in general, the union of all unbalanced blocks, the exception being that when there is only one *unbalanced block in Ω and it is an unbalanced plain necklace of blocks Ω_i, then each Ω_i is an irreducible separator of $L(\Omega)$.*

Proof. We rely on the circuit definition of irreducible separators of a matroid, according to which they are the equivalence classes of edges under the relation of belonging to a common circuit, and thus on Theorem 3.1(e). It is clear that a balanced block is a separator and is irreducible, so let us assume there are no such blocks. We also assume E is not null.

In $L_0(\Omega)$ the circuits on e_0 connect the unbalanced blocks (by Proposition I.3.9), so we are done there.

In $L(\Omega)$, if there are two or more unbalanced blocks, then their unbalanced circuits connect them into an irreducible separator (by Proposition I.3.9). Suppose then that Ω is a single unbalanced block and that it has a nontrivial irreducible separator S.

We first show that S is balanced. If it were not, say it had an unbalanced circle C in a block S_1 of $\Omega \mid S$. Let $e \in S^c$. By Menger's theorem there exist disjoint paths from the ends of e to distinct points on C. Since the theta graph so formed meets S and its complement, it cannot be a lift circuit. But neither can either of its circles through e. This is an absurdity. So there can be no such C.

We may conclude that S is connected, indeed a block graph. We show that S^c is also connected. If it had components T_1 and T_2, let Q_i be a path in T_i between two points of attachment v_i and w_i, and let P be a tree in S whose end nodes are v_1, w_1, v_2, w_2. Then either $P \cup Q_1 \cup Q_2$ is a lift circuit, or it contains a lift circuit composed of two circles containing Q_1 and Q_2 and edges in P, or it contains a balanced circle composed of a Q_i and edges in P. Every one of these possibilities contradicts the character of S as a separator. So S^c is connected.

If S and S^c have three (or more) nodes of attachment, then there is a theta graph through those nodes which contains edges of S and S^c in every circle. This again contradicts the separateness of S.

Since there are exactly two nodes v and w at which S meets S^c and since Γ is 2-connected, the block/cutpoint graph of S^c must be an open path with v in one end block and w in the other, neither being a cutpoint of S^c. Each of the blocks $\Omega_2, \Omega_3, ..., \Omega_l$ of S^c is then balanced (by the same construction used to show balance of S, since we now know that any circle not contained in an Ω_i must include an edge of S), hence an irreducible separator of S^c. We have found the irreducible separators of $L(\Omega)$: they are $S = \Omega_1$ and the other Ω_i. (Since Ω is unbalanced, the unbalanced circles are exactly the "long" circles that pass through every Ω_i.) ▮

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Suppose Ω is connected. Then every lift circuit lies in a bias circuit and every bias circuit contains a lift circuit. Therefore, the separators of $G(\Omega)$ are unions of separators of $L(\Omega)$. This observation permits us to deduce Theorem 2.8 from 3.8.

The lift and complete lift matroids of a biased graph pose questions like those raised for the bias matroid in the previous section.

Problem 3.9. (a) Determine the minor-minimal matroids not of the form $L(\Omega)$ for some biased graph Ω. (b) Determine those not of the form $L_0(\Omega)$ for some Ω.

Let us call these *minor-minimal nonlift* and *complete nonlift* matroids. For part (a) we have $U_{3,7}$ again.

PROPOSITION 3.10. *$U_{3,7}$ is a minor-minimal nonlift matroid.*

Proof. Suppose $L(\Omega) = U_{3,7}$. Then Ω is unbalanced, so has order 3 and is contrabalanced. If it has no two node-disjoint circles, then $L(\Omega) = G(\Omega)$ and Proposition 2.11 applies. Otherwise it has a loop and Proposition 3.11 applies. ▮

We mention that $L(\Omega) = U_{3,6} \Leftrightarrow \Omega = (2K_3, \varnothing)$ and $L(\Omega) = U_{3,5} \Leftrightarrow \Omega = (\Gamma_{5a}, \varnothing)$; these facts too follow from Propositions 2.11 and 3.11 and the observation that, if Ω has no loops (and $L(\Omega) = U_{3,k}$ for $k \geqslant 5$), then $L(\Omega) = G(\Omega)$.

