



Problems: 10606-10612

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PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, and Douglas B. West**

with the collaboration of Paul T. Bateman, Duane M. Broline, Ezra A. Brown, Richard T. Bumby, Underwood Dudley, Michael A. Filaseta, Ira M. Gessel, Bart Goddard, Jerrold R. Griggs, Douglas A. Hensley, John R. Isbell, Robert Israel, Murray S. Klamkin, Daniel J. Kleitman, Fred Kochman, Frederick W. Luttman, Frank B. Miles, Richard Pfeifer, Leonard Smiley, John Henry Steelman, Kenneth Stolarsky, Richard Stong, Charles Vanden Eynden, and William E. Watkins.

Proposed problems and solutions should be sent in duplicate to the MONTHLY problems address on the inside front cover. Submitted problems should include solutions and relevant references. Submitted solutions should arrive at that address before February 28, 1998; Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An acknowledgement will be sent only if a mailing label is provided. An asterisk () after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

10606. *Proposed by Thomas Zaslavsky, Binghamton University, Binghamton, NY.* Given a positive integer m , show that there is a positive integer n such that, for any group G of order at least n , it is possible to choose m elements g_1, g_2, \dots, g_m of G so that no product of the form $g_{i_1}^{\pm 1} g_{i_2}^{\pm 1} \dots g_{i_k}^{\pm 1}$ with $1 \leq k \leq m$ and distinct subscripts i_1, i_2, \dots, i_k in $\{1, 2, \dots, m\}$ equals the identity.

10607. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.* Evaluate

$$\lim_{n \rightarrow \infty} \left(\frac{2^x + 4^x + \dots + (2n)^x}{1^x + 3^x + \dots + (2n-1)^x} \right)^n$$

for $x > 0$.

10608. *Proposed by Victor Zalgaller, Steklov Mathematical Institute, St. Petersburg, Russia.* Let S be a compact convex set in the plane. If l is any line of support for S , let $f(l)$ be the length of the shortest curve that begins and ends on l and that together with l surrounds S . Prove that if $f(l)$ is independent of l , then S is a circle.

10609. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* Let

$$a(l, m, n) = \sum_{k=0}^l \binom{n}{k} (l+m-k)^{n-k} (k-l)^k.$$

Prove that

$$\sum_{l=1}^n a(l, m, n) = \frac{m+n+1}{2} a(n, m, n) - \frac{m+1}{2} m^n.$$

10610. *Proposed by Richard Hall, University of Portsmouth, Portsmouth, England.* Given a positive integer m , let $C(m)$ be the greatest positive integer k such that, for some set S of m integers, every integer from 1 to k belongs to S or is a sum of two not necessarily distinct elements of S . For example, $C(3) = 8$ with $S = \{1, 3, 4\}$.

(a) Show that, for all $\epsilon > 0$, $1/4 < C(m)/m^2 < 1/2 + \epsilon$ for all sufficiently large m .

(b)* Improve the asymptotic bounds in part (a).

10611. Proposed by Zoltán Sasvári, Technical University of Dresden, Dresden, Germany. Find the largest value of a and the smallest value of b for which the inequalities

$$\frac{1 + \sqrt{1 - e^{-ax^2}}}{2} < \Phi(x) < \frac{1 + \sqrt{1 - e^{-bx^2}}}{2}$$

hold for all $x > 0$, where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy.$$

10612. Proposed by John P. Robertson, Anistics/Aon, New York, NY. Fermat proved that there are no nontrivial 4-term arithmetic progressions all of whose terms are integer squares.

- (a) Find all 5-term arithmetic progressions such that all terms but the fourth are squares.
 (b) Call two arithmetic progressions *essentially different* if the ratios of corresponding terms differ. For each integer $m \geq 6$ show that there are infinitely many essentially different m -term arithmetic progressions such that the first 3 terms and the m th term are squares.

SOLUTIONS

A Fairly General Family of Integrals

10393 [1994, 573]. Proposed by Jean Anglesio, Garches, France. Show that

$$\int_0^\infty \frac{e^{-ax}(1 - e^{-x})^n}{x^r} dx = \frac{(-1)^r}{(r-1)!} \sum_{k=0}^n \binom{n}{k} (-1)^k (a+k)^{r-1} \log(a+k)$$

where $a \geq 0$ and $1 \leq r \leq n$ (except for $a = 0, r = 1$).

Solution by Jet Wimp, Drexel University, Philadelphia, PA. The Gamma function $\Gamma(\sigma)$ is analytic for $\sigma \neq 0, -1, -2, \dots$; it satisfies $\Gamma(\sigma + 1) = \sigma \Gamma(\sigma)$, so $\Gamma(r) = (r-1)!$ when r is a positive integer. Also, we have

$$\begin{aligned} \lim_{\sigma \rightarrow 1-r} \frac{d}{d\sigma} \frac{1}{\Gamma(\sigma)} &= \lim_{\sigma \rightarrow 1-r} \frac{d}{d\sigma} \left((\sigma+r-1) \frac{\sigma(\sigma+1)\dots(\sigma+r-2)}{\Gamma(\sigma+r)} \right) \\ &= \lim_{\sigma \rightarrow 1-r} \left((\sigma+r-1) \frac{d}{d\sigma} \left(\frac{\sigma(\sigma+1)\dots(\sigma+r-2)}{\Gamma(\sigma+r)} \right) + \frac{\sigma(\sigma+1)\dots(\sigma+r-2)}{\Gamma(\sigma+r)} \right) \\ &= (-1)^{r-1} (r-1)! \end{aligned} \tag{A}$$

for $r = 1, 2, \dots$.

Expanding $(1 - e^{-x})^n$ by the binomial theorem, integrating term by term, and using $\int_0^\infty e^{-px} x^{\sigma-1} dx = \Gamma(\sigma) p^{-\sigma}$ for $p, \sigma > 0$ yields

$$\int_0^\infty e^{-ax} (1 - e^{-x})^n x^{\sigma-1} dx = \frac{\sum_{k=0}^n \binom{n}{k} (-1)^k (a+k)^{-\sigma}}{1/\Gamma(\sigma)} \tag{B}$$

for $a > 0$. The range of validity of the integral in (B) may be extended from $\sigma > 0$ to $\sigma > -n$. In fact, this integral is analytic for complex σ with real part greater than $-n$.

For $\sigma = 1 - r, r = 1, 2, \dots, n$, the right side of (B) is indeterminate: the sum in the numerator is zero (it is an n th difference of $a+k$ to a nonnegative integral power less than n) and the denominator must be zero in the limit since the integral is clearly not. We now take the limit of (B) as $\sigma \rightarrow 1 - r, r = 1, 2, \dots, n$ using L'Hospital's rule. Differentiating the numerator using $\frac{d}{d\sigma} (a+k)^{-\sigma} = -(a+k)^{-\sigma} \ln(a+k)$ and the denominator using (A) gives the result.