Reprinted from JOURNAL OF COMBINATORIAL THEORY, Series B
All Rights Reserved by Academic Press, New York and London

Vol. 47, No. 1, August 1989
Printed in Belgium

Biased Graphs. I. Bias, Balance, and Gains*

THOMAS ZASLAVSKY

State University of New York,
University Center at Binghamton, Binghamton, New York 13901

Communicated by U. S. R. Murty

Received March 27, 1986

A *biased graph* is a graph together with a class of circles (simple closed paths), called *balanced*, such that no theta subgraph contains exactly two balanced circles. A *gain graph* is a graph in which each edge has a *gain* (a label from a group so that reversing the direction inverts the gain); a circle is *balanced* if its edge gain product is 1; this defines a biased graph. We initiate a series devoted to biased graphs and their matroids. Here we study properties of balance and also subgraphs and contractions of biased and gain graphs. © 1989 Academic Press, Inc.

INTRODUCTION

Perhaps the way to introduce biased graphs is through an example. Take a graph Γ and a group $\mathfrak{G}$. Orient the edges of Γ; to each edge assign a value in $\mathfrak{G}$, the *gain* of the edge. If e has gain g, the gain of e^{-1} (e traversed in the opposite direction) is g^{-1}. Let $e_1 e_2 \cdots e_k$ be a circle (the edge set of a closed walk with no repeated nodes or edges). Its gain value is $g_1 g_2 \cdots g_k$; if this equals 1 the circle is *balanced.* Call the set of balanced circles $\mathscr{B}$. The pair $(\Gamma, \mathscr{B})$ is a biased graph.

In full generality, a *biased graph* is a graph Γ with a designated *linear subclass* of "balanced" circles: a subclass of the circles of Γ having the property that, whenever the union of two balanced circles is a theta graph, the third circle in the union is also balanced.[1] It so happens that the balanced circles of any *gain graph* (graph with group gains) are a linear class; indeed, this is the principal source of examples. But not all biased graphs arise from gains and several that do are more conveniently studied directly as biased graphs.

The theory of biased graphs is a combinatorial abstraction of the notion of balance in a gain graph. It grew out of an attempt to understand certain

* Research substantially assisted by a grant from the National Science Foundation in 1976–1977.

[1] The definition in the introduction of [20], which differs from that above, is incorrect.

0095-8956/89 $3.00

matroids and to calculate their invariants; these matroids turned out to be those of certain signed graphs [13]. (A *signed graph* is a gain graph where the gain group has order 2.) It turned out to be easy and, from the axiomatic standpoint, natural to generalize many results to biased graphs, although the proofs are sometimes more complicated.

The matroids of biased graphs can be described with more precision than can most matroids. In this series we develop the general structural and enumerative theory of biased-graphic matroids, including the fundamentals of balance and minors (subgraphs and contractions) for biased and gain graphs (in Part I, the present article) and of the bias and lift matroids (in Part II [18]), and formulas for invariants like the Whitney numbers and the characteristic and Tutte polynomials (Part III [19]). We plan in later parts to treat general examples and representations and to characterize modular flats of the matroids.

The series lays the foundation for separate treatments of some of the interesting examples. Among them are: Signed graphs, introduced along with the notion of balance by Harary in [5, 6]. Their bias matroids were treated in [15–17], where many of the results of this series appear, restricted to the simpler case of signed graphs. The bicircular matroid, introduced by Simões-Pereira. It is based on the bias in which no circle is balanced. Dowling's lattices of a group [4], which for the two-element group are related to the classical root systems [13]. Matthews' two digraph matroids. Networks with gains, also known as "generalized networks." They are gain graphs with gain group the multiplicative (and usually, positive) reals, having an associated optimal flow problem and side conditions like costs and capacities. See for instance [9].

In this first article we concentrate on the elementary theory presupposed by later parts. In Section 2 we define the fundamental concepts of biased graphs and balance. Section 3 develops technical lemmas. In Section 4 we define minors of biased graphs and show that, formally, they behave like minors of ordinary graphs. Section 5 concerns gain graphs, their minors, and their relationship to biased graphs. In the catalog of Section 6 we describe some of the more interesting general types of biased and gain graphs and in Section 7 we thoroughly examine seven small examples.

1. Definitions without Bias

Underlying every biased or gain graph is a graph Γ. Throughout this work Γ will be a graph with node set $N = N(\Gamma)$ of cardinality $n = \#N$ (the *order* of Γ), edge set $E = E(\Gamma)$, and endpoint mapping ν_Γ, which assigns to each edge e a multiset of at most two nodes, not necessarily distinct. (This definition allows multiple edges and loops.) We may say "(N, E) is a matroids and to calculate their invariants; these matroids turned out to be those of certain signed graphs [13]. (A *signed graph* is a gain graph where the gain group has order 2.) It turned out to be easy and, from the axiomatic standpoint, natural to generalize many results to biased graphs, although the proofs are sometimes more complicated.

The matroids of biased graphs can be described with more precision than can most matroids. In this series we develop the general structural and enumerative theory of biased-graphic matroids, including the fundamentals of balance and minors (subgraphs and contractions) for biased and gain graphs (in Part I, the present article) and of the bias and lift matroids (in Part II [18]), and formulas for invariants like the Whitney numbers and the characteristic and Tutte polynomials (Part III [19]). We plan in later parts to treat general examples and representations and to characterize modular flats of the matroids.

The series lays the foundation for separate treatments of some of the interesting examples. Among them are: Signed graphs, introduced along with the notion of balance by Harary in [5, 6]. Their bias matroids were treated in [15–17], where many of the results of this series appear, restricted to the simpler case of signed graphs. The bicircular matroid, introduced by Simões-Pereira. It is based on the bias in which no circle is balanced. Dowling's lattices of a group [4], which for the two-element group are related to the classical root systems [13]. Matthews' two digraph matroids. Networks with gains, also known as "generalized networks." They are gain graphs with gain group the multiplicative (and usually, positive) reals, having an associated optimal flow problem and side conditions like costs and capacities. See for instance [9].

In this first article we concentrate on the elementary theory presupposed by later parts. In Section 2 we define the fundamental concepts of biased graphs and balance. Section 3 develops technical lemmas. In Section 4 we define minors of biased graphs and show that, formally, they behave like minors of ordinary graphs. Section 5 concerns gain graphs, their minors, and their relationship to biased graphs. In the catalog of Section 6 we describe some of the more interesting general types of biased and gain graphs and in Section 7 we thoroughly examine seven small examples.

1. Definitions without Bias

Underlying every biased or gain graph is a graph Γ. Throughout this work Γ will be a graph with node set $N=N(\Gamma)$ of cardinality $n=\#N$ (the *order* of Γ), edge set $E=E(\Gamma)$, and endpoint mapping v_Γ, which assigns to each edge e a multiset of at most two nodes, not necessarily distinct. (This definition allows multiple edges and loops.) We may say "(N, E) is a graph"; this means N is the node set and E is the edge set of the graph. An edge is a *link* if it has two distinct endpoints, a *loop* if two coincident endpoints, a *half edge* if one endpoint, a *loose edge* ("free loop" in [15]) if no endpoints. A loop or link is an *ordinary edge*; an *ordinary graph* has only ordinary edges. The set of ordinary edges of Γ is E_*.

We digress for some definitions about sets and partitions. The *power set* $\mathscr{P}(X)$ of a set X is a group under the operation of symmetric difference or "set sum," denoted by $+$. The disjoint union of sets is denoted by $X \cup Y$.

A *partition* π is a class of pairwise-disjoint, nonempty sets, called the *blocks* or *parts* of π. The *support* is $\operatorname{supp} \pi = \bigcup \pi = \bigcup \{B: B \in \pi\}$. We write $\pi(v) =$ the block of π containing v, if $v \in \operatorname{supp} \pi$; $\pi(v)$ is otherwise undefined. A *partition of* X is a partition whose support is X; the set of partitions of X is Π_X and that of an n-element set is Π_n. A *partial partition of* X is a partition whose support is a subset of X; the set of partial partitions of X is $\Pi_X^\dagger$ and that of an n-element set is $\Pi_n^\dagger$. Two partitions π and τ, possibly of different sets, are ordered by $\pi \leqslant \tau$ (π *refines* τ) if $\operatorname{supp} \pi \supseteq \operatorname{supp} \tau$ and every block of π is either disjoint from $\operatorname{supp} \tau$ or lies within a block of τ. Under the refinement partial ordering the least element of Π_X and of $\Pi_X^\dagger$ is the *total partition* $0_X = \{\{x\}: x \in X\}$; the greatest element of Π_X is the *trivial partition* $1_X = \{X\}$ (if $X \neq \varnothing$), and that of $\Pi_X^\dagger$ is $0_\varnothing = \varnothing$. Notice that $\Pi_n^\dagger \cong \Pi_{n+1}$.

Returning to graphs, let $X \subseteq N$ and $S \subseteq E$ in what follows. We write $X^c = N \backslash X$ and $S^c = E \backslash S$. By $N(S)$ we mean the set $v_\Gamma(S)$ of all endpoints of edges in S. By $\Delta \subseteq \Gamma$ we mean Δ is a subgraph of Γ; by $(X, S) \subseteq \Gamma$ we mean (X, S) is a subgraph of Γ. The union of subgraphs (X_1, S_1), (X_2, S_2),... of a graph is the subgraph $(X_1 \cup X_2 \cup ..., S_1 \cup S_2 \cup ...)$. A subgraph Δ *spans* Γ if its node set $N(\Delta) = N$. We shall frequently use S as shorthand for the spanning subgraph (N, S) (and *never* for the subgraph $(N(S), S)$), relying on context to clarify the meaning. A *component* of Δ is a maximal connected subgraph which is not a loose edge; thus every component has at least one node, although it need have no edges. By $\pi(\Delta)$ we mean $\{N(D): D$ is a component of $\Delta\}$; according to our shorthand therefore $\pi(S) = \pi(N, S)$ is a partition of N. We let

$$\Pi(\Gamma) = \{\pi(S): S \subseteq E\},$$

$$\Pi^\dagger(\Gamma) = \{\pi(X, S): (X, S) \subseteq \Gamma\}.$$

Particular subgraphs of Γ are the *induced subgraph* on a subset X of N, which is $\Gamma{:}X = (X, E{:}X)$ where

$$E{:}X = \{e \in E: v_\Gamma(e) \subseteq X \text{ and } v_\Gamma(e) \neq \varnothing\},$$

and the subgraph induced by a partial partition τ of N, which is

$$\Gamma{:}\tau = \bigcup \{(\Gamma{:}B): B \in \tau\}.$$

We call X *stable* if $E{:}X = \varnothing$. The *node deletion* $\Gamma \backslash X$ has node set X^c and edge set $\{e \in E{:} v_\Gamma(e) \cap X = \varnothing\}$. Thus $\Gamma \backslash X = (\Gamma{:}X^c) \cup \{\text{loose edges}\}$. For a single-node deletion $\Gamma \backslash \{v\}$ we write $\Gamma \backslash v$.

For edges and walks we employ some further shorthand. To indicate that an edge e has endpoints v and w, or v only, or is a loose edge, we may refer to it as $e{:}vw$, $e{:}v$, or $e{:}\varnothing$, respectively. If we are concerned about direction we write $e{:}v \to w$. A *walk* is a chain of nodes and edges,

$$P = (v_0, e_1, v_1, e_2, ..., e_l, v_l),$$

where $v_i \in N$, $e_i \in E$, and $v_\Gamma(e_i) = \{v_{i-1}, v_i\}$; its *length* is l. To indicate its endpoints we may write $P{:}\ v_0 \to v_l$. With minor exceptions, P is determined by its edge sequence, so it may be written as a word

$$P = e_1 e_2 \cdots e_l$$

in the free group $\mathfrak{F}(E)$ generated by E. Then we regard e^{-1} as not merely a formal inverse but as the edge e traversed in the opposite direction. A walk is a *path* if it has no repeated nodes except possibly for $v_l = v_0$ if $l > 0$ (then it is *closed*, otherwise *open*). A *circle* is the edge set of a closed path. (The widely used term "circuit" we reserve for matroid circuits; "cycle" we prefer to reserve for coherently oriented circles.) The set of all circles in Γ is written $\mathscr{C} = \mathscr{C}(\Gamma)$.