For part (b) we have two examples. The Bixby–Seymour matroid R_{10} is the linear dependence matroid of the ten 5-dimensional binary vectors having exactly three ones. It equals $L(-K_5)$ [33, Sect. 6].

PROPOSITION 3.11. *$U_{3,5}$ and R_{10} are minor-minimal complete nonlift matroids.*

Proof. Suppose $L_0(\Omega) = U_{3,5}$. Then $G(\|\Omega\|) = L_0(\Omega)/e_0 = U_{2,4}$, which is impossible. On the other hand, $U_{3,4} = L_0(K_3, \varnothing)$ and $U_{2,4} = L_0(3K_2, \varnothing)$.

That $L_0(\Omega) = R_{10}$ is impossible is the conjunction of [32, Theorem 1] and [33, Proposition 6A]. On the other hand, $R_{10}\backslash\text{point} = G(K_{3,3})$, which is graphic, and $R_{10}/\text{point} = G(K_{3,3})^{\perp}$, the dual, which is signed-graphic [33, Proposition 5A]. ▮

Problem 3.12. Characterize the biased graphs having the same lift or complete lift matroid.

This problem has been solved for the complete lift matroid [34] but is otherwise open.

Problem 3.13. Characterize the biased graphs whose (complete) lift matroids are of various types, e.g., binary, ternary, regular, graphic.

The solution for binary type is given in [32]. For regular, graphic, or cographic type, [32] reduces the problem to the sign-biased case solved by

Shih [21] (for graphic type only) and by Lovász and Schrijver [15]. To cover ternary type and more, I propose

Conjecture 3.14. Let K be a field and Ω a biased graph. If $L(\Omega)$ is linearly representable over K, then (i) $L_0(\Omega)$ is representable, or (ii) Ω is decomposable into balanced pieces, or (iii) Ω is indecomposable, $L(\Omega)$ is graphic, and Ω has no node v such that $\Omega \backslash v$ is balanced.

It is known that $L_0(\Omega)$ is representable over K if and only if Ω is K^+-biased (Proposition IV.4.3).

The method of decomposition in (ii) I leave intentionally vague. In [32] it is shown that when $K = GF(2)$, Ω satisfies either (i) or (ii). The decomposable cases for $K = GF(2)$ are unbalanced 2-sums of balanced graphs. This description of case (ii) may well carry over to larger fields. I expect at least that any exception Ω to (i) has $L(\Omega)$ isomorphic to $L(\Omega_1)$, where $\Omega_1 \not\cong \Omega$ and $L_0(\Omega_1)$ is K-representable.

In Theorems 2.1 and 3.1 it is necessary that $\mathscr{B}$ be a linear class. Otherwise G and L will not be matroids. We state this for two cryptomorphisms.

PROPOSITION 3.15. *Let Γ be a graph and $\mathscr{B} \subseteq \mathscr{C}(\Gamma)$; $\mathscr{B}$ need not be a linear subclass. If either $G(\Gamma, \mathscr{B})$ or $L(\Gamma, \mathscr{B})$ [as defined by* (e) *or* (j)] *is a matroid, then $\mathscr{B}$ is a linear class.*

Proof. Suppose that (e) defines the circuits of a matroid. Let C_1, C_2, C_3 be the circles of a theta subgraph. If $C_1, C_2 \in \mathscr{B}$, they are circuits. Let $e \in C_1 \cap C_2$; then $(C_1 \cup C_2) \backslash e$ is dependent (by circuit exchange), and this requires that C_3 be balanced.

