PROBLEMS AND SOLUTIONS

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Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted problems should include solutions and relevant references. Submitted solutions should arrive at that address before February 28, 1998; Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An acknowledgement will be sent only if a mailing label is provided. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

10606. *Proposed by Thomas Zaslavsky, Binghamton University, Binghamton, NY.* Given a positive integer m, show that there is a positive integer n such that, for any group G of order at least n, it is possible to choose m elements $g_1, g_2, \ldots, g_m$ of G so that no product of the form $g_{i_1}^{\pm 1} g_{i_2}^{\pm 1} \ldots g_{i_k}^{\pm 1}$ with $1 \le k \le m$ and distinct subscripts $i_1, i_2, \ldots, i_k$ in $\{1, 2, \ldots, m\}$ equals the identity.

10607. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.* Evaluate

$$\lim_{n\to\infty} \left(\frac{2^x + 4^x + \cdots + (2n)^x}{1^x + 3^x + \cdots + (2n-1)^x} \right)^n$$

for $x > 0$.

10608. *Proposed by Victor Zalgaller, Steklov Mathematical Institute, St. Petersburg, Russia.* Let S be a compact convex set in the plane. If l is any line of support for S, let $f(l)$ be the length of the shortest curve that begins and ends on l and that together with l surrounds S. Prove that if $f(l)$ is independent of l, then S is a circle.

10609. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* Let

$$a(l, m, n) = \sum_{k=0}^{l} \binom{n}{k} (l + m - k)^{n-k} (k - l)^k.$$

Prove that

$$\sum_{l=1}^{n} a(l, m, n) = \frac{m + n + 1}{2} a(n, m, n) - \frac{m + 1}{2} m^n.$$

10610. *Proposed by Richard Hall, University of Portsmouth, Portsmouth, England.* Given a positive integer m, let $C(m)$ be the greatest positive integer k such that, for some set S of m integers, every integer from 1 to k belongs to S or is a sum of two not necessarily distinct elements of S. For example, $C(3) = 8$ with $S = \{1, 3, 4\}$.
(a) Show that, for all $\epsilon > 0$, $1/4 < C(m)/m^2 < 1/2 + \epsilon$ for all sufficiently large m.
(b)* Improve the asymptotic bounds in part **(a)**.

10611. *Proposed by Zoltán Sasvári, Technical University of Dresden, Dresden, Germany.* Find the largest value of a and the smallest value of b for which the inequalities

$$\frac{1+\sqrt{1-e^{-ax^2}}}{2} < \Phi(x) < \frac{1+\sqrt{1-e^{-bx^2}}}{2}$$

hold for all $x > 0$, where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy.$$

10612. *Proposed by John P. Robertson, Anistics/Aon, New York, NY.* Fermat proved that there are no nontrivial 4-term arithmetic progressions all of whose terms are integer squares.
(**a**) Find all 5-term arithmetic progressions such that all terms but the fourth are squares.
(**b**) Call two arithmetic progressions *essentially different* if the ratios of corresponding terms differ. For each integer $m \geq 6$ show that there are infinitely many essentially different m-term arithmetic progressions such that the first 3 terms and the mth term are squares.

SOLUTIONS

A Fairly General Family of Integrals

10393 [1994, 573]. *Proposed by Jean Anglesio, Garches, France.* Show that

$$\int_0^\infty \frac{e^{-ax}\left(1-e^{-x}\right)^n}{x^r}\, dx = \frac{(-1)^r}{(r-1)!} \sum_{k=0}^{n} \binom{n}{k} (-1)^k (a+k)^{r-1} \log(a+k)$$

where $a \geq 0$ and $1 \leq r \leq n$ (except for $a = 0, r = 1$).

Solution by Jet Wimp, Drexel University, Philadelphia, PA. The Gamma function $\Gamma(\sigma)$ is analytic for $\sigma \neq 0, -1, -2, \ldots$; it satisfies $\Gamma(\sigma+1) = \sigma\Gamma(\sigma)$, so $\Gamma(r) = (r-1)!$ when r is a positive integer. Also, we have

$$\begin{aligned}
\lim_{\sigma\to 1-r} \frac{d}{d\sigma}\frac{1}{\Gamma(\sigma)} &= \lim_{\sigma\to 1-r} \frac{d}{d\sigma}\left((\sigma+r-1)\frac{\sigma(\sigma+1)\ldots(\sigma+r-2)}{\Gamma(\sigma+r)}\right) \\
&= \lim_{\sigma\to 1-r}\left((\sigma+r-1)\frac{d}{d\sigma}\left(\frac{\sigma(\sigma+1)\ldots(\sigma+r-2)}{\Gamma(\sigma+r)}\right) + \frac{\sigma(\sigma+1)\ldots(\sigma+r-2)}{\Gamma(\sigma+r)}\right) \\
&= (-1)^{r-1}(r-1)! \qquad (A)
\end{aligned}$$

for $r = 1, 2, \ldots$.

Expanding $\left(1-e^{-x}\right)^n$ by the binomial theorem, integrating term by term, and using $\int_0^\infty e^{-px}x^{\sigma-1}\, dx = \Gamma(\sigma)p^{-\sigma}$ for $p, \sigma > 0$ yields

$$\int_0^\infty e^{-ax}\left(1-e^{-x}\right)^n x^{\sigma-1}\, dx = \frac{\sum_{k=0}^{n}\binom{n}{k}(-1)^k(a+k)^{-\sigma}}{1/\Gamma(\sigma)} \qquad (B)$$

for $a > 0$. The range of validity of the integral in (B) may be extended from $\sigma > 0$ to $\sigma > -n$. In fact, this integral is analytic for complex σ with real part greater than $-n$.

For $\sigma = 1-r$, $r = 1, 2, \ldots, n$, the right side of (B) is indeterminate: the sum in the numerator is zero (it is an nth difference of $a+k$ to a nonnegative integral power less than n) and the denominator must be zero in the limit since the integral is clearly not. We now take the limit of (B) as $\sigma \to 1-r$, $r = 1, 2, \ldots, n$ using L'Hospital's rule. Differentiating the numerator using $\frac{d}{d\sigma}(a+k)^{-\sigma} = -(a+k)^{-\sigma}\ln(a+k)$ and the denominator using (A) gives the result.