A *cutpoint* of Γ is a node whose removal topologically disconnects a component of Γ. In particular, a node which supports a loop or half edge is a cutpoint. A *block graph* is a graph with no cutpoints. A *block* of Γ is a maximal block graph contained in Γ. A node incident to a loop, half edge, or isthmus is a cutpoint and a loose edge, loop, half edge, or isthmus is a block of Γ, for example. The *block/cutpoint graph* of Γ has as nodes all the cutpoints and blocks; an edge joins a block B and a cutpoint p whenever p is a node of B. This graph is a tree if Γ is connected.

A *theta graph* is a subdivision of a triple link, that is, three open paths meeting only at their endpoints. A *handcuff* consists of a pair of edge sets, C_1 and C_2, each of which is a circle or a half-edge singleton set, and the edge set of a connecting open path $P{:}\ u_1 \to u_2$ such that P meets C_i at u_i and nowhere else and C_1 meets C_2 only at $\{u_1\} \cap \{u_2\}$. If P has positive length the handcuff is *loose*. Otherwise it is *tight*. Thetas and handcuffs (excluding half edges) are called *bicircular graphs* by Simões-Pereira (and "bicycles" by some other authors).

The *complete graph* on vertex set X is denoted by K_X. It is *simple*: all edges are links and there are no multiple edges. In particular K_n denotes K_N.

Coalescing Γ by a partial partition π of N means coalescing each block of π to a single node and discarding the nodes outside the support of π, We call X *stable* if $E{:}X=\varnothing$. The *node deletion* $\Gamma\backslash X$ has node set X^c and edge set $\{e\in E{:}v_\Gamma(e)\cap X=\varnothing\}$. Thus $\Gamma\backslash X=(\Gamma{:}X^c)\cup\{\text{loose edges}\}$. For a single-node deletion $\Gamma\backslash\{v\}$ we write $\Gamma\backslash v$.

For edges and walks we employ some further shorthand. To indicate that an edge e has endpoints v and w, or v only, or is a loose edge, we may refer to it as $e{:}vw$, $e{:}v$, or $e{:}\varnothing$, respectively. If we are concerned about direction we write $e{:}v\to w$. A *walk* is a chain of nodes and edges,

$$P=(v_0, e_1, v_1, e_2, ..., e_l, v_l),$$

where $v_i\in N$, $e_i\in E$, and $v_\Gamma(e_i)=\{v_{i-1}, v_i\}$; its *length* is l. To indicate its endpoints we may write $P{:}\ v_0\to v_l$. With minor exceptions, P is determined by its edge sequence, so it may be written as a word

$$P=e_1e_2\cdots e_l$$

in the free group $\mathfrak{F}(E)$ generated by E. Then we regard e^{-1} as not merely a formal inverse but as the edge e traversed in the opposite direction. A walk is a *path* if it has no repeated nodes except possibly for $v_l=v_0$ if $l>0$ (then it is *closed*, otherwise *open*). A *circle* is the edge set of a closed path. (The widely used term "circuit" we reserve for matroid circuits; "cycle" we prefer to reserve for coherently oriented circles.) The set of all circles in Γ is written $\mathscr{C}=\mathscr{C}(\Gamma)$.

A *cutpoint* of Γ is a node whose removal topologically disconnects a component of Γ. In particular, a node which supports a loop or half edge is a cutpoint. A *block graph* is a graph with no cutpoints. A *block* of Γ is a maximal block graph contained in Γ. A node incident to a loop, half edge, or isthmus is a cutpoint and a loose edge, loop, half edge, or isthmus is a block of Γ, for example. The *block/cutpoint graph* of Γ has as nodes all the cutpoints and blocks; an edge joins a block B and a cutpoint p whenever p is a node of B. This graph is a tree if Γ is connected.

A *theta graph* is a subdivision of a triple link, that is, three open paths meeting only at their endpoints. A *handcuff* consists of a pair of edge sets, C_1 and C_2, each of which is a circle or a half-edge singleton set, and the edge set of a connecting open path $P{:}\ u_1\to u_2$ such that P meets C_i at u_i and nowhere else and C_1 meets C_2 only at $\{u_1\}\cap\{u_2\}$. If P has positive length the handcuff is *loose*. Otherwise it is *tight*. Thetas and handcuffs (excluding half edges) are called *bicircular graphs* by Simões-Pereira (and "bicycles" by some other authors).

The *complete graph* on vertex set X is denoted by K_X. It is *simple*: all edges are links and there are no multiple edges. In particular K_n denotes K_N.

Coalescing Γ by a partial partition π of N means coalescing each block of π to a single node and discarding the nodes outside the support of π, while retaining all the edges. The coalesced graph is written Γ/π. Formally, $N(\Gamma/\pi)=\pi$, $E(\Gamma/\pi)=E$, and the new endpoints are given by $v_{\Gamma/\pi}(e)=\{\pi(v){:}\ v\in v_\Gamma(e)\}$, where we recall that $\pi(v)$ is undefined if $v\notin\operatorname{supp}\pi$.

The *restriction* of Γ to an edge set $S\subseteq E$ is just the spanning subgraph (N, S). We sometimes write $\Gamma|S$ for the restriction. The *deletion* of S is $\Gamma\backslash S=(N, S^c)$. The *contraction* of Γ by an edge set A is

$$\Gamma/A=(\Gamma/\pi(A))\backslash A.$$

A *minor* of Γ is any graph resulting from a sequence of contractions and taking of subgraphs. A *proper minor* is any minor except Γ itself.

It is a well-known theorem that any minor of Γ is a contraction of a subgraph; but to justify this, given our definitions, requires some discussion. Suppose $\pi\leqslant\tau$ in $\Pi_N^\dagger$. We want to be able to say that $\Gamma/\tau=(\Gamma/\pi)/\tau$. But technically, τ cannot coalesce Γ/π because it is not a partial partition of $N(\Gamma/\pi)=\pi$. Let us agree that τ acts as a partition of π in the following way (we call it $\tau_\pi\in\Pi_\pi^\dagger$ if we need to stress the distinction between its actions on N and on π): the blocks of τ_π are the sets $C_\pi=\{B\in\pi{:}\ B\subseteq C\}$ for $C\in\tau$. Then $N(\Gamma/\tau)=\tau$ and $N((\Gamma/\pi)/\tau_\pi)=\tau_\pi$ can be considered identical, so several successive coalescences or contractions may be combined into a single one. This suffices to justify the statement that any minor is a contraction of a subgraph, or a subgraph of a contraction (these two being obviously equivalent).

2. Definitions about Bias

A class $\mathscr{B}$ of circles of a graph is a *linear (sub)class* if it has the property:

If C_1 and $C_2\in\mathscr{B}$ and $C_1\cup C_2$ is a theta graph, then $C_1+C_2\in\mathscr{B}$.

In other words, in no theta subgraph do exactly two circles belong to $\mathscr{B}$. A *biased graph* Ω consists of an underlying graph $\|\Omega\|$ and a linear subclass $\mathscr{B}(\Omega)$ of circles of $\|\Omega\|$, called *balanced circles*. We will always let $\Omega=(\Gamma, \mathscr{B})=(N, E, \mathscr{B})$ denote a biased graph with underlying graph $\Gamma=(N, E)$ and balanced circle class $\mathscr{B}$.

A subgraph or edge set of Γ is *balanced* if it has no half edges and every circle in it is balanced. It is *contrabalanced* if it has no balanced circles and no loose edges. An *unbalanced figure* is an unbalanced circle or a half edge. (In using the term "balance" I follow Harary, whose study [5] of criteria for balance in signed graphs foreshadowed the theory of gains and bias. I consider "bias" to be complementary to balance: the less balanced a graph is, the more biased.)

A stronger property than linearity of a subclass of circles of Γ is *additivity* ("circle additivity" in [14]): in any theta subgraph, an odd number of circles belong to the subclass. An *additively biased graph* is a pair $(\Gamma, \mathscr{B})$ where $\mathscr{B}$ is an additive subclass of circles of Γ.

A subgraph Δ of $\|\Omega\|$ is biased in the obvious way: with balanced circle class $\mathscr{B}(\Omega) \cap \mathscr{C}(\Delta)$. Particular subgraphs are $\Omega{:}X$, $\Omega|S$, etc., with the obvious meanings. Any subgraph Δ is a union of balanced and unbalanced components and loose edges. Its *balanced partial partition* is

$$\pi_b(\Delta) = \{B \in \pi(\Delta): (\Delta{:}B) \text{ is balanced}\},$$

its *balanced component number* is

$$b(\Delta) = \# \pi_b(\Delta),$$

and its *unbalanced node set* is

$$N_0(\Delta) = \bigcup \{B \in \pi(\Delta): (\Delta{:}B) \text{ is unbalanced}\}.$$

Its *balanced part* is

$$\Delta{:}N_0(\Delta)^c = \text{union of the balanced components of } \Delta.$$

In particular for $S \subseteq E$ we have (regarding S as a spanning subgraph of Ω) $\pi_b(S)$, $b(S)$, and $N_0(S)$.

A *bias circuit* in Ω is a balanced circle, a loose edge (considered as a singleton set), or a contrabalanced theta or handcuff. A *lift circuit* is a balanced circle, a loose edge, a contrabalanced theta or tight handcuff, or the union of two nodedisjoint unbalanced figures.

A *full* biased graph has an unbalanced edge (a half edge or unbalanced loop) at every node. If Ω is a biased graph, $\Omega^{\bullet}$ denotes Ω made full: Ω with a half edge or unbalanced loop added to every node not already carrying one.

An unbiased graph Δ can be regarded as a biased graph in which every circle is balanced; then it is denoted by $[\Delta]$.

Suppose Ω_1 and Ω_2 are biased graphs. Their *biased union* $\Omega_1 \sqcup \Omega_2$ is the biased graph with vertex set $N_1 \cup N_2$, edge set $E_1 \uplus E_2$ (disjoint union), and balanced circle class $\mathscr{B}_1 \uplus \mathscr{B}_2$.

Let A be an edge set in Ω. The *contraction* of Ω by A is the biased graph Ω/A whose underlying graph is $\|\Omega/A\| = (\Gamma/\pi_b(A))\backslash A$ and whose balanced circle class $\mathscr{B}(\Omega/A)$ consists of all circles

$$C = e_1 e_2 \cdots e_k \in \mathscr{C}(\|\Omega/A\|)$$

such that $C = C' \backslash A$ for some balanced circle C' of Ω. We will see in the

next section that Ω/A is really
graph obtained from Ω by taki
in the next section that any mi
(what is obviously equivalent)
Contraction of a biased gra
worthwhile to describe contrac
following rules:

If $e{:}pq$ is not a loop, dele
$C \not\ni e$ remains balanced if it re
Ω, $C\backslash e$ is a balanced circle of

If e is a balanced loop or

If e is an unbalanced loop
or half edge at p (except e)
replaced by a half edge at q.
remains balanced. There are n

A special kind of minor (bu
the *unbalanced coalescence* Ω
underlying graph is Γ/π;
$\mathscr{C}(\Gamma/\pi) \cap \mathscr{B}(\Omega)$. It is easy t
It is also easy to see that
$[K_n^{\bullet}{:}(\operatorname{supp} \pi)^c]$.

Let $S \subseteq E$. The *balance-clos*

bcl $S = S \cup \{e \in S^c$: there is a

$\cup$ {loose edges}.

An edge set is *balance-closed*
Two biased graphs Ω_1 and
$\nu: N_1 \to N_2$ and $\varepsilon: E_1 \to E_2$ w
$\mathscr{B}_2 = \{\varepsilon(C_1): C_1 \in \mathscr{B}_1\}$. We ca
Finally, a *subdivision* of Ω
a subdivision of $\|\Omega\|$ and w
those of Ω. Biased graphs Ω
isomorphic to subdivisions o

3.

In certain circumstances it
that a biased graph is balan
balance.

("circle additivity" in [14]): in any theta subgraph, an odd number of circles belong to the subclass. An *additively biased graph* is a pair $(\Gamma, \mathscr{B})$ where $\mathscr{B}$ is an additive subclass of circles of Γ.