Supposing that (j) defines a matroid, we can proceed similarly. ▌

4. BETWEEN THE BIAS AND LIFT MATROIDS

Comparing the set $\mathrm{Lat}^b\,\Omega$ of balanced flats and the lattices of flats of the various matroids associated with Ω and its underlying graph, we see that $\mathrm{Lat}^b\,\Omega$ is an order ideal and a meet subsemilattice of all three matroid lattices of Ω. We also have the relations given by

COROLLARY 4.1. *For a biased graph Ω without half edges we have*

$$\mathrm{Lat}^b\,\Omega = \mathrm{Lat}\,G(\Omega) \cap \mathrm{Lat}\,L_0(\Omega),$$

$$\mathrm{Lat}\,L(\Omega) = \mathrm{Lat}^b(\Omega) \cup \mathrm{Lat}\,G(\|\Omega\|),$$

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$$\mathrm{Lat}\,L_0(\Omega) \cong \mathrm{Lat}^b(\Omega) \cup \mathrm{Lat}\,G(\|\Omega\|),$$

the latter a disjoint union where $S \in \mathrm{Lat}\,G(\|\Omega\|)$ corresponds to $S \cup e_0 \in$ $\mathrm{Lat}\,L_0(\Omega)$.

The maximal elements of $\mathrm{Lat}^b\,\Omega$ *are all of rank $n - c(\Omega)$ and have corank $c(\Omega) - b(\Omega)$ in $G(\Omega)$.*

Proof. This is all clear except possibly the last remark. A maximal balanced flat is the closure of a maximal forest, hence has $c(\Omega)$ components (counting isolated nodes), all balanced. ▮

One would like to find a common theoretical basis for the bias and lift matroids in which Theorems 2.1 and 3.1(B) can be combined in a single statement and proof. The lack of a common framework is apparent in [35], where all the examples come in pairs that must be treated separately although they are derived from the same biased graphs by the bias and lift matroid constructions. I have not found a unified description but it is possible to show that the two matroids have a certain uniqueness. That is the intent of Theorem 4.5, below.

A natural partial ordering of matroids on the same set is by their independent sets. We say M is *weaker* than N, $M \geqslant N$, if every independent set of N is independent in M. From part (c) of Theorems 2.1 and 3.1 we have at once

COROLLARY 4.2. *If Ω is a biased graph, then $L(\Omega) \leqslant G(\Omega)$.*

We can therefore characterize $G(\Omega)$ as the strongest possible matroid $M \geqslant L(\Omega)$ whose circuits are connected. What if we required only that M be *intermediate* between L and G, in other words that $L \leqslant M \leqslant G$?

Problem 4.3. (a) Given Ω, what matroids M on E may exist that are intermediate between $L(\Omega)$ and $G(\Omega)$? (b) Is there a systematic way to construct intermediate matroids?

I think these are difficult questions, in part because it is not clear what ought to be considered a "systematic" construction. A reasonable interpretation is that it is a mapping $\mathbf{M}$ from the class of biased graphs or some subclass (the *domain*) to matroids such that $\mathbf{M}(\Omega)$ is a matroid on $E(\Omega)$ and $\mathbf{M}(\Omega \mid S)$ is defined and equal to $\mathbf{M}(\Omega) \mid S$ for each $S \subseteq E(\Omega)$. For instance, $\mathbf{G}$ and $\mathbf{L}$ are the constructions assigning to Ω the bias and lift matroids, respectively. We may call an *intermediate-matroid construction* any $\mathbf{M}$ such that $L(\Omega) \leqslant \mathbf{M}(\Omega) \leqslant G(\Omega)$ for all Ω in the domain of $\mathbf{M}$.

A priori, it is plausible to suppose that almost any biased graph with $L \neq G$ supports intermediate matroids in which some of the circuits with two disjoint circles are connected and others are not, and the larger Ω is, the more intermediate matroids it has. This turns out not to be the case. A biased graph is *full* if every node supports an unbalanced edge.