A subgraph Δ of $\|\Omega\|$ is biased in the obvious way: with balanced circle class $\mathscr{B}(\Omega) \cap \mathscr{C}(\Delta)$. Particular subgraphs are $\Omega{:}X$, $\Omega|S$, etc., with the obvious meanings. Any subgraph Δ is a union of balanced and unbalanced components and loose edges. Its *balanced partial partition* is

$$\pi_b(\Delta) = \{B \in \pi(\Delta)\colon (\Delta{:}B) \text{ is balanced}\},$$

its *balanced component number* is

$$b(\Delta) = \#\pi_b(\Delta),$$

and its *unbalanced node set* is

$$N_0(\Delta) = \bigcup \{B \in \pi(\Delta)\colon (\Delta{:}B) \text{ is unbalanced}\}.$$

Its *balanced part* is

$$\Delta{:}N_0(\Delta)^c = \text{union of the balanced components of } \Delta.$$

In particular for $S \subseteq E$ we have (regarding S as a spanning subgraph of Ω) $\pi_b(S)$, $b(S)$, and $N_0(S)$.

A *bias circuit* in Ω is a balanced circle, a loose edge (considered as a singleton set), or a contrabalanced theta or handcuff. A *lift circuit* is a balanced circle, a loose edge, a contrabalanced theta or tight handcuff, or the union of two nodedisjoint unbalanced figures.

A *full* biased graph has an unbalanced edge (a half edge or unbalanced loop) at every node. If Ω is a biased graph, $\Omega^\bullet$ denotes Ω made full: Ω with a half edge or unbalanced loop added to every node not already carrying one.

An unbiased graph Δ can be regarded as a biased graph in which every circle is balanced; then it is denoted by $[\Delta]$.

Suppose Ω_1 and Ω_2 are biased graphs. Their *biased union* $\Omega_1 \sqcup \Omega_2$ is the biased graph with vertex set $N_1 \cup N_2$, edge set $E_1 \cup E_2$ (disjoint union), and balanced circle class $\mathscr{B}_1 \cup \mathscr{B}_2$.

Let A be an edge set in Ω. The *contraction* of Ω by A is the biased graph Ω/A whose underlying graph is $\|\Omega/A\| = (\Gamma/\pi_b(A)) \backslash A$ and whose balanced circle class $\mathscr{B}(\Omega/A)$ consists of all circles

$$C = e_1 e_2 \cdots e_k \in \mathscr{C}(\|\Omega/A\|)$$

such that $C = C' \backslash A$ for some balanced circle C' of Ω. We will see in the next section that Ω/A is really a biased graph. A *minor* of Ω is any biased graph obtained from Ω by taking subgraphs and contractions. We will see in the next section that any minor is a subgraph of a contraction of Ω, or (what is obviously equivalent) a contraction of a subgraph.

Contraction of a biased graph is sufficiently complicated that it seems worthwhile to describe contraction by a single edge. Ω/e is described by the following rules:

If $e{:}pq$ is not a loop, delete e and coalesce p and q. A balanced circle $C \not\ni e$ remains balanced if it remains a circle; if $C \ni e$ is a balanced circle of Ω, $C \backslash e$ is a balanced circle of Ω/e. There are no other balanced circles.

If e is a balanced loop or loose edge, $\Omega/e = \Omega \backslash e$.

If e is an unbalanced loop or half edge at p, delete e and p. Every loop or half edge at p (except e) becomes a loose edge. Every edge $f{:}pq$ is replaced by a half edge at q. Any balanced circle not passing through p remains balanced. There are no other balanced circles.

A special kind of minor (but which is not, technically, a minor of Ω) is the *unbalanced coalescence* Ω/π of Ω by a partial partition π of N. The underlying graph is Γ/π; the balanced circle class is $\mathscr{B}(\Omega/\pi) = \mathscr{C}(\Gamma/\pi) \cap \mathscr{B}(\Omega)$. It is easy to see directly that Ω/π is a biased graph. It is also easy to see that $\Omega/\pi = (\Omega \sqcup \Psi)/E(\Psi)$ where $\Psi = [K_n{:}\pi] \cup [K_n^\bullet{:}(\operatorname{supp} \pi)^c]$.

Let $S \subseteq E$. The *balance-closure* of S is

$$\operatorname{bcl} S = S \cup \{e \in S^c\colon \text{there is a balanced circle } C \text{ such that } e \in C \subseteq S \cup \{e\}\} \cup \{\text{loose edges}\}.$$

An edge set is *balance-closed* if it is its own balance-closure.

Two biased graphs Ω_1 and Ω_2 are *isomorphic* when there are bijections $\nu\colon N_1 \to N_2$ and $\varepsilon\colon E_1 \to E_2$ which are a graph isomorphism and such that $\mathscr{B}_2 = \{\varepsilon(C_1)\colon C_1 \in \mathscr{B}_1\}$. We call the pair (ν, ε) an *isomorphism* $\Omega_1 \to \Omega_2$.

Finally, a *subdivision* of Ω is a biased graph whose underlying graph is a subdivision of $\|\Omega\|$ and whose balanced circles are the subdivisions of those of Ω. Biased graphs Ω_1 and Ω_2 are *homeomorphic* if they are both isomorphic to subdivisions of the same biased graph.

3. Basics of Balance

In certain circumstances it is not necessary to test every circle to find out that a biased graph is balanced. In this section we discuss some criteria for balance.

PROPOSITION 3.1. *Let B be an edge set in a biased graph. Then* bcl B *is balanced if and only if B is balanced.*

Proof. We prove the nontrivial half (the "if") by means of Tutte's path theorem (see [2, p. 15.2] or [12, Theorem 4.34]), which for graphs says that if $\mathscr{L}$ is a linear class of circles and C_0, C are circles such that $C \notin \mathscr{L}$, then there is a "path of circles"

$$C_0, C_1, ..., C_k = C \qquad (*)$$

such that $C_0 \cup C_1$, $C_1 \cup C_2$, ... are theta graphs and $C_1, ..., C_{k-1} \notin \mathscr{L}$.

Let C be a hypothetical unbalanced circle in bcl B. In order to guarantee finiteness we will shrink the example. Write $C \backslash B = \{e_0, ..., e_m\}$. By definition each e_i belongs to a balanced circle $C_i \subseteq B \cup \{e_i\}$. Discard all of B except $(B \cap C) \cup \bigcup_i (C_i \backslash e_i)$, leaving a new set B which is finite, and let $A = B \cup C$.

Now we can proceed by induction on the size of A. The induction assumption is that $B_0 = A \backslash e_0$ is balanced. We also know that e_0 belongs to a balanced circle $C_0 \subseteq B \cup \{e_0\}$. The linear class $\mathscr{L}$ we need is that consisting of all circles in B_0. By hypothesis every $C' \in \mathscr{L}$ is balanced. By Tutte's theorem, there is a path $(*)$ such that each $C_i \ni e_0$. But then $C_{i-1} + C_i$ is a circle in B_0, so it is balanced. From the axiom of bias, $C_0 \in \mathscr{B}$ and $C_0 + C_1 \in \mathscr{B}$ imply $C_1 \in \mathscr{B}$; continuing in this fashion we find that $C_k = C$ is balanced, contrary to the assumption. Therefore $B_0 \cup \{e_0\}$ is balanced, if B is balanced and finite. Since we have already reduced the problem to the finite case, the proposition is proved. ∎

COROLLARY 3.2. *Let Ω be a biased ordinary graph and T a maximal forest in Ω. For Ω to be balanced, it is necessary and sufficient that the unique circle $C_e \subseteq T \cup \{e\}$ be balanced, for every edge $e \notin T$.*

Proof. Necessity is obvious. For sufficiency note that the hypothesis implies bcl $T = E$. Then Ω is balanced by Proposition 3.1. ∎

A class $\mathscr{D} \subseteq \mathscr{C}(\Gamma)$ *spans* if every circle is representable as $C_1 + C_2 + \cdots + C_\gamma$ for $C_1, C_2, ..., C_\gamma$ in the class. A *basis of circles* is a minimal spanning class of circles. Given T as in Corollary 3.2, the circles C_e, $e \notin T$, are a basis. But not every basis is of this form and it is not true of an arbitrary basis of circles in a biased ordinary graph Ω that, if all its circles are balanced, so is Ω. For example, let $\Omega = (K_n, \mathscr{H})$ where $\mathscr{H}$ is the set of Hamiltonian circles and n is odd and at least 5. Then Ω is a biased ordinary graph and $\mathscr{H}$ spans, so there is a basis of balanced circles (contained in $\mathscr{H}$). But Ω is unbalanced. On the other hand, any biasing of K_4 which has a balanced basis is balanced, and I suspect this may be true for any even-order K_n. This suggests

PROPOSITION 3.1. *Let B be an edge set in a biased graph. Then* bcl B *is balanced if and only if B is balanced.*

Proof. We prove the nontrivial half (the "if") by means of Tutte's path theorem (see [2, p. 15.2] or [12, Theorem 4.34]), which for graphs says that if $\mathscr{L}$ is a linear class of circles and C_0, C are circles such that $C \notin \mathscr{L}$, then there is a "path of circles"

$$C_0, C_1, ..., C_k = C \qquad (*)$$

such that $C_0 \cup C_1$, $C_1 \cup C_2$, ... are theta graphs and $C_1, ..., C_{k-1} \notin \mathscr{L}$.

Let C be a hypothetical unbalanced circle in bcl B. In order to guarantee finiteness we will shrink the example. Write $C \backslash B = \{e_0, ..., e_m\}$. By definition each e_i belongs to a balanced circle $C_i \subseteq B \cup \{e_i\}$. Discard all of B except $(B \cap C) \cup \bigcup_i (C_i \backslash e_i)$, leaving a new set B which is finite, and let $A = B \cup C$.

Now we can proceed by induction on the size of A. The induction assumption is that $B_0 = A \backslash e_0$ is balanced. We also know that e_0 belongs to a balanced circle $C_0 \subseteq B \cup \{e_0\}$. The linear class $\mathscr{L}$ we need is that consisting of all circles in B_0. By hypothesis every $C' \in \mathscr{L}$ is balanced. By Tutte's theorem, there is a path $(*)$ such that each $C_i \ni e_0$. But then $C_{i-1} + C_i$ is a circle in B_0, so it is balanced. From the axiom of bias, $C_0 \in \mathscr{B}$ and $C_0 + C_1 \in \mathscr{B}$ imply $C_1 \in \mathscr{B}$; continuing in this fashion we find that $C_k = C$ is balanced, contrary to the assumption. Therefore $B_0 \cup \{e_0\}$ is balanced, if B is balanced and finite. Since we have already reduced the problem to the finite case, the proposition is proved. ∎

COROLLARY 3.2. *Let Ω be a biased ordinary graph and T a maximal forest in Ω. For Ω to be balanced, it is necessary and sufficient that the unique circle $C_e \subseteq T \cup \{e\}$ be balanced, for every edge $e \notin T$.*

Proof. Necessity is obvious. For sufficiency note that the hypothesis implies bcl $T = E$. Then Ω is balanced by Proposition 3.1. ∎

A class $\mathscr{D} \subseteq \mathscr{C}(\Gamma)$ *spans* if every circle is representable as $C_1 + C_2 + \cdots + C_\gamma$ for $C_1, C_2, ..., C_\gamma$ in the class. A *basis of circles* is a minimal spanning class of circles. Given T as in Corollary 3.2, the circles C_e, $e \notin T$, are a basis. But not every basis is of this form and it is not true of an arbitrary basis of circles in a biased ordinary graph Ω that, if all its circles are balanced, so is Ω. For example, let $\Omega = (K_n, \mathscr{H})$ where $\mathscr{H}$ is the set of Hamiltonian circles and n is odd and at least 5. Then Ω is a biased ordinary graph and $\mathscr{H}$ spans, so there is a basis of balanced circles (contained in $\mathscr{H}$). But Ω is unbalanced. On the other hand, any biasing of K_4 which has a balanced basis is balanced, and I suspect this may be true for any even-order K_n. This suggests

Problem 3.3. Find necessary and/or sufficient conditions on a graph Γ for it to have no other spanning linear subclass of circles than $\mathscr{C}(\Gamma)$.

COROLLARY 3.4. *In a biased graph Ω let B_1 and B_2 be balanced edge sets for which $X = N(B_1) \cap N(B_2)$ has $\# X \leqslant 2$. If $\# X \leqslant 1$, or if $X = \{u, v\}$ where u and v are distinct nodes and there exist paths $P_i\colon u \to v$ in B_i for $i = 1$ and 2 such that $P_1 \cup P_2$ is balanced, then $B_1 \cup B_2$ is balanced.*

Proof. If $\# X \leqslant 1$ the result is obvious.

If $\# X = 2$, suppose u is not a cut node of B_2 and let $B' = B_1 \cup (B_2 \backslash u)$. Then B' is balanced by the case $\# X \leqslant 1$ with $B_2 \backslash u$ instead of B_2. Since $P_2 \subseteq$ bcl B', $B' \cup P_2$ is balanced. Since $B_2 \subseteq$ bcl$(B' \cup P_2)$, $B_1 \cup B_2$ is balanced.

If u is a cut node, let B_2^* be the part of B_2 separated by u and containing v. Then $B_1 \cup B_2^*$ is balanced by the above argument and $B_1 \cup B_2$ is balanced by the case $\# X \leqslant 1$. ∎

PROPOSITION 3.5. *Let S be an edge set in a biased graph Ω. If S is balanced, then* bcl S *is balance-closed. But if S is unbalanced,* bcl S *may not be balance-closed.*

Proof. Suppose S is balanced. Let T be a maximal forest in S. Obviously, it is a maximal forest in bcl(bcl S), which is balanced by Proposition 3.1. Therefore,

$$\text{bcl(bcl } S) \subseteq \text{bcl } T \subseteq \text{bcl } S,$$

from which the desired result is immediate.

We present an example of a biased graph Ω and an unbalanced set S whose second balance-closure is larger than bcl S. Let $N = \{v, u_1, u_2, w)$ and $E = \{e_i\colon vu_i, f_i\colon vw, g_i\colon u_iw, h\colon v_1v_2\}$, where $i = 1, 2$. The balanced circles are $e_i f_i g_i$, $e_i f_i g_j h$ (where $j \neq i$), and $g_1 g_2 h$. For the set S take $\{e_1, e_2, f_1, f_2\}$. Then bcl $S = S \cup \{g_1, g_2\}$ and bcl(bcl $S) = E$. ∎

We see from this proposition that balance-closure is not an abstract closure operation [1].

A different way of stating the positive half of Proposition 3.5 is

COROLLARY 3.6. *Let S be a balanced edge set in Ω. The balance-closure* bcl S *is characterized as the largest balanced edge set containing S in $\Omega{:}\pi_b(S)$.*

Two more easy consequences of Proposition 3.1 are the following criteria for balance.

COROLLARY 3.7. *Let B be a balanced, connected edge set and let S be a balanced edge set with connected components S_i, $i \in I$. Then $B \cup S$ is balanced if and only if all the $B \cup S_i$ such that $N(B) \cap N(S_i) \neq \varnothing$ are balanced.*

Proof. The nontrivial part is the "if." We may assume $B \cup S$ is connected. Let T_B be a spanning tree of B and extend it to a spanning tree T_i of each $B \cup S_i$. Let T be the union of all T_i. Then T is a spanning tree of $B \cup S$. Since all $B \cup S_i \subseteq \text{bcl}\, T_i \subseteq \text{bcl}\, T$, $B \cup S$ is balanced. ▮

COROLLARY 3.8. *Let B_1 and B_2 be balanced edge sets such that $N(B_1) \cap N(B_2)$ is connected in $B_1 \cap B_2$. Then $B_1 \cup B_2$ is balanced.*

Proof. Choose a spanning tree of $B_1 \cap B_2$ and extend it to maximal forests T_1 of B_1 and T_2 of B_2. Then $T_1 \cup T_2$ is a maximal forest in $B_1 \cup B_2$. Since $B_i \subseteq \text{bcl}\, T_i$, we have $B_1 \cup B_2 \subseteq \text{bcl}(T_1 \cup T_2)$. So $B_1 \cup B_2$ is balanced. ▮

We conclude with a simple result that will be useful later in this series.

PROPOSITION 3.9. *Let Ω be a biased graph whose underlying graph is a block graph. Either Ω is balanced or else every edge belongs to an unbalanced figure (unbalanced circle or half edge).*

Proof. We need treat only the case where Ω has at least two edges. Suppose Ω is unbalanced, C is an unbalanced circle, and e is an edge not in C. By Menger's theorem, we can join the endpoints of e by paths to nodes of C so as to form a theta graph H. Since C is unbalanced and the balanced circles are a linear class, at least one of the circles in H on e is unbalanced. ▮

4. BALANCE AND MINORS

Now we can show that a contraction of a biased graph is biased and that all minors are obtained by one contraction of a subgraph, as well as other good things about the relationship between balance and contraction.

LEMMA 4.1. *Let Ω be a biased graph, $A \subseteq E$, and $C \in \mathscr{C}(\Omega/A)$. For C to be balanced in Ω/A it is necessary and sufficient that $C \cup (A{:}N_0(A)^c)$ be balanced in Ω.*

Proof. The sufficiency is obvious from the definition of contraction. The necessity follows as follows. Let C be balanced in Ω/A; hence by definition, there is a balanced circle C^* of Ω such that $C \subseteq C^* \subseteq C \cup A_{\text{b}}$, where $A_{\text{b}} = A{:}N_0(A)^c$, the balanced part of A. Let A_i, $i \in I$, be the balanced components of A. By Corollary 3.4, each $A_i \cup C^*$ is balanced. Then $A_{\text{b}} \cup C^*$ is balanced, by Corollary 3.7 applied to $S = A_{\text{b}}$, $B = C^*$. ▮

THEOREM 4.2. *A contraction of a biased graph is a biased graph.*

Proof. We need conside
C_1, C_2, C_3 be the three circ
are balanced. According t
Let B_i consist of the comp
Then $B_1 \cap B_2$ consists of th
components of A, whenc
$B_1 \cap B_2$. By Corollary 3.8
follows that $C_3 \in \mathscr{B}(\Omega/A)$.

The next results indicat
convenient to let S/A den
$S \subseteq E$.

LEMMA 4.3. *Let A be a
is balanced in Ω/A if and o*

Proof. If S/A is unbala
must have been a half edg
which case $C \cup A$ is unba
unbalanced.

Suppose S/A is balance
it to a maximal forest T in
$e \in S \backslash (A \cup T)$. The funda
$C_e/A \subseteq (T/A) \cup \{e\} \subseteq S/A$,
is balanced by Lemma 4.1

LEMMA 4.4. *Let Ω be
of Ω. Then*

$N_0(A \cup S) = N$

Proof. Let $A_{\text{b}} = A{:}N_0($
First we prove the l
$N_0(A) \subseteq N_0(A \cup S)$. Suppo
$C \cup A_{\text{b}}$ is unbalanced, by
containing C is unbalance
Then either e is a half edg
of A. In any case, if $X \in \pi$
S/A, then it is contained i
To prove the right-har
$N_0(A \cup S) \backslash N_0(A)$ and l
$(A \cup S){:}Y$ is an unbalance
is connected in S/A to a
$Y \backslash N_0(A)$ to $Y \cap N_0(A)$. C
is unbalanced; that is, $X \in$

Proof. The nontrivial part is the "if." We may assume $B \cup S$ is connected. Let T_B be a spanning tree of B and extend it to a spanning tree T_i of each $B \cup S_i$. Let T be the union of all T_i. Then T is a spanning tree of $B \cup S$. Since all $B \cup S_i \subseteq \text{bcl}\, T_i \subseteq \text{bcl}\, T$, $B \cup S$ is balanced. ∎

COROLLARY 3.8. *Let B_1 and B_2 be balanced edge sets such that $N(B_1) \cap N(B_2)$ is connected in $B_1 \cap B_2$. Then $B_1 \cup B_2$ is balanced.*

Proof. Choose a spanning tree of $B_1 \cap B_2$ and extend it to maximal forests T_1 of B_1 and T_2 of B_2. Then $T_1 \cup T_2$ is a maximal forest in $B_1 \cup B_2$. Since $B_i \subseteq \text{bcl}\, T_i$, we have $B_1 \cup B_2 \subseteq \text{bcl}(T_1 \cup T_2)$. So $B_1 \cup B_2$ is balanced. ∎

We conclude with a simple result that will be useful later in this series.

PROPOSITION 3.9. *Let Ω be a biased graph whose underlying graph is a block graph. Either Ω is balanced or else every edge belongs to an unbalanced figure (unbalanced circle or half edge).*

Proof. We need treat only the case where Ω has at least two edges. Suppose Ω is unbalanced, C is an unbalanced circle, and e is an edge not in C. By Menger's theorem, we can join the endpoints of e by paths to nodes of C so as to form a theta graph H. Since C is unbalanced and the balanced circles are a linear class, at least one of the circles in H on e is unbalanced. ∎

4. BALANCE AND MINORS

Now we can show that a contraction of a biased graph is biased and that all minors are obtained by one contraction of a subgraph, as well as other good things about the relationship between balance and contraction.

LEMMA 4.1. *Let Ω be a biased graph, $A \subseteq E$, and $C \in \mathscr{C}(\Omega/A)$. For C to be balanced in Ω/A it is necessary and sufficient that $C \cup (A{:}N_0(A)^c)$ be balanced in Ω.*

Proof. The sufficiency is obvious from the definition of contraction. The necessity follows as follows. Let C be balanced in Ω/A; hence by definition, there is a balanced circle C^* of Ω such that $C \subseteq C^* \subseteq C \cup A_b$, where $A_b = A{:}N_0(A)^c$, the balanced part of A. Let A_i, $i \in I$, be the balanced components of A. By Corollary 3.4, each $A_i \cup C^*$ is balanced. Then $A_b \cup C^*$ is balanced, by Corollary 3.7 applied to $S = A_b$, $B = C^*$. ∎

THEOREM 4.2. *A contraction of a biased graph is a biased graph.*

Proof. We need consider only contraction by a balanced edge set A. Let C_1, C_2, C_3 be the three circles of a theta graph in Ω/A and suppose C_1, C_2 are balanced. According to the lemma, $C_1 \cup A$ and $C_2 \cup A$ are balanced. Let B_i consist of the component of $C_i \cup A$ that contains C_i, for $i = 1, 2, 3$. Then $B_1 \cap B_2$ consists of the common edges of C_1 and C_2 and the incident components of A, whence $N(B_1 \cap B_2) = N(B_1) \cap N(B_2)$ is connected by $B_1 \cap B_2$. By Corollary 3.8, $B_1 \cup B_2$ is balanced. Thus B_3 is balanced; it follows that $C_3 \in \mathscr{B}(\Omega/A)$. ∎

The next results indicate how balance interacts with contraction. It is convenient to let S/A denote $S \setminus A$ considered as an edge set in Ω/A, for $S \subseteq E$.

LEMMA 4.3. *Let A be a balanced edge set of Ω and let $S \subseteq E \setminus A$. Then S is balanced in Ω/A if and only if $S \cup A$ is balanced in Ω.*

Proof. If S/A is unbalanced, then either it contains a half edge, which must have been a half edge in S, or it contains an unbalanced circle C, in which case $C \cup A$ is unbalanced by Lemma 4.1. In either case $S \cup A$ is unbalanced.

Suppose S/A is balanced. Let T_A be a maximal forest of A and extend it to a maximal forest T in $S \cup A$. Then T/A is a maximal forest of S/A. Let $e \in S \setminus (A \cup T)$. The fundamental circle $C_e \subseteq T \cup \{e\}$ is balanced; for $C_e/A \subseteq (T/A) \cup \{e\} \subseteq S/A$, which is balanced, and C_e/A is a circle, so C_e is balanced by Lemma 4.1. By Corollary 3.2, $S \cup A$ is balanced. ∎

LEMMA 4.4. *Let Ω be a biased graph and let A, S be disjoint edge sets of Ω. Then*

$$N_0(A \cup S) = N_0(A) \cup \bigcup \{X \in \pi_b(A) : X \in N_0(S/A)\}.$$

Proof. Let $A_b = A{:}N_0(A)^c$, the balanced part of A.

First we prove the left-hand side contains the right. Obviously, $N_0(A) \subseteq N_0(A \cup S)$. Suppose C/A is an unbalanced circle in S/A. Then $C \cup A_b$ is unbalanced, by Lemma 4.1. Therefore, the component of $S \cup A$ containing C is unbalanced. Suppose $X \in \pi_b(A)$ carries a half edge e of S/A. Then either e is a half edge in S or it joins X to an unbalanced component of A. In any case, if $X \in \pi_b(A)$ is a node of an unbalanced component of S/A, then it is contained in $N_0(A \cup S)$.