PROPOSITION 4.4. *Let Ω be a finite biased graph that is full and complete. Let M be an intermediate matroid on E; that is, $L(\Omega) \leqslant M \leqslant G(\Omega)$. Then $M = L(\Omega)$ or $G(\Omega)$.*

Proof. Let Ω be full and complete; we may assume there is one unbalanced loop h_i at each node v_i and that all other edges are links. Let M be an intermediate matroid, $L = L(\Omega)$, and $G = G(\Omega)$. Let $H_{ij} = \{h_i, h_j\}$ and $C(e_{ij}) = H_{ij} \cup e_{ij}$, where e_{ij} denotes a link $v_i v_j$.

We show that either all H_{ij} are circuits or all $C(e_{ij})$ are. Suppose H_{ij} and H_{jk} are circuits; then by circuit exchange (and since $M \geqslant L$) H_{ik} is a circuit. Suppose $C(e_{ij})$ and $C(e_{jk})$ are circuits. By circuit exchange there is a circuit $D \subseteq \{h_i, e_{ij}, e_{jk}, h_k\}$ containing e_{ij}. Since D contains a lift circuit, which can only be H_{ik}, H_{ik} itself is not a circuit in M. Therefore $C(e_{ik})$ is a circuit for each link between v_i and v_k.

We show next that each circuit C of M is a bias or lift circuit. Suppose a circuit C is neither a bias nor a lift circuit. Then C contains no bias circuit, so its components are r unbalanced unicycles $U_1, U_2, ..., U_r$ and perhaps some trees, where $r \geqslant 2$ because $M \geqslant L$. Let C_i be the circle in U_i, $v_i \in N(C_i)$, and $e_1 \in C_1$ and let $e: v_1 v_2$ be a link. $C_1 \cup C_2$ is independent in M, for otherwise C would be a lift circuit. The set $D = C_1 \cup C_2 \cup e$, being a bias circuit, is dependent in M, hence a circuit. By circuit exchange between C and D, $(C \cup e) \backslash e_1$ contains a circuit C', whose cyclomatic number is necessarily lower than that of C. Since C' cannot be a lift circuit (because if $C' \not\subseteq C$, it contains the isthmus e) or a bias circuit, it in turn can be modified as above. Eventually one gets a circuit with at most one circle, but that contradicts $M \geqslant L$. Hence after all C must have been a bias or lift circuit.

Suppose now that M has a circuit C which is a lift circuit but not a bias circuit; that is, $C = C_1 \cup C_2$, where C_1 and C_2 are node-disjoint unbalanced circles. Let $v_i \in N(C_i)$ and $e_1 \in C_1$. By exchange with the circuit $C_1 \cup h_1$, there is a circuit $C' \subseteq C_1 \cup C_2 \cup h_1 \backslash e_1$. C' can only be $C_2 \cup h_1$. Exchange with $C_2 \cup h_2$ leads to the conclusion that H_{12} and consequently all H_{ij} are circuits.

On the other hand suppose M has a circuit C which is a bias but not a lift circuit. Thus C is a loose handcuff $C_1 \cup C_2 \cup P$, P being a path connecting $v_1 \in N(C_1)$ to $v_2 \in N(C_2)$. By circuit exchange with $C_1 \cup h_1$, there is a bias or lift circuit $C' \subseteq C \cup h_1 \backslash e_1$ (where $e_1 \in C_1$), which can only be the bias circuit $C_2 \cup P \cup h_1$ or the lift circuit $C_2 \cup h_1$. Actually the former obtains, for if $C_2 \cup h_1$ were a circuit, exchange with $C_1 \cup h_1$ would imply that $C_1 \cup C_2$ is dependent, contradicting C's being a circuit. By exchanging with $C_2 \cup h_2$, we deduce that $P \cup h_1 \cup h_2$ is a circuit. Thus H_{12} is not; it follows that all $C(e_{ij})$ are circuits.

From the last two paragraphs we conclude that M equals L or G. ∎