To prove the right-hand side contains the left, let v be a vertex of $N_0(A \cup S) \setminus N_0(A)$ and let $v \in X \in \pi_b(A)$ and $X \subseteq Y \in \pi(A \cup S)$. Thus, $(A \cup S){:}Y$ is an unbalanced component of $A \cup S$. If Y meets $N_0(A)$, then X is connected in S/A to a half edge which (in Ω) was an edge of S linking $Y \setminus N_0(A)$ to $Y \cap N_0(A)$. Consequently, the component of S/A containing X is unbalanced; that is, $X \in N_0(S/A)$. If on the other hand Y does not meet

$N_0(A)$, then $A{:}Y$ is balanced and $(S{:}Y)/A$ is a component of S/A. By Lemma 4.3 applied to $A{:}Y$ and $S{:}Y$, $(S{:}Y)/A$ is unbalanced. It follows that $X \in N_0(S/A)$. ▮

PROPOSITION 4.5. *Let Ω be a biased graph and $A \subseteq E$. For an edge set $S \subseteq E(\Omega/A)$ the following properties are equivalent:*

(i) *S is balanced in Ω/A.*

(ii) *$(S \cup A){:}N_0(A)^c$ is balanced (in Ω) and no edge of S links $N_0(A)$ to $N_0(A)^c$.*

(iii) *$N_0(S \cup A) = N_0(A)$ in Ω.*

Proof. The equivalence of (i) and (ii) follows from Lemma 4.4. That (ii) and (iii) are equivalent is obvious from the definitions. ▮

PROPOSITION 4.6. *In a biased graph Ω, let A and S be disjoint edge sets such that A is balanced and S is balanced in Ω/A. Then* $\mathrm{bcl}_{\Omega/A}(S) = \mathrm{bcl}_\Omega(A \cup S)\backslash A$.

Proof. Let $B = \mathrm{bcl}_\Omega(A \cup S)$ and $B' = A \cup \mathrm{bcl}_{\Omega/A}(S)$, so that $B'/A = \mathrm{bcl}_{\Omega/A}(S)$. We rely on Lemma 4.3 and Corollary 3.6.

Since B is balanced, so is B/A. By definition, $\pi_b(B) = \pi_b(A \cup S)$; thus $\pi_b(B/A) = \pi_b(S/A)$. It follows from Corollary 3.6 that $B/A \subseteq B'/A$.

Since B'/A is balanced, B' is balanced. Also, $\pi_b(B'/A) = \pi_b(S/A)$ implies $\pi_b(B') = \pi_b(S \cup A)$. It follows that $B' \subseteq B$.

Combining these deductions, we have $B = B'$. ▮

THEOREM 4.7. *Let Ω be a biased graph. If A_1 and A_2 are disjoint edge sets of Ω, then $(\Omega/A_1)/A_2 = \Omega/(A_1 \cup A_2)$. If $T \subseteq S \subseteq E(\Omega)$, then $(\Omega|S)|T = \Omega|T$ and $(\Omega|S)/T = (\Omega/T)|(S\backslash T)$.*

Proof. The latter two equations are clear. The former, we note, implicitly identifies the partial partition $\pi_b(A_2; \Omega/A_1)$ of $N(\Omega/A_1) = \pi_b(A_1)$ with a partial partition of N as discussed in Section 1.

Consider $(\Omega/A_1)/A_2$ and $\Omega/(A_1 \cup A_2)$. It is clear that $\pi(A_2; \Omega/A_1) = \pi(A_1 \cup A_2; \Omega)$. According to Lemma 4.4, we can further state that $\pi_b(A_2; \Omega/A_1) = \pi_b(A_1 \cup A_2; \Omega)$. (All this assumes the standard identifications.) Thus one can see that $\|(\Omega/A_1)/A_2\| = \|\Omega/(A_1 \cup A_2)\|$. We have to show that the two biased graphs have the same balance. Let $S \subseteq E\backslash(A_1 \cup A_2)$. Then S is balanced in $\Omega/(A_1 \cup A_2)$ if and only if $S \cup (A_1 \cup A_2)$ is balanced in Ω. At the same time, S is balanced in $(\Omega/A_1)/A_2 \Leftrightarrow S \cup A_2$ is balanced in $\Omega/A_1 \Leftrightarrow (S \cup A_2) \cup A_1$ is balanced in Ω. Evidently, balance in $(\Omega/A_1)/A_2$ and in $\Omega/(A_1 \cup A_2)$ do agree. ▮

$N_0(A)$, then $A{:}Y$ is balanced and $(S{:}Y)/A$ is a component of S/A. By Lemma 4.3 applied to $A{:}Y$ and $S{:}Y$, $(S{:}Y)/A$ is unbalanced. It follows that $X \in N_0(S/A)$. ▮

PROPOSITION 4.5. *Let Ω be a biased graph and $A \subseteq E$. For an edge set $S \subseteq E(\Omega/A)$ the following properties are equivalent:*

(i) *S is balanced in Ω/A.*

(ii) *$(S \cup A){:}N_0(A)^c$ is balanced (in Ω) and no edge of S links $N_0(A)$ to $N_0(A)^c$.*

(iii) *$N_0(S \cup A) = N_0(A)$ in Ω.*

Proof. The equivalence of (i) and (ii) follows from Lemma 4.4. That (ii) and (iii) are equivalent is obvious from the definitions. ▮

PROPOSITION 4.6. *In a biased graph Ω, let A and S be disjoint edge sets such that A is balanced and S is balanced in Ω/A. Then* $\mathrm{bcl}_{\Omega/A}(S) = \mathrm{bcl}_{\Omega}(A \cup S)\backslash A$.

Proof. Let $B = \mathrm{bcl}_{\Omega}(A \cup S)$ and $B' = A \cup \mathrm{bcl}_{\Omega/A}(S)$, so that $B'/A = \mathrm{bcl}_{\Omega/A}(S)$. We rely on Lemma 4.3 and Corollary 3.6.

Since B is balanced, so is B/A. By definition, $\pi_b(B) = \pi_b(A \cup S)$; thus $\pi_b(B/A) = \pi_b(S/A)$. It follows from Corollary 3.6 that $B/A \subseteq B'/A$.

Since B'/A is balanced, B' is balanced. Also, $\pi_b(B'/A) = \pi_b(S/A)$ implies $\pi_b(B') = \pi_b(S \cup A)$. It follows that $B' \subseteq B$.

Combining these deductions, we have $B = B'$. ▮

THEOREM 4.7. *Let Ω be a biased graph. If A_1 and A_2 are disjoint edge sets of Ω, then $(\Omega/A_1)/A_2 = \Omega/(A_1 \cup A_2)$. If $T \subseteq S \subseteq E(\Omega)$, then $(\Omega|S)|T = \Omega|T$ and $(\Omega|S)/T = (\Omega/T)|(S\backslash T)$.*

Proof. The latter two equations are clear. The former, we note, implicitly identifies the partial partition $\pi_b(A_2; \Omega/A_1)$ of $N(\Omega/A_1) = \pi_b(A_1)$ with a partial partition of N as discussed in Section 1.

Consider $(\Omega/A_1)/A_2$ and $\Omega/(A_1 \cup A_2)$. It is clear that $\pi(A_2; \Omega/A_1) = \pi(A_1 \cup A_2; \Omega)$. According to Lemma 4.4, we can further state that $\pi_b(A_2; \Omega/A_1) = \pi_b(A_1 \cup A_2; \Omega)$. (All this assumes the standard identifications.) Thus one can see that $\|(\Omega/A_1)/A_2\| = \|\Omega/(A_1 \cup A_2)\|$. We have to show that the two biased graphs have the same balance. Let $S \subseteq E\backslash(A_1 \cup A_2)$. Then S is balanced in $\Omega/(A_1 \cup A_2)$ if and only if $S \cup (A_1 \cup A_2)$ is balanced in Ω. At the same time, S is balanced in $(\Omega/A_1)/A_2 \Leftrightarrow S \cup A_2$ is balanced in $\Omega/A_1 \Leftrightarrow (S \cup A_2) \cup A_1$ is balanced in Ω. Evidently, balance in $(\Omega/A_1)/A_2$ and in $\Omega/(A_1 \cup A_2)$ do agree. ▮

COROLLARY 4.8. *Any minor of a biased graph is a subgraph of a contraction and is also a contraction of a subgraph.*

5. BIAS FROM GAINS

A *gain graph* (also known as "voltage graph") $\Phi = (\Gamma, \phi)$ consists of an underlying graph $\|\Phi\| = \Gamma = (N, E)$ and a *gain mapping* $\phi: E_* \to \mathfrak{G}$ from the ordinary edges of Γ into a *gain group* $\mathfrak{G}$. To be precise we may call Φ a $\mathfrak{G}$*-gain graph.* It is understood that $\phi(e^{-1}) = \phi(e)^{-1}$, where e^{-1} means e with its orientation reversed. (This applies to loops as well as links.) Thus $\phi(e)$ depends on the orientation of e but neither orientation is preferred.

Formally, we may say that ϕ defines a homomorphism $\mathfrak{F}(E_*) \to \mathfrak{G}$ from the free group on E_* into the gain group. A walk $P = e_1 e_2 \cdots e_k$ thus has the gain value $\phi(P) = \phi(e_1)\,\phi(e_2) \cdots \phi(e_k)$ under ϕ. If P is a circle, its value depends on the starting point and direction, but whether or not the value equals the identity element 1 is an absolute. A circle whose value is 1 is called *balanced*; the class of balanced circles is $\mathscr{B}(\Phi)$. We write $[\Phi] = (\Gamma, \mathscr{B}(\Phi))$. In what follows, Φ will always be a gain graph on underlying graph Γ, with gain mapping ϕ and group $\mathfrak{G}$.

PROPOSITION 5.1. *If Φ is a gain graph, $[\Phi]$ is a biased graph.*

Proof. In a theta graph with two balanced circles, all three constituent paths have the same gain value. As a consequence, the third circle is also balanced. ▮

So every gain graph is a biased graph; but the converse is false: see Example 5.8.

Let $\lambda: N \to \mathfrak{G}$ be any function. *Switching* Φ by λ means replacing $\phi(e)$ by $\phi^{\lambda}(e) = \lambda(v)^{-1}\phi(e)\,\lambda(w)$, where e is oriented from v to w. The switched graph, $\Phi^{\lambda} = (\Gamma, \phi^{\lambda})$. is called *switching equivalent* to Φ.

LEMMA 5.2. $[\Phi^{\lambda}] = [\Phi]$. ▮

Since our interest is in the bias rather than the particular gains, from now on we will consider switching-equivalent gain graphs to be essentially the same. The fundamental lemma on switching (for our purpose) is

LEMMA 5.3. *Φ is balanced if and only if it has no half edges and ϕ switches to the identity gain.*

Proof. We may assume Φ is connected. Let T be a spanning tree and u a root node. For $v, w \in N$, let T_{vw} be the unique path in T from v to w.

Switching by $\lambda(v) = \phi(T_{vu})$ reduces the gains on T to 1, and no other switching function will achieve this reduction. Considering the fundamental circles C_e of ordinary edges $e \notin T$, it is clear that Φ^λ is balanced if and only if $\phi^\lambda \equiv 1$. ▮

A *subgraph* of Φ is a subgraph of Γ with the same gain mapping, restricted of course to the subgraph's edges. In particular the *restriction* $\Phi | S$, where $S \subseteq E$, is a spanning subgraph. The *contraction* Φ/A by an edge set A is defined as follows. Let B be the union of the balanced components of A and switch so $\phi^\lambda|_B \equiv 1$. Coalesce $\|\Phi\|$ by $\pi_b(A)$ and delete all edges in A. The gain of an ordinary edge e in the resulting graph is $\phi^\lambda(e)$. This defines the contracted gain graph Φ/A. Of course, the gain mapping of Φ/A is only determined up to switching by this construction, but that is quite satisfactory here. A *minor* of Φ is any gain graph resulting from switching, contracting, and taking subgraphs as often as desired.

THEOREM 5.4. *Let Φ be a gain graph and let $S, A \subseteq E(\Phi)$. Then $[\Phi]|S = [\Phi|S]$ and $[\Phi]/A = [\Phi/A]$.*

Proof. The former statement is obvious. As for the latter, from the construction clearly $\|\Phi/A\| = \|[\Phi]/A\|$. Suppose ϕ switched so $\phi|_B \equiv 1$, where B is the balanced part of A. Consider a circle $C = e_1 e_2 \cdots e_k$ in the contracted graph. There is a circle D in Φ of the form $e_1 P_1 e_2 P_2 \cdots e_k P_k$, where P_i is a path in a component of B. We have $\phi_{\Phi/A}(C) = \phi(D)$ since $\phi|_B \equiv 1$. We know that C is balanced in $[\Phi]/A$ precisely when D is, by definition of biased contraction. Therefore Φ/A and $[\Phi]/A$ have the same balanced circles. ▮

COROLLARY 5.5. *Any minor of a gain graph Φ is (up to switching) a subgraph of a contraction and also a contraction of a subgraph.*

Proof. Let Ψ be the minor. We know the corresponding minor of $[\Phi]$ is $[\Psi]$, which is a subgraph Δ of a contraction $[\Phi]/A$. So Ψ has the same underlying graph as the corresponding subgraph $(\Phi/A)|\Delta$ of Φ/A, by Theorem 5.4. It is easy to see that, by switching ϕ beforehand to be 1 on the balanced part of A, we have the same gains on Ψ and on $(\Phi/A)/\Delta$. ▮

We call a biased graph *gain biased*, or more precisely $\mathfrak{G}$-*biased*, if it equals $[\Phi]$ for some gain graph, or $\mathfrak{G}$-gain graph, Φ. A result that will be useful later is

LEMMA 5.6. *If Ω_1 and Ω_2 are gain-biased graphs, then so is their biased union $\Omega_1 \sqcup \Omega_2$.*

Proof. Suppose $\Omega_i = [\Phi_i]$. First, enlarge $\mathfrak{G}_1$ to $\mathfrak{G}_1' = \mathfrak{G}_1 \times \mathfrak{F}(N)$. Define $\lambda(v) = (1, v) \in \mathfrak{G}_1'$ for $v \in N$, treat ϕ_1 as mapping into $\mathfrak{G}_1 \times \{1\} \subseteq \mathfrak{G}_1'$, and

switch ϕ_1 to ϕ_1^λ. Next,
$\mathfrak{G}$ in the obvious way.
for $\Omega_1 \sqcup \Omega_2$. ▮

COROLLARY 5.7. *Th*
$\mathfrak{G}$-biased graphs for an

Corollary 5.7 shows
graphs can be characte
not in the class. This t
we merely show by exa
not gain biased.

EXAMPLE 5.8. Let
pairs $e_{i-1,i}$, $f_{i-1,i}$ fo
endpoints and are take
$e_{12}e_{23}f_{34}f_{41}$, and $f_{12}f_{23}$
biased. However, every

Proof. Ω_4 is obviou
may use the edge name
script order (e_{12} from v
all $f_{ij} \neq 1, f_{12}f_{23} = 1, f_{34}$
The symmetry of Ω_4
if $\Omega_4 \backslash e_{12}$ and $\Omega_4 \backslash f_{12}$
and that $\Omega_4 \backslash e_{12} = \Psi \sqcup$
$\Omega_4 \backslash f_{12} = \Psi' \sqcup f_{23}$. So,
A similar approach
aside from the loop f
The former is sign-bias
biased union of $(\Omega_4/$
(although not with gro
We conclude that ev

We list some biased
in the literature for the
the matroids although
treat many of these ex

EXAMPLE 6.1. *Bala*
without half edges. Fo
graphs.

Switching by $\lambda(v)=\phi(T_{vu})$ reduces the gains on T to 1, and no other switching function will achieve this reduction. Considering the fundamental circles C_e of ordinary edges $e \notin T$, it is clear that Φ^{λ} is balanced if and only if $\phi^{\lambda} \equiv 1$. ∎

A *subgraph* of Φ is a subgraph of Γ with the same gain mapping, restricted of course to the subgraph's edges. In particular the *restriction* $\Phi|S$, where $S \subseteq E$, is a spanning subgraph. The *contraction* Φ/A by an edge set A is defined as follows. Let B be the union of the balanced components of A and switch so $\phi^{\lambda}|_B \equiv 1$. Coalesce $\|\Phi\|$ by $\pi_b(A)$ and delete all edges in A. The gain of an ordinary edge e in the resulting graph is $\phi^{\lambda}(e)$. This defines the contracted gain graph Φ/A. Of course, the gain mapping of Φ/A is only determined up to switching by this construction, but that is quite satisfactory here. A *minor* of Φ is any gain graph resulting from switching, contracting, and taking subgraphs as often as desired.

THEOREM 5.4. *Let Φ be a gain graph and let $S, A \subseteq E(\Phi)$. Then $[\Phi]|S=[\Phi|S]$ and $[\Phi]/A=[\Phi/A]$.*

Proof. The former statement is obvious. As for the latter, from the construction clearly $\|\Phi/A\|=\|[\Phi]/A\|$. Suppose ϕ switched so $\phi|_B \equiv 1$, where B is the balanced part of A. Consider a circle $C=e_1 e_2 \cdots e_k$ in the contracted graph. There is a circle D in Φ of the form $e_1 P_1 e_2 P_2 \cdots e_k P_k$, where P_i is a path in a component of B. We have $\phi_{\Phi/A}(C)=\phi(D)$ since $\phi|_B \equiv 1$. We know that C is balanced in $[\Phi]/A$ precisely when D is, by definition of biased contraction. Therefore Φ/A and $[\Phi]/A$ have the same balanced circles. ∎

COROLLARY 5.5. *Any minor of a gain graph Φ is (up to switching) a subgraph of a contraction and also a contraction of a subgraph.*

Proof. Let Ψ be the minor. We know the corresponding minor of $[\Phi]$ is $[\Psi]$, which is a subgraph Δ of a contraction $[\Phi]/A$. So Ψ has the same underlying graph as the corresponding subgraph $(\Phi/A)|\Delta$ of Φ/A, by Theorem 5.4. It is easy to see that, by switching ϕ beforehand to be 1 on the balanced part of A, we have the same gains on Ψ and on $(\Phi/A)/\Delta$. ∎

We call a biased graph *gain biased*, or more precisely *$\mathfrak{G}$-biased*, if it equals $[\Phi]$ for some gain graph, or $\mathfrak{G}$-gain graph, Φ. A result that will be useful later is

LEMMA 5.6. *If Ω_1 and Ω_2 are gain-biased graphs, then so is their biased union $\Omega_1 \sqcup \Omega_2$.*

Proof. Suppose $\Omega_i=[\Phi_i]$. First, enlarge $\mathfrak{G}_1$ to $\mathfrak{G}_1'=\mathfrak{G}_1 \times \mathfrak{F}(N)$. Define $\lambda(v)=(1, v) \in \mathfrak{G}_1'$ for $v \in N$, treat ϕ_1 as mapping into $\mathfrak{G}_1 \times \{1\} \subseteq \mathfrak{G}_1'$, and switch ϕ_1 to ϕ_1^{λ}. Next, let $\mathfrak{G}=\mathfrak{G}_1' \times \mathfrak{G}_2$ and redefine ϕ_1^{λ} and ϕ_2 to map into $\mathfrak{G}$ in the obvious way. Then ϕ, defined by $\phi|_{E_1}=\phi_1^{\lambda}$ and $\phi|_{E_2}=\phi_2$, is a gain for $\Omega_1 \sqcup \Omega_2$. ∎

COROLLARY 5.7. *The class of gain-biased graphs, and the class of $\mathfrak{G}$-biased graphs for any group $\mathfrak{G}$, are closed under taking of minors.* ∎

Corollary 5.7 shows that the class of gain-biased graphs or of $\mathfrak{G}$-biased graphs can be characterized by finding the minor-minimal biased graphs not in the class. This theme will be developed in a future article. For now we merely show by example that there are indeed biased graphs which are not gain biased.

EXAMPLE 5.8. Let Ω_4 have node set $\{v_1, v_2, v_3, v_4\}$ and parallel edge pairs $e_{i-1,i}$, $f_{i-1,i}$ for $i=1, 2, 3, 4$, where the subscripts indicate the endpoints and are taken modulo 4. Let the balanced circles be $e_{12}e_{23}e_{34}e_{41}$, $e_{12}e_{23}f_{34}f_{41}$, and $f_{12}f_{23}e_{34}e_{41}$. Then Ω_4 is a biased graph which is not gain biased. However, every proper minor is gain biased.

Proof. Ω_4 is obviously a biased graph. Suppose it were gain-biased. We may use the edge names to denote gain values, with edges oriented in subscript order (e_{12} from v_1 to v_2, etc.), and we may switch so all $e_{ij}=1$. Then all $f_{ij} \neq 1$, $f_{12}f_{23}=1$, $f_{34}f_{41}=1$, and $f_{12}f_{23}f_{34}f_{41} \neq 1$. This is a contradiction.

The symmetry of Ω_4 implies that every proper subgraph is gain-biased if $\Omega_4 \backslash e_{12}$ and $\Omega_4 \backslash f_{12}$ are. Note that $\Psi=\Omega_4 \backslash \{e_{12}, e_{23}\}$ is gain-biased and that $\Omega_4 \backslash e_{12}=\Psi \sqcup e_{23}$. Also, $\Psi'=\Omega_4 \backslash \{f_{12}, f_{23}\}$ is sign-biased and $\Omega_4 \backslash f_{12}=\Psi' \sqcup f_{23}$. So, by Lemma 5.6, we are done with subgraphs.

A similar approach works for contractions. Here we note that Ω_4/e_{12} is, aside from the loop f_{12}, the biased union of $(\Omega_4/e_{12}) \backslash \{f_{12}, f_{23}\}$ and f_{23}. The former is sign-biased. Moreover, Ω_4/f_{12} is, neglecting the loop e_{12}, the biased union of $(\Omega_4/f_{12}) \backslash \{e_{12}, e_{23}\}$ and e_{23}. The former is gain-biased (although not with group $\mathbb{Z}_2$).

We conclude that every proper minor of Ω_4 is gain-biased. ∎

6. A CATALOG OF EXAMPLES

We list some biased graphs of particular interest. Most have been studied in the literature for the sake of their bias or lift matroids; hence we mention the matroids although we do not define them until Part II. We plan to treat many of these examples in detail in future articles.

EXAMPLE 6.1. *Balanced graphs.* These have the form $[\Gamma]$ for a graph Γ without half edges. For most purposes they behave exactly like ordinary graphs.

EXAMPLE 6.2. *Contrabalanced graphs.* These were introduced by Simões-Pereira in the form of the bias matroid $G(\Gamma, \varnothing)$, which he christened the *bicircular matroid* of Γ [10, 11].

EXAMPLE 6.3. *Parity bias.* Parity-biased graphs are $(\Gamma, \mathscr{B}_2)$ where $\mathscr{B}_2$ is the set of even-length circles in Γ. They are the biased graphs of all-negative signed graphs (next example). The bias matroid, sometimes called the *even-circle matroid* of Γ, arose in Doob's study [3] of the eigenspace of -2 of a line graph. The lift matroid appeared in recent work of Lovász and Schrijver [7] concerning graphs with no two vertex-disjoint odd cycles.

EXAMPLE 6.4. *Sign bias.* A *signed graph* is a gain graph whose gain group has order two. It was proved in [14] that a biased graph is sign biased if and only if its bias is additive (see Section 2).

EXAMPLE 6.5. *Poise bias.* In a directed graph D let $\mathscr{B}$ be the linear class consisting of all circles with the same number of edges directed each way. We call such a circle *poised* and the resulting bias the *poise bias* of D. The poise bias matroid was discovered by Matthews [8]. Observe that the poised circles are the balanced circles in the $\mathbb{Z}$-gain graph which assigns gain $+1$ to an edge when oriented as in D, so -1 in the opposite orientation.

If M is a positive integer we can define *poise modulo* M of a circle: it means that the numbers of edges directed either way around the circle differ by a multiple of M. Matthews also discussed the bias matroids of modular poise. Notice that modular poise derives from the gains above with group $\mathbb{Z}_M$. If $M=1$ we get $\mathscr{B}=\mathscr{C}(\Gamma)$; if $M=2$ we get the parity bias, regardless of the orientations in D. If $M>n$, poise modulo M is the same as nonmodular poise.

Poise generalizes to mixed graphs, which have directed and undirected edges. In determining whether a circle is poised we ignore undirected edges. Mixed poise is also a gain bias: a directed edge has gain as above, and the gain of an undirected edge is zero. Mixed-graph poise modulo 2 or 3 is equivalent to having gains with gain group $\mathbb{Z}_2$ or $\mathbb{Z}_3$, respectively.

EXAMPLE 6.6. *Antidirection bias.* Let $\mathscr{B}$ consist of all circles in a digraph D which are *antidirected*; that is, no two consecutive edges are directed the same way. Then $\mathscr{B}$ is a linear class. Matthews discovered the antidirection bias matroid. We observe that antidirection is a gain bias. Let $\mathfrak{G}$ be the free abelian group generated by the nodes and let the gain of an edge e, directed from v to w in D, be $\phi(e)=v+w$ when oriented from v to w. We may instead take $\mathfrak{G}$ to be the free $\mathbb{Z}_M$-module generated by N, where $M \geqslant 3$.

EXAMPLE 6.2. *Contrabalanced graphs.* These were introduced by Simões-Pereira in the form of the bias matroid $G(\Gamma, \varnothing)$, which he christened the *bicircular matroid* of Γ [10, 11].

EXAMPLE 6.3. *Parity bias.* Parity-biased graphs are $(\Gamma, \mathscr{B}_2)$ where $\mathscr{B}_2$ is the set of even-length circles in Γ. They are the biased graphs of all-negative signed graphs (next example). The bias matroid, sometimes called the *even-circle matroid* of Γ, arose in Doob's study [3] of the eigenspace of -2 of a line graph. The lift matroid appeared in recent work of Lovász and Schrijver [7] concerning graphs with no two vertex-disjoint odd cycles.

EXAMPLE 6.4. *Sign bias.* A *signed graph* is a gain graph whose gain group has order two. It was proved in [14] that a biased graph is sign biased if and only if its bias is additive (see Section 2).

EXAMPLE 6.5. *Poise bias.* In a directed graph D let $\mathscr{B}$ be the linear class consisting of all circles with the same number of edges directed each way. We call such a circle *poised* and the resulting bias the *poise bias* of D. The poise bias matroid was discovered by Matthews [8]. Observe that the poised circles are the balanced circles in the $\mathbb{Z}$-gain graph which assigns gain $+1$ to an edge when oriented as in D, so -1 in the opposite orientation.

If M is a positive integer we can define *poise modulo* M of a circle: it means that the numbers of edges directed either way around the circle differ by a multiple of M. Matthews also discussed the bias matroids of modular poise. Notice that modular poise derives from the gains above with group $\mathbb{Z}_M$. If $M=1$ we get $\mathscr{B}=\mathscr{C}(\Gamma)$; if $M=2$ we get the parity bias, regardless of the orientations in D. If $M>n$, poise modulo M is the same as nonmodular poise.

Poise generalizes to mixed graphs, which have directed and undirected edges. In determining whether a circle is poised we ignore undirected edges. Mixed poise is also a gain bias: a directed edge has gain as above, and the gain of an undirected edge is zero. Mixed-graph poise modulo 2 or 3 is equivalent to having gains with gain group $\mathbb{Z}_2$ or $\mathbb{Z}_3$, respectively.

EXAMPLE 6.6. *Antidirection bias.* Let $\mathscr{B}$ consist of all circles in a digraph D which are *antidirected*; that is, no two consecutive edges are directed the same way. Then $\mathscr{B}$ is a linear class. Matthews discovered the antidirection bias matroid. We observe that antidirection is a gain bias. Let $\mathfrak{G}$ be the free abelian group generated by the nodes and let the gain of an edge e, directed from v to w in D, be $\phi(e)=v+w$ when oriented from v to w. We may instead take $\mathfrak{G}$ to be the free $\mathbb{Z}_M$-module generated by N, where $M\geqslant 3$.

A *bidirected graph* has two direction arrows on each edge, one at each end. (This generalization of digraphs originated with Edmonds.) The antidirected circles of a bidirected graph form a linear class. This is a gain bias since we can assign gains $\phi(e)=\varepsilon(w)\,w-\varepsilon(v)\,v$ in the same group as before, where $\varepsilon(v)=+1$ if the arrow at the v end of e points toward v, -1 otherwise.

EXAMPLE 6.7. *Group expansions.* Let $\Gamma=(N, E)$ be an ordinary graph and $\mathfrak{G}$ a group. By $\mathfrak{G}\Gamma$ we mean the gain graph derived from Γ by replacing each edge of Γ by $\#\mathfrak{G}$ new edges, one bearing each possible gain value. We call $\mathfrak{G}\Gamma$ the *$\mathfrak{G}$-expansion* of Γ and the corresponding full graph $\mathfrak{G}\Gamma^{\bullet}$ the *full $\mathfrak{G}$-expansion.* The matroid and invariant theory of these is particularly elegant. Dowling initiated it with his article on the bias matroid of $\mathfrak{G}K_n^{\bullet}$, whose lattice of closed sets is known as the rank *n Dowling lattice of* $\mathfrak{G}$. In [13] we studied the signed expansions of arbitrary graphs.

EXAMPLE 6.8. *k-gon-Generated bias.* Suppose we take the class $\mathscr{B}$ of circles in Γ generated under set sum by a fixed class $\mathscr{D}$, say all k-gons. Then $(\Gamma, \mathscr{B})$ is a gain-biased graph. Let $\mathscr{P}(E)$ be the binary vector space generated by the edges and V the vector subspace spanned by $\mathscr{D}$. Thus $\mathscr{B}=V\cap\mathscr{C}(\Gamma)$. Let $\mathfrak{G}$ be the additive group $\mathscr{P}(E)/V$. The natural mapping $\phi\colon E\to\mathfrak{G}$ is a gain mapping for the bias.

For instance, the class of triangle-generated circles is a linear class. So is the class of circles that are generated by Hamiltonian circles.

EXAMPLE 6.9. *Bias from matroids.* Suppose Γ is a graph and M is a matroid on E. Let $\mathscr{B}=\{C\in\mathscr{C}\colon C \text{ is dependent in } M\}$. If every forest is independent in M, and any connected subgraph with cyclomatic number one whose sole circle is independent is also independent, then $\mathscr{B}$ is a linear class.

Proof. If not, then Γ contains a theta subgraph that is the union of dependent circles C and D but whose third circle is independent. Let $e\in C\cap D$. Then $(C\cup D)\backslash\{e\}$ is dependent by circuit exchange but independent by the hypothesis on M. ∎

7. SEVEN DWARVES: THE BIASED K_4'S

To illustrate our theory we examine K_4. There are, up to isomorphism, seven different biased graphs based on K_4. All are gain biased and most are also derivable from poise and antidirection bias. We call these examples $\Omega_i(K_4)$, briefly Ω_i, for $i=1, 2, \ldots, 7$. Each $\Omega_i(K_4)$ is defined in Example 7.1 below.

To show there are seven biasings of K_4 we study the balance of triangles. If every triangle is balanced, so is the whole graph. (See Example 7.1.) If three triangles are balanced it is easy to deduce that the fourth is. If only two are balanced, then the quadrilateral contained in their union is balanced but that is the only balanced quadrilateral (Example 7.2). If just one triangle is balanced, no quadrilateral can be balanced (Example 7.3). If no triangle is balanced, any number of quadrilaterals can be balanced (Examples 7.4–7.7).

To facilitate the analysis of the possible gain groups of each example we switch so the edges at a particular node v_1 have the identity gain. We let $N(K_4)=\{v_1, v_2, v_3, v_4\}$. If ϕ is a gain mapping, we let $a=\phi(v_2v_3)$, $b=\phi(v_3v_4)$, and $c=\phi(v_2v_4)$, where v_iv_j denotes an edge oriented, for gain calculations, from v_i to v_j.

EXAMPLE 7.1. The balanced graph $\Omega_1=[K_4]$. It is gain biased with gains in any group, since a gain mapping is the constant function $\phi\equiv 1$. Up to switching this is the only gain mapping (Lemma 5.3).

EXAMPLE 7.2. The biased union $\Omega_2=[\Delta]\sqcup e$, where $e\in E(K_4)$ and $\Delta=K_4\backslash e$. As a gain group we can take any nontrivial group; we let $\phi(f)=1$ if $f\in E(\Delta)$ and $\phi(e)\neq 1$.

EXAMPLE 7.3. Let the balanced triangle be $v_1v_2v_3v_1$, so $a=1$. Imbalance of the other three triangles implies that 1, b, c must all be different group elements. This is enough to make every quadrilateral unbalanced. Therefore any group having order at least 3 can be a gain group, but $\mathbb{Z}_2$ cannot. This example is Ω_3.

EXAMPLE 7.4. If no triangles are balanced but all quadrilaterals are, we have the parity bias on K_4. Since that is the bias derived from the all-negative edge signing, this example may be called $[-K_4]$. Switching v_1 gives signs $\sigma'(v_1v_i)=+$ and $\sigma'(v_iv_j)=-$ if $i,j\neq 1$. Now let ϕ be any gain mapping for $[-K_4]$ (switched so $\phi(v_1v_i)=1$) and $\mathfrak{G}$ its gain group. The imbalance of triangles entails a, b, $c\neq 1$ and $c\neq ab$. From the balance of quadrilaterals we obtain $ab=1$ and $a=c=b$. Therefore $a^2=1$, so $\mathfrak{G}$ can only be a group containing $\mathbb{Z}_2$. The gain mapping is essentially unique in the sense that it must switch to a composition $\gamma\circ\sigma$ where $\sigma\colon E(K_4)\to\mathbb{Z}_2$ maps every edge to the nonidentity element and γ is a monomorphism $\mathbb{Z}_2\to\mathfrak{G}$.

EXAMPLE 7.5. If just two quadrilaterals and no triangles are balanced, say $v_1v_2v_3v_4v_1$ and $v_1v_2v_4v_3v_1$ are the balanced quadrilaterals, then $a^{-1}=b=c$. The imbalance of the triangles entails $c\neq 1$ and that of the third quadrilateral implies $a^{-1}c\neq 1$, that is, $c^2\neq 1$. Therefore we may take

To show there are seven biasings of K_4 we study the balance of triangles. If every triangle is balanced, so is the whole graph. (See Example 7.1.) If three triangles are balanced it is easy to deduce that the fourth is. If only two are balanced, then the quadrilateral contained in their union is balanced but that is the only balanced quadrilateral (Example 7.2). If just one triangle is balanced, no quadrilateral can be balanced (Example 7.3). If no triangle is balanced, any number of quadrilaterals can be balanced (Examples 7.4–7.7).

To facilitate the analysis of the possible gain groups of each example we switch so the edges at a particular node v_1 have the identity gain. We let $N(K_4) = \{v_1, v_2, v_3, v_4\}$. If ϕ is a gain mapping, we let $a = \phi(v_2 v_3)$, $b = \phi(v_3 v_4)$, and $c = \phi(v_2 v_4)$, where $v_i v_j$ denotes an edge oriented, for gain calculations, from v_i to v_j.

EXAMPLE 7.1. The balanced graph $\Omega_1 = [K_4]$. It is gain biased with gains in any group, since a gain mapping is the constant function $\phi \equiv 1$. Up to switching this is the only gain mapping (Lemma 5.3).

EXAMPLE 7.2. The biased union $\Omega_2 = [\Delta] \sqcup e$, where $e \in E(K_4)$ and $\Delta = K_4 \backslash e$. As a gain group we can take any nontrivial group; we let $\phi(f) = 1$ if $f \in E(\Delta)$ and $\phi(e) \neq 1$.

EXAMPLE 7.3. Let the balanced triangle be $v_1 v_2 v_3 v_1$, so $a = 1$. Imbalance of the other three triangles implies that $1, b, c$ must all be different group elements. This is enough to make every quadrilateral unbalanced. Therefore any group having order at least 3 can be a gain group, but $\mathbb{Z}_2$ cannot. This example is Ω_3.

EXAMPLE 7.4. If no triangles are balanced but all quadrilaterals are, we have the parity bias on K_4. Since that is the bias derived from the all-negative edge signing, this example may be called $[-K_4]$. Switching v_1 gives signs $\sigma'(v_1 v_i) = +$ and $\sigma'(v_i v_j) = -$ if $i, j \neq 1$. Now let ϕ be any gain mapping for $[-K_4]$ (switched so $\phi(v_1 v_i) = 1$) and $\mathfrak{G}$ its gain group. The imbalance of triangles entails $a, b, c \neq 1$ and $c \neq ab$. From the balance of quadrilaterals we obtain $ab = 1$ and $a = c = b$. Therefore $a^2 = 1$, so $\mathfrak{G}$ can only be a group containing $\mathbb{Z}_2$. The gain mapping is essentially unique in the sense that it must switch to a composition $\gamma \circ \sigma$ where $\sigma\colon E(K_4) \to \mathbb{Z}_2$ maps every edge to the nonidentity element and γ is a monomorphism $\mathbb{Z}_2 \to \mathfrak{G}$.

EXAMPLE 7.5. If just two quadrilaterals and no triangles are balanced, say $v_1 v_2 v_3 v_4 v_1$ and $v_1 v_2 v_4 v_3 v_1$ are the balanced quadrilaterals, then $a^{-1} = b = c$. The imbalance of the triangles entails $c \neq 1$ and that of the third quadrilateral implies $a^{-1} c \neq 1$, that is, $c^2 \neq 1$. Therefore we may take for gain group any group containing an element of order at least three, but not any group of involutions.

EXAMPLE 7.6. If the only balanced circle is a quadrilateral, say $v_1 v_2 v_3 v_4 v_1$, then we have $[\Delta] \sqcup \{v_1 v_3, v_2 v_4\}$ where $\Delta = K_4 \backslash \{v_1 v_3, v_2 v_4\}$. From the balanced quadrilateral we deduce $a = b$; then we need $a, c \neq 1$ to make the triangles unbalanced and $c \neq a, a^{-1}$ to make the other two quadrilaterals unbalanced. Consider a potential gain group. Clearly, it cannot be $\mathbb{Z}_2$ or $\mathbb{Z}_3$. If it has an element of order at least 4, take that to be a and let $c = a^2$. If it has an element of order 2 or 3, take that to be a and let c be any element not a power of a. Thus, any group of four or more elements can be a gain group for this example, but $\mathbb{Z}_2$ and $\mathbb{Z}_3$ cannot.

EXAMPLE 7.7. The contrabalanced graph $(K_4, \varnothing)$, in which every circle is unbalanced. We deduce that $a \neq 1$; $b \neq 1, a^{-1}$; and $c \neq 1, a, b, ab$. It follows that a gain group requires at least four elements. If a group $\mathfrak{G}$ has an element of order at least 4, let that be a, let $b = a$, and let $c = a^3$. If $\mathfrak{G}$

TABLE 7.1

The Biases of K_4 which Are Obtained from Gains, Poise, Modular Poise, and Antidirection

Type of bias		Example						
		Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_7
Gain group:	$\{1\}$	G	X	X	X	X	X	X
	$\mathbb{Z}_2$	G	G	X	G	X	X	X
	$\mathbb{Z}_3$	G	G	G	X	G	X	X
	$\mathbb{Z}_2 \times \mathbb{Z}_2$	G	G	G	G	X	G	X
	Any other group	G	G	G	G^2	G^3	G	G
Poise (mod M), $5 \leqslant M \leqslant \infty$		M	M	M	X	D	D^*	D
Poise (mod 4)		M	M	M	M	D	M	D
Poise (mod 3)		M	D	D	X	D	X	X
Poise (mod 2)		M	M	X	D	X	X	X
Antidirection		B	B	B	X	X	D	D

Note. Key to the first part: the bias is obtainable from: (G) gains in any group of the specified kind, (G^2) gains in a group having an involutory element, (G^3) gains in a group having a nonidentity element which is not an involution, (X) no gains in any group of the specified kind. Key to the second part: the bias is obtainable from: (D) a digraph and also from a strictly mixed graph (if poise) or strictly bidirected graph (if antidirection), (D^*) a digraph but not from any strictly mixed graph, (M) a mixed graph but not from a digraph, (B) a bidirected graph but not from a digraph; or (X) it is not obtainable from any mixed graph (if poise) or bidirected graph (if antidirection). (A mixed or bidirected graph is *strict* if it is not a digraph. Poise (mod ∞) means nonmodular poise.)

has an element of order 3, but is not $\mathbb{Z}_3$, let a be that element and $b=a$ and let c be any element not a power of a. But suppose every nonidentity element of $\mathfrak{G}$ has order 2. Then b cannot equal a, so 1, a, b, ab are four distinct elements. For c we need a fifth element. Therefore $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not a possible gain group for $(K_2, \varnothing)$, but every other group of order four or more is.

These results are summarized in Table 7.1.

The biases on K_4 that arise from poise, modular poise, and antidirection are also displayed in Table 7.1. The proofs are easy. For instance, to find out where to place D's in the table one examines the three essentially different orientations of K_4. (Converse orientations are equivalent because they produce the same bias.) To handle mixed-graph poise modulo 2 and 3 one treats the directed edges as having gain 1 in $\mathbb{Z}_2$ and $\mathbb{Z}_3$. We omit the details.

Acknowledgments

This work began on the cruise of the *Rachael and Ebenezer* out of Rockland, Maine, and blossomed in the M.I.T. combinatorics group, assisted by my colleagues Louise Balzarini, Joseph Kung, Richard P. Stanley, and Jay Sulzberger, whom I thank for their support. The present series is a much-modified and extended version of the original unpublished manuscript "Biased graphs" of 1977, to which I have referred occasionally in print. References to the manuscript will be satisfied by this series, which omits no essential parts of the original. For her speedy and capable assistance in preparing the manuscripts I thank Marge Pratt of SUNY.

References

1. G. Birkhoff, "Lattice Theory," Amer. Math. Soc. Colloq. Publ., Vol. 25, Amer. Math. Soc., Providence, RI, 1967. [MR 37 #2638]
2. H. H. Crapo and G.-C. Rota, "On the Foundations of Combinatorial Theory: Combinatorial Geometries," preliminary ed., M.I.T. Press, Cambridge, MA, 1970. [MR 45 #74]
3. M. Doob, An interrelation between line graphs, eigenvalues, and matroids, *J. Combin. Theory Ser. B* **15** (1973), 40–50. [MR 55 #12573]
4. T. A. Dowling, A class of geometric lattices based on finite groups, *J. Combin. Theory Ser. B* **14** (1973), 61–86 [MR 46 #7066]; Erratum, *J. Combin. Theory Ser. B* **15** (1973), 211. [MR 47 #8369]
5. F. Harary, On the notion of balance of a signed graph, *Michigan Math. J.* **2** (1953–1954), 143–146. Addendum, *Michigan Math. J.* **2** (1953–1954), preceding p. 1. [MR 16, 733]
6. F. Harary, On local balance and N-balance in signed graphs, *Michigan Math. J.* **3** (1955–1956), 37–41. [MR 17, 394]
7. L. Lovász and A. Schrijver, in preparation (communications from P. D. Seymour, July 1985, and L. Lovász, October 1985).

8. L. R. Matthews, Matr
81e:05055]
9. E. Minieka, "Optimizat
1978. [MR 80a:90066]
10. J. M. S. Simões-Pereira
[MR 47 #6522]
11. J. M. S. Simões-Pereira
as circuits, II, *Discrete*
12. W. T. Tutte, Lectures
(1965), 1–47. [MR 31 #
13. T. Zaslavsky, The geom
No. 2 (1981), 88–105. [
14. T. Zaslavsky, Charact
[MR 83a:05122]
15. T. Zaslavsky, Signed
Appl. Math. **5** (1983), 2
16. T. Zaslavsky, Signe
[MR 84h:05050a]
17. T. Zaslavsky, Chromat
[MR 84h:05050b]
18. T. Zaslavsky, Biased g
19. T. Zaslavsky, Biased
publication.
20. T. Zaslavsky, The bia
42 (1987), 337–347. [M

Printe

has an element of order 3, but is not $\mathbb{Z}_3$, let a be that element and $b=a$ and let c be any element not a power of a. But suppose every nonidentity element of $\mathfrak{G}$ has order 2. Then b cannot equal a, so 1, a, b, ab are four distinct elements. For c we need a fifth element. Therefore $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not a possible gain group for $(K_2, \varnothing)$, but every other group of order four or more is.

These results are summarized in Table 7.1.

The biases on K_4 that arise from poise, modular poise, and antidirection are also displayed in Table 7.1. The proofs are easy. For instance, to find out where to place D's in the table one examines the three essentially different orientations of K_4. (Converse orientations are equivalent because they produce the same bias.) To handle mixed-graph poise modulo 2 and 3 one treats the directed edges as having gain 1 in $\mathbb{Z}_2$ and $\mathbb{Z}_3$. We omit the details.

Acknowledgments

This work began on the cruise of the *Rachael and Ebenezer* out of Rockland, Maine, and blossomed in the M.I.T. combinatorics group, assisted by my colleagues Louise Balzarini, Joseph Kung, Richard P. Stanley, and Jay Sulzberger, whom I thank for their support. The present series is a much-modified and extended version of the original unpublished manuscript "Biased graphs" of 1977, to which I have referred occasionally in print. References to the manuscript will be satisfied by this series, which omits no essential parts of the original. For her speedy and capable assistance in preparing the manuscripts I thank Marge Pratt of SUNY.

References

1. G. Birkhoff, "Lattice Theory," Amer. Math. Soc. Colloq. Publ., Vol. 25, Amer. Math. Soc., Providence, RI, 1967. [MR 37 #2638]
2. H. H. Crapo and G.-C. Rota, "On the Foundations of Combinatorial Theory: Combinatorial Geometries," preliminary ed., M.I.T. Press, Cambridge, MA, 1970. [MR 45 #74]
3. M. Doob, An interrelation between line graphs, eigenvalues, and matroids, *J. Combin. Theory Ser. B* **15** (1973), 40–50. [MR 55 #12573]
4. T. A. Dowling, A class of geometric lattices based on finite groups, *J. Combin. Theory Ser. B* **14** (1973), 61–86 [MR 46 #7066]; Erratum, *J. Combin. Theory Ser. B* **15** (1973), 211. [MR 47 #8369]
5. F. Harary, On the notion of balance of a signed graph, *Michigan Math. J.* **2** (1953–1954), 143–146. Addendum, *Michigan Math. J.* **2** (1953–1954), preceding p. 1. [MR 16, 733]
6. F. Harary, On local balance and N-balance in signed graphs, *Michigan Math. J.* **3** (1955–1956), 37–41. [MR 17, 394]
7. L. Lovász and A. Schrijver, in preparation (communications from P. D. Seymour, July 1985, and L. Lovász, October 1985).

8. L. R. Matthews, Matroids from directed graphs, *Discrete Math.* **24** (1978), 47–61. [MR 81e:05055]
9. E. Minieka, "Optimization Algorithms for Networks and Graphs," Dekker, New York, 1978. [MR 80a:90066]
10. J. M. S. Simões-Pereira, On subgraphs as matroid cells, *Math. Z.* **127** (1972), 315–322. [MR 47 #6522]
11. J. M. S. Simões-Pereira, On matroids on edge sets of graphs with connected subgraphs as circuits, II, *Discrete Math.* **12** (1975), 55–78. [MR 54 #7298]
12. W. T. Tutte, Lectures on matroids, *J. Res. Nat. Bur. Standards (U.S.A.) Sect. B* **69B** (1965), 1–47. [MR 31 #4023]
13. T. Zaslavsky, The geometry of root systems and signed graphs, *Amer. Math. Monthly* **88**, No. 2 (1981), 88–105. [MR 82g:05012]
14. T. Zaslavsky, Characterizations of signed graphs, *J. Graph Theory* **5** (1981), 401–406. [MR 83a:05122]
15. T. Zaslavsky, Signed graphs, *Discrete Appl. Math.* **4** (1982), 47–74. Erratum, *Discrete Appl. Math.* **5** (1983), 248. [MR 84e:05095]
16. T. Zaslavsky, Signed graph coloring, *Discrete Math.* **39** (1982), 215–228. [MR 84h:05050a]
17. T. Zaslavsky, Chromatic invariants of signed graphs, *Discrete Math.* **42** (1982), 287–312. [MR 84h:05050b]
18. T. Zaslavsky, Biased graphs, II. The three matroids, *J. Combin. Theory Ser. B*, in press.
19. T. Zaslavsky, Biased graphs, III. Chromatic and dichromatic invariants, submitted for publication.
20. T. Zaslavsky, The biased graphs whose matroids are binary, *J. Combin. Theory Ser. B* **42** (1987), 337–347. [MR 88h:05082]

Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium