ASSOCIATIVITY IN MULTIARY QUASIGROUPS: THE WAY OF BIASED EXPANSIONS

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Abstract. A *multiary* (*polyadic, n-ary*) *quasigroup* is an n-ary operation which is invertible with respect to each of its variables. A *biased expansion* of a graph is a kind of branched covering graph with additional structure similar to combinatorial homotopy of circles. A biased expansion of a circle with chords encodes a multiary quasigroup, the chords corresponding to factorizations, i.e., associative structure.

Some but not all biased expansions are constructed from groups (*group expansions*); these include all biased expansions of complete graphs (with at least four nodes), which correspond to Dowling's lattices of a group and encode an iterated group operation. We show that any biased expansion of a 3-connected graph (with at least four nodes) is a group expansion, and that all 2-connected biased expansions are constructed by identification of edges from group expansions and irreducible multiary quasigroups. If a 2-connected biased expansion covers every base edge at most three times, or if every four-node minor that contains a fixed edge is a group expansion, then the whole biased expansion is a group expansion.

We deduce that, if a multiary quasigroup has a factorization graph that is 3-connected, or if every ternary principal retract is an iterated group isotope, it is isotopic to an iterated group.

We mention applications to generalizing Dowling geometries and to transversal designs of high strength.

Contents

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1. Biased graphs and the associative law

A *multiary* or *polyadic quasigroup*[1] is a set $\mathfrak{Q}$ with a multiary operation $(\cdots)$ or $f : \mathfrak{Q}^n \to \mathfrak{Q}$ such that the equation

$$f(x_1, \dots, x_n) = x_0 \tag{1.1}$$

is uniquely solvable for any one variable x_i given the values of the n remaining variables. Multiary quasigroups were implicit in early work of H.A. Thurston [31, 32]; ternary quasigroups (called "N-algebras") were studied explicitly by Rado [29]; and finally the general study of multiary quasigroups as such was initiated by Belousov and Sandik [7]. A *Dowling geometry of a group*, $Q_n(\mathfrak{G})$, is a certain matroid of rank $n \geq 1$ associated with a group $\mathfrak{G}$; it was invented by Dowling [13] and shown by Kahn and Kung [24] to have a central role in matroid theory. These two structures are both equivalent to particular kinds of the same general object, something I call a *biased expansion of a graph*. Associativity in multiary quasigroups, and quasigroup generalizations of Dowling geometries, both depend on and can be analyzed through the structure of biased expansions.

1.1. Associativity.

The customary view of the associative law is that it describes a relationship between two different ways of carrying out a binary operation on three arguments:

$$(xy)z = x(yz).$$

We look at it differently: we regard associativity as a property of factorizability or reducibility of a multiary product. For instance, letting $(\cdots)$ denote a ternary or binary product, we think of ordinary associativity as the combination of

$$(xyz) = ((xy)z) \tag{1.2}$$

and

$$(xyz) = (x(yz)). \tag{1.3}$$

[1]The most common term is "n-ary", where n is left unspecified, which is unsatisfactory. Since "polyadic" has not become popular, I propose "multiary" as a generic adjective.

The two factorizations (1.2) and (1.3) constitute associativity in the usual sense (if all the binary multiplications are the same), but to us this is a secondary phenomenon. We are more interested in association as factorization and specifically in the consequences of the approach through biased expansions, which is suited only to operations that come from multiary quasigroups.

Our generalized associativity is (consecutive) factorization of f:

$$f(x_1, \ldots, x_n) = g(x_1, \ldots, x_i, h(x_{i+1}, \ldots, x_j), x_{j+1}, \ldots, x_n), \tag{1.4}$$

where $1 \leq i{+}1 < j \leq n$ (and $(i, j) \neq (0, n)$) and g and h are multiary quasigroups of suitable arity. Such a factorization is an $(i{+}1, j)$-*factorization* or $(i{+}1, j)$-*reduction* of f. As an associativity property we call it *reductive associativity.* (Reductive associativity of multiary quasigroups was studied first by Thurston [31, 32] and next, independently, by Belousov and colleagues in many papers beginning with [7].) At one extreme of factorizability is an *irreducible* multiary quasigroup, whose operation has no factorizations at all. At the other extreme are *iterated groups*: multiary quasigroups whose operation has the form $f(x_1, \ldots, x_n) = x_1x_2 \cdots x_n$ computed in a group. Such a quasigroup has every factorization. Every multiary quasigroup (of arity $n > 2$) that factors in all possible ways is known to be essentially an iterated group.

By "essentially" we mean up to *isotopy.* Multiary operations $f : \mathfrak{Q}^n \to \mathfrak{Q}$ and $g : \mathfrak{Q}_1^n \to \mathfrak{Q}_1$ are called *isotopic* if there exist bijections $\alpha_0, \alpha_1, \ldots, \alpha_n : \mathfrak{Q} \to \mathfrak{Q}_1$ such that

$$g(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}) = f(x_1, \ldots, x_n)^{\alpha_0}.$$

From our abstract graph-theoretic standpoint we cannot distinguish isotopic operations. Nor can we distinguish between operations that are related by circular permutation of the $n + 1$ variables, i.e., replacing the operation $a_0 = f(a_1, a_2, \ldots, a_n)$ by any operation g defined by $a_i = g(a_{i+1}, a_{i+2}, \ldots, a_n, a_0, \ldots, a_{i-1})$ or $a_i = g(a_{i-1}, a_{i-2}, \ldots, a_0, a_n, \ldots, a_{i+1})$ for some i, where the subscripts are taken modulo $n + 1$. We call the combination of isotopy and circular permutation *circular paratopy* (or *cycloparatopy*): that is, multiary operations are *circularly paratopic* if one can be obtained from the other by a combination of isotopy and circular permutation of all the variables. Our method does not distinguish multiary quasigroups that are circularly paratopic.

Belousov treated Equation (1.4) as representing a binary operation on functions, written $f = g +^{i+1} h$ (this operation is called *Mann superposition*). The resulting algebra led to many theorems. A simple example is the equation

$$g +^i h = g' +^j h', \tag{1.5}$$

where $i < j$, which Belousov called (i, j)-*associativity.* (Thurston's work on multiary quasigroups concerned a version of this equation.) Belousov solved it in [5] (see Corollary 9.5) by axiomatizing the algebra of multiary quasigroups with composition operations $+^i$.

However, he left outstanding an important question. The *factorization graph* $\Delta(\mathfrak{Q})$ of a multiary quasigroup $\mathfrak{Q}$ with operation f is the circle graph C_{n+1} on node set $\{v_0, v_1, \ldots, v_n\}$, whose edges are $v_{i-1}v_i$ for $i = 1, \ldots, n, n+1$ (we take $v_{n+1} = v_0$), together with an added chord v_iv_j whenever f has a factorization as in Equation (1.4). The question left open by Belousov is whether, whenever the factorization graph of $\mathfrak{Q}$ is 3-connected, then $\mathfrak{Q}$ is isotopic to an iterated group.[2] We prove this (Theorem 9.2), as well as reproducing the solution of (1.5), as corollaries of our structure theorem for biased expansions.

We also prove a second criterion for a multiary quasigroup to be an iterated group isotope. A multiary quasigroup obtained from $\mathfrak{Q}$ by fixing the values of some set of variables in Equation (1.1) is called a *retract* of $\mathfrak{Q}$. If the fixed variables are all independent variables (i.e., they are chosen from $x_1, \ldots, x_n$), the retract is *principal*. (If x_0 is fixed, we may take any unfixed variable to be the dependent variable; that is a consequence of the fact that circular permutation of the variables produces a quasigroup operation.) We show in Section 7, as interpreted in Theorem 9.8, that $\mathfrak{Q}$ is isotopic to an iterated group if its arity is at least three and every principal retract that is a ternary quasigroup is an iterated group isotope.

The traditional multiary generalization of associativity is a stronger form of (i,j)-associativity, due to Dörnte [12] and extensively studied (see, e.g., [28, 20, 23, 14]). An n-ary operation $f : \mathfrak{Q}^n \to \mathfrak{Q}$ is called *associative* if it satisfies all the n identities

$$\hat{f}(x_1, \ldots, x_{2n-1}) = \hat{f}_i(x_1, \ldots, x_{2n-1}) \quad \text{for } i = 1, \ldots, n, \tag{1.6}$$

where $\hat{f}_i$ is the operation

$$\hat{f}_i(x_1, \ldots, x_{2n-1}) := f(x_1, \ldots, x_{i-1}, f(x_i, \ldots, x_{i+n-1}), x_{i+n}, \ldots, x_{2n-1}). \tag{1.7}$$

(That is, (1.6) consists of $n-1$ identities and one definition of $\hat{f}$.) We might call this *substitutive associativity* by way of contrast with reductive associativity. A multiary quasigroup with substitutive associativity for all i is called an *n-ary group* (or *n-group*, or *multiary* or *polyadic group*), a 2-group being an ordinary group. Evidently, $\hat{f}$ is an example of a multiary quasigroup operation that is reducible in a multiplicity of ways. By our general theorem just mentioned, f is isotopic to an iterated group. That is part of the n-group structure theorem of Post [28, pp. 245–246] (which is generally known as the Hosszú–Gluskin, or Gluskin–Hosszú, Theorem because of the converses proved by Hosszú [23] and Gluskin [20]). Post's complete theorem is an explicit formula for every multiary group in terms of a group. It should be possible to refine our method so as to obtain the formula, but we do not do so here.[3]

[2] Dudek [15] has heard that Belousov conjectured this to be true, but I have not been able to confirm that statement nor to find any published reference to such a conjecture.

[3] Post taught at the City College of New York. Several years later, as an undergraduate at City College in 1965, I received from the mathematics department the "Emil L. Post Memorial

There is an extensive literature, of which n-groups are only one example, on criteria for a multiary quasigroup to be isotopic to an iterated group. I do not attempt to survey this literature since my theorems and proofs do not make use of it. (The reader may consult the 1982 bibliography of n-ary groups by Głazek [19], which includes early papers on n-ary quasigroups, and the introduction and references of [16].) It would be interesting to see which existing criteria are, and which are not, deducible from our graph theoretic approach. I hope this question will be not be left unexamined.

1.2. **Introduction to expansions.** Our fundamental observation is that biased expansions of circles are equivalent to circular paratopy classes of multiary quasigroups. First we need definitions.

A *biased graph* $\Omega = (\|\Omega\|, \mathcal{B})$ consists of a graph $\|\Omega\|$, which may be finite or infinite, and a *linear class* $\mathcal{B}$ of circles (circuits, cycles) of $\|\Omega\|$, meaning that in each theta subgraph the number of circles that belong to $\mathcal{B}$ is different from two. (A *theta graph* consists of three paths with the same two endpoints but no other nodes or edges in common.) The circles in $\mathcal{B}$ are called the *balanced circles* of Ω.

A *biased expansion* of a graph Δ is a biased graph Ω together with a *projection* mapping $p : \|\Omega\| \to \Delta$ that is surjective, is the identity on nodes, maps no balanced digon to a single edge, and has the *circle lifting property*: for each circle $C = e_1 e_2 \cdots e_l$ in Δ and each $\tilde{e}_1 \in p^{-1}(e_1)$, $\ldots$, $\tilde{e}_{l-1} \in p^{-1}(e_{l-1})$, there is a unique $\tilde{e}_l \in p^{-1}(e_l)$ for which $\tilde{e}_1 \tilde{e}_2 \cdots \tilde{e}_l$ is balanced. In addition, no edge fiber $p^{-1}(e)$ may contain a balanced digon; but this is implied by the other properties if e is not an isthmus. One can think of Ω as a kind of branched covering of Δ. We write $\Omega \downarrow_p \Delta$ to mean that Ω is a biased expansion of Δ with projection p; though usually we omit p from the notation. We call Ω a *regular* or γ*-fold* biased expansion if each $p^{-1}(e)$ has the same cardinality γ; then γ is the *multiplicity* of the expansion. $\gamma \cdot \Delta$ denotes a γ-fold biased expansion of Δ. Clearly, a biased expansion of an inseparable graph must be regular (see Figure 1). A biased expansion is *trivial* if it is regular with multiplicity 1. In defining a biased expansion of a circle it is not necessary to require Ω to be a biased graph; that follows from the rest of the definition because a theta graph exists only by containing a digon.

A simple kind of biased expansion is a *group expansion* [37, Example I.6.7]. The expansion of a graph Δ by a group $\mathfrak{G}$, in brief the $\mathfrak{G}$*-expansion* of Δ, is the gain graph (definition in Sect. 1.2), denoted by $\mathfrak{G}\Delta$, whose node set is $N(\Delta)$ and whose edge set is $\mathfrak{G} \times E(\Delta)$, the endpoints of an edge ge (this is shorthand for (g, e)) being the same as those of e. The projection $p : \mathfrak{G}\Delta \to \Delta$ maps ge to e. We associate with ge the group element g, called the *gain* of ge; in order to define gains in a technically correct manner we orient Δ arbitrarily and orient ge similarly to e, so the gain of ge in the chosen direction is g and in the opposite direction g^{-1}. A circle in $\mathfrak{G}\Delta$ is balanced if the product of the gains of its edges, taken in a

Award". Little did I think I would ever cite Post's work; but now I am pleased to have that opportunity.

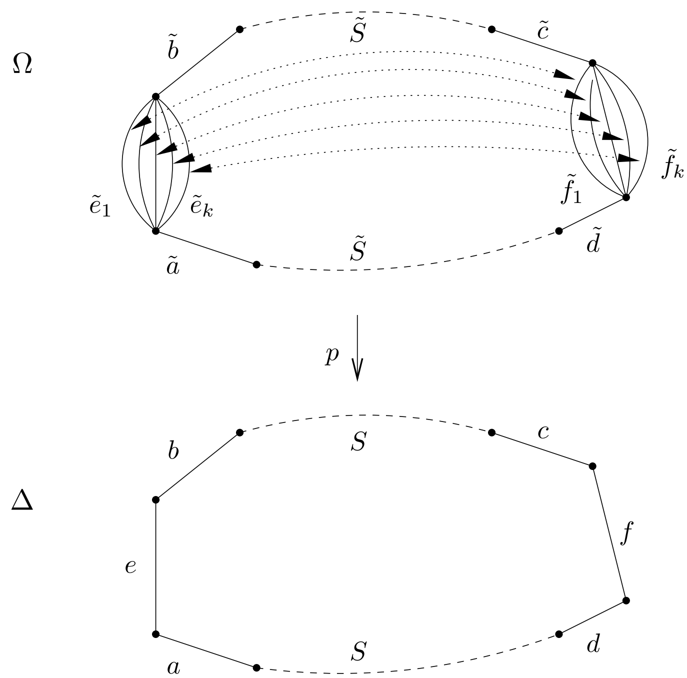


Figure 1. Two fibers $p^{-1}(e)$ and $p^{-1}(f)$ such that e and f are in a common circle C have the same cardinality. Lift $C \setminus \{e, f\}$ to $\tilde{S}$ and use the circle lifting property to establish inverse mappings $p^{-1}(e) \to p^{-1}(f)$ and $p^{-1}(f) \to p^{-1}(e)$. The mappings are indicated by the dotted lines. A different choice of $\tilde{S}$ may lead to different mappings.

consistent direction around the circle, equals 1, the group identity. This defines a biased graph, which we write $\langle \mathfrak{G}\Delta \rangle$. A biased graph which is not a group expansion is called a *non-group biased expansion.*

If Δ is simple with n nodes, then $\mathfrak{G}\Delta$ is contained in $\mathfrak{G}K_n$. Thus, group expansions of complete graphs are basic.

Figure 2 shows the expansion of C_3 by the group $\mathbb{Z}_3$ (written additively, so the identity is 0) and the expansion of C_4 by the group $\mathbb{Z}_4$ (also written additively).

A very different kind of biased expansion is the expansion of a circle C_{n+1} of length $n+1$ by an n-ary quasigroup $\mathfrak{Q}$. In the *quasigroup expansion* $\mathfrak{Q}C_{n+1}$, the nodes are $v_0, v_1, \ldots, v_n$. There is an edge $ae_{i-1,i}$ for every $a \in \mathfrak{Q}$ and $i = 1, 2, \ldots, n$ as well as an edge ae_{0n}. The balanced circles are the circles $\{a_1e_{12}, a_2e_{12}, \ldots, a_ne_{n-1,n}, a_0e_{0n}\}$ such that $(a_1a_2 \cdots a_n) = a_0$ in $\mathfrak{Q}$. A quasigroup expansion need not be contained in a biased expansion of a complete graph; see Theorem 9.1.

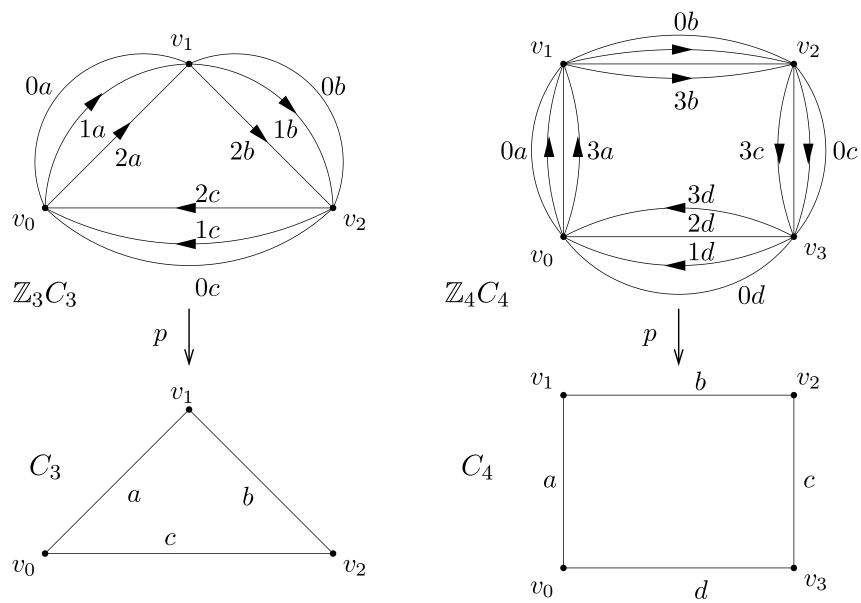
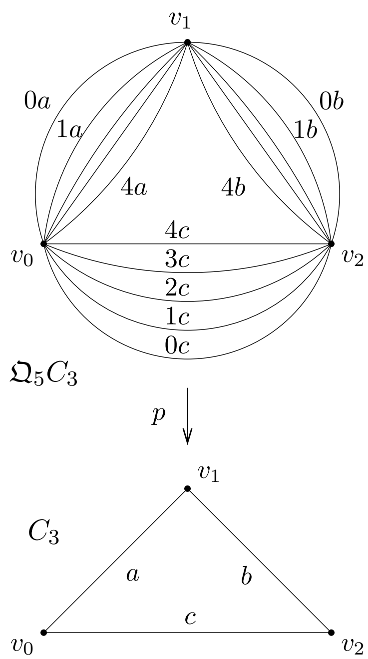


FIGURE 2. *Left*: $\mathbb{Z}_3C_3$, the expansion of C_3 by the group $\mathbb{Z}_3$. The balanced circles are the triangles $\{ia, jb, kc\}$ such that $i+j+k=0$. The arrows show the orientation that has the indicated gain; the 0-edges do not need orientation since 0 is self-inverse. *Right*: The group expansion $\mathbb{Z}_4C_4$. The balanced circles are the quadrilaterals $\{ia, jb, kc, ld\}$ such that $i+j+k+l=0$, where a denotes an edge v_0v_1 in the expansion, b denotes v_1v_2, etc. The edges with gain 0 and 2 do not need to be oriented since 0 and 2 are self-inverse.

Figure 3 shows the expansion of C_3 by a binary quasigroup $\mathfrak{Q}_5$ which is not isotopic to a group. (It has a subquasigroup of order 2, which cannot be a subgroup of a group of order 5.)

In the quasigroup expansion $\mathfrak{Q}C_{n+1}$ the edges are labelled by the members of $\mathfrak{Q}$, so that $\mathfrak{Q}$ is determined by $\mathfrak{Q}C_{n+1}$. If we forget that labelling, we have the biased expansion $\langle \mathfrak{Q}C_{n+1} \rangle$; we still call it a quasigroup expansion, but it is no longer possible to recover the quasigroup from $\langle \mathfrak{Q}C_{n+1} \rangle$. What we do recover is an equivalence class of n-ary quasigroups under circular paratopy.

Start with a biased expansion $\gamma \cdot C_{n+1}$. Let C_{n+1} have nodes $v_0, v_1, \ldots, v_n$ and edges $e_{01}, e_{12}, \ldots, e_{n-1,n}, e_{0n}$. Set $E_{ij} = E(\gamma \cdot C_{n+1}){:}\{v_i, v_j\}$, the set of edges between v_i and v_j, and fix bijections $\beta_i : \mathfrak{Q} \to E_{i-1,i}$ for $i = 1, \ldots, n$ and $\beta_0 : \mathfrak{Q} \to E_{0n}$, where $\mathfrak{Q}$ is a set that will be the set of elements of the n-ary quasigroup. The multiary operation is $(x_1 \cdots x_n) = \beta_0^{-1}(\tilde{e})$ where $\tilde{e}$ is the unique edge in E_{0n} that forms a balanced circle with $\beta_1(x_1), \ldots, \beta_n(x_n)$. (We do not



$\mathfrak{Q}_5$

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	4	2
2	2	3	4	0	1
3	3	4	1	2	0
4	4	2	0	1	3

Figure 3. $\mathfrak{Q}_5 C_3$, the expansion of C_3 by the quasigroup $\mathfrak{Q}_5$, and the multiplication table of $\mathfrak{Q}_5$. The balanced circles are the triangles $\{ia, jb, kc\}$ such that $i \cdot j = k$.

get a single quasigroup; that is why the $\mathfrak{Q}$-labelling of the edges of a quasigroup expansion $\mathfrak{Q}C_{n+1}$, which—together with the node indices $0, 1, \dots, n$—determines $\mathfrak{Q}$ uniquely, makes it different from the corresponding biased expansion $\langle \mathfrak{Q}C_{n+1} \rangle$. The arbitrary choice of the bijections is what makes $\mathfrak{Q}$ defined only up to isotopy. The arbitrariness of the distinguished edge e_{0n} and the direction of reading the circle is what leaves $\mathfrak{Q}$ well defined only up to circular permutation of the variables.) Thus we have the first two parts of Proposition 1.1.

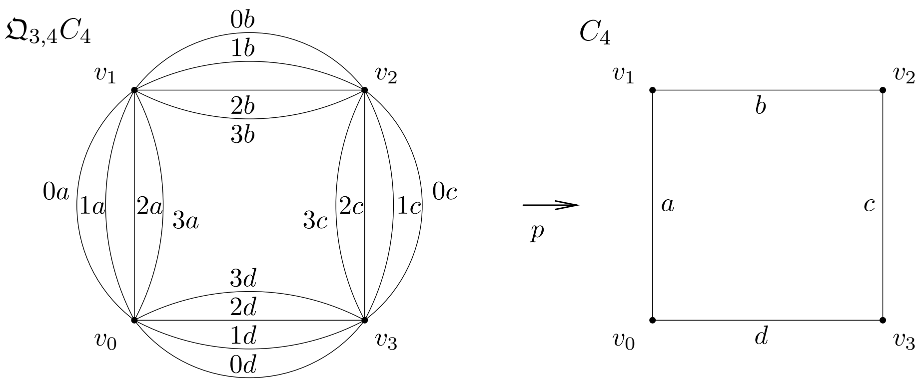
The third part is proved at Theorem 9.1. We say $\langle \mathfrak{Q}C_{n+1} \rangle$ *extends* to e_{ij} if there is a biased expansion $\Omega \downarrow (\Delta \cup \{e_{ij}\})$ such that $p^{-1}(C_{n+1}) = \langle \mathfrak{Q}C_{n+1} \rangle$. (Section 3 has a fuller treatment.)

Proposition 1.1. *An n-ary quasigroup expansion $\langle \mathfrak{Q}C_{n+1} \rangle$ is a biased expansion of C_{n+1}. Conversely, every biased expansion of C_{n+1} has the form $\langle \mathfrak{Q}C_{n+1} \rangle$ for an n-ary quasigroup $\mathfrak{Q}$.*

Furthermore, two n-ary quasigroup expansions $\langle \mathfrak{Q}_1 C_{n+1} \rangle$ and $\langle \mathfrak{Q}_2 C_{n+1} \rangle$ are isomorphic if and only if $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are circularly paratopic.

Moreover, $\langle \mathfrak{Q}C_{n+1} \rangle$ extends to a chord e_{ij} of C_{n+1} if and only if the operation f of $\mathfrak{Q}$ factors as in (1.4).

Figure 4 shows the multiplication table of the irreducible ternary quasigroup $\mathfrak{Q}_{3,4}$ (from [7, §5]) and the corresponding quasigroup expansion $\mathfrak{Q}_{3,4}C_4$.



$\mathfrak{Q}_{3,4}$

$(0jk)$

$k =$	0	1	2	3
$j = 0$	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$(1jk)$

$k =$	0	1	2	3
$j = 0$	1	0	3	2
1	0	1	2	3
2	3	2	1	0
3	2	3	0	1

$(2jk)$

$k =$	0	1	2	3
$j = 0$	2	3	0	1
1	3	0	1	2
2	0	1	2	3
3	1	2	3	0

$(3jk)$

$k =$	0	1	2	3
$j = 0$	3	2	1	0
1	2	3	0	1
2	1	0	3	2
3	0	1	2	3

FIGURE 4. The biased expansion $\mathfrak{Q}_{3,4}C_4$, of $C_4 = abcd$ by $\mathfrak{Q}_{3,4}$, and the multiplication table $(ijk) = l$ of $\mathfrak{Q}_{3,4}$. The balanced circles are those of the form $\{ia, jb, kc, ld\}$ such that $(ijk) = l$.

Taking $n = 2$, we see that biased expansions of a triangle are (as Dowling knew in terms of his geometries; see [13, pp. 78–79]) graph-theoretic realizations of circular paratopy classes of binary quasigroups and, therefore, of Latin squares and

3-nets. The quadrangle criterion of Latin squares [10, Theorem 1.2.1(2)] tells us when a binary quasigroup is isotopic to a group; its translation into the language of expansions is the following, applicable to any biased expansion of a triangle:

> *Quadrangle Criterion.* For any twelve distinct edges $e_{12}^{\alpha,\beta}$, $e_{23}^{\alpha,\beta}$, $e_{13}^{\alpha,\beta}$, where $e_{ij}^{\alpha,\beta} \in p^{-1}(e_{ij})$ for $\alpha, \beta = 1, 2$ and $ij = 12, 23, 13$, if seven of the triangles of the form $e_{12}^{\alpha,\delta} e_{23}^{\beta,\delta} e_{13}^{\alpha,\beta}$ are balanced, then so is the eighth.

Proposition 1.2. *A biased expansion of K_3 has the form $\langle \mathfrak{G} K_3 \rangle$ for some group $\mathfrak{G}$ if and only if it satisfies the quadrangle criterion.* □

A γ-fold biased expansion of C_{n+1}, where $n \geq 3$, giving us an n-ary quasigroup according to Proposition 1.1, can be interpreted as the n-dimensional generalization of a Latin square that is called a *permutation hypercube* [10, p. 181] or *Latin hypercube*, defined up to paratopy (isotopy and arbitrary permutation of the variables). As far as I know, no analog of the quadrangle criterion has been formulated for such objects and therefore for biased expansions of larger graphs such as C_{n+1} for $n \geq 3$.

In a sense indicated by Proposition 1.1, biased expansion graphs are a graphical generalization of groups and multiary quasigroups. That they truly are a generalization is shown by the fact that a biased expansion need not have a Hamiltonian circle. If the base graph is Hamiltonian, then we have a multiary operation—which in general depends on the choice of Hamiltonian circle—from Proposition 1.1, but if it is not, then we have something that, from an algebraic standpoint, is more complicated; it might be thought of as a combinatorial complex of multiary quasigroups.

If biased expansion graphs generalize groups, it is natural to ask how far a given biased expansion is from being a group. Biased expansions of complete graphs of order at least 4 are group expansions, essentially because K_4, as the base of a biased expansion, encodes the associative law (see Section 4). Thus, a more precise version of the question is: How far is the base graph from being complete? But this is still not quite right, because it might be possible to extend the expansion to new edges between nonadjacent nodes. If the expansion extends to a complete graph, then it is contained in a group expansion so it itself is a group expansion and any corresponding multiary quasigroup is an iterated group isotope. In general, there is always a maximal extension of the given biased expansion graph, that has the most pairs of adjacent nodes (see Section 3); it is of this extended expansion that we should ask the refined question, and indeed it makes sense to think of the number of nonadjacent node pairs in the base graph as a measure of how much a biased expansion fails to represent a group.

It is perhaps noteworthy that nongroup expansions of large incomplete graphs, and in particular irreducible multiary quasigroups, exist at all. However, all examples are 2-separable, for, as we prove in Sections 3 and 4, every biased expansion of a 3-connected simple graph having more than three nodes is a group expansion. From this and other work we can deduce the complete structure of a biased expansion (Section 6) and answer the question raised in [37, Example III.3.8] of exactly which graphs have a nongroup biased expansion (Corollary 6.7).

1.3. **Dowling geometries.** Biased expansion graphs were inspired by Dowling's matroids of a group—though no matroids were used in the preparation of this article. One way to construct the rank-n Dowling matroid (or "geometry") $Q_n(\mathfrak{G})$ of a group $\mathfrak{G}$ is to take the group expansion $\mathfrak{G}K_n$, adjoin a half edge to each node, and take the frame matroid (or "bias matroid") [37, Section II.2]. We sketch this construction (from [37, Example III.5.7 or Part V]) to suggest how biased expansions of simple graphs, especially those that are maximal on the given node set, are a natural generalization of Dowling geometries. The unexplained terms can be found in [37, Part III or V].

Given any biased expansion $\Omega \downarrow \Delta$, one can add an unbalanced loop at each node and take the frame matroid; call this matroid $G^\bullet(\Omega)$. The operator $G^\bullet$, applied to maximal biased expansions, generalizes the construction of Dowling geometries. The Dowling geometries are the only examples derived from groups because the only maximal group expansions are those of the complete graphs K_n. Given that nongroup biased expansions exist, one asks what other matroids can be obtained from maximal biased expansions; they are natural candidates for generalized Dowling geometries. That question motivated this work. We will see (via Theorem 9.1) that an expansion $\mathfrak{Q}C_{n+1}$ of C_{n+1} by an irreducible n-ary quasigroup $\mathfrak{Q}$ is maximal, since it can have no chords; thus these are part of the answer. The question is completely answered if we can classify all maximal biased expansions. That is our Theorem 6.2—the solution of a problem that had puzzled me since 1976.

1.4. **Transversal designs.** A final way to look upon a multiary quasigroup, or a biased expansion of a circle, is as a kind of transversal design. A *transversal t-design* consists of a set of points partitioned into l *point classes* L_i (usually called "groups", but they have nothing to do with algebra) of k points each, and a class of *blocks*, which are subsets of points satisfying

(TD1) no two points in a class are contained in a common block, and

(TD2) any t points, no two in a class together, are contained in exactly λ common blocks,

where λ, the *index*, is a fixed positive integer. As t is called the *strength*, we refer to an $(l-1)$-design as having *high strength.* A k-fold biased expansion Ω of C_l is equivalent to a transversal design T of high strength with $\lambda = 1$. The points of T

are the edges of Ω, the class L_i consists of all edges with endpoints v_{i-1} and v_i, and the blocks are the balanced circles.

A group expansion thus generates a design based on the group. The construction of the design is easy to describe directly. The classes are copies of the group and a block is any set $\{x_1, x_2, \ldots, x_l\}$, composed of one element of each class L_j, such that $x_1 x_2 \cdots x_l = 1$. Here we need to assume the classes are ordered; we let the first class be L_1, the second L_2, etc. The analog of factorization is consecutive composition: supposing T' and T'' are two transversal designs of high strength with index 1, we form their *i-composition* T by identifying L'_i with L''_1 and then defining the classes of T to be those of T' and T'', with the exception of L'_i $(= L''_1)$, and the blocks to be the sets of the form $B' \oplus B''$ where B' and B'' are overlapping blocks of T' and T''. The inverse operation to i-composition is *i-decomposition.* The analog of the factorization graph is defined by the existence of i-decompositions of T. We have the theorems, for instance, that if this graph is 3-connected and the number of classes is at least four, then the design is derived from a group, and that if every four-class transversal design induced by T and including the class L_1 is derived from a group, then T is so derived. (Precise statements can be obtained by translating results of Section 9.)

1.5. **Overview.** Here is a summary of our main results. Most of them were announced in [38], which can serve as a readable précis of this work.

Biased expansion graphs.

- A 3-connected expansion is a group expansion (Theorem 4.1).
- Edge amalgamation and edge sum for 2-separable expansions (Theorem 5.3).
- Decomposition into group and irreducible multiary quasigroup expansions (Theorems 6.2, 6.3).
- Characterization of base graphs having nongroup biased expansions (Corollary 6.7).
- Uniqueness and structure of maximal biased expansions (Theorems 3.2, 6.2, Corollary 6.6).
- An expansion of multiplicity at most 3 is a group expansion (Theorem 8.1).
- A 2-connected biased expansion with at least four nodes is a group expansion if every minor of order four is gainable (Theorem 7.2).

Multiary quasigroups.

- The factorization graph of $\mathfrak{Q}$ corresponds to the maximal extension of $\mathfrak{Q}C_{n+1}$ (Theorem 9.1).
- A nonbinary multiary quasigroup whose factorization graph is 3-connected is an iterated group isotope (Theorem 9.2).
- Characterization of possible factorization graphs (Theorem 9.3).
- A multiary quasigroup of which every ternary principal retract is an iterated group isotope is itself isotopic to an iterated group (Theorem 9.8).

2. Preliminary remarks

Here we collect a few old and new definitions and some elementary observations about expansions.

2.1. **Basic concepts.** Formally, a graph Γ is a pair (N, E) consisting of a node set N and an edge set E. The *order* of Γ is $|N|$. An edge whose two endpoints are distinct is a *link*; one whose endpoints coincide is a *loop*. A graph without loops or parallel edges is called *simple*.

An *induced subgraph* is a subgraph $\Gamma{:}X$, where $X \subseteq N$, whose node set is X and whose edge set consists of all edges of Γ with both endpoints in X. If $S \subseteq E$, $S{:}X$ is the set of edges in S that have both endpoints in X.

The *degree* of a node is the number of edges of which it is an endpoint. A *circle*, also known as a polygon, circuit, or cycle, is a connected graph with degree 2 at every node, or the edge set of such a graph. A *path* is a connected graph with degree 2 except at two nodes, which have degree 1; or it is a single node. The *length* of a circle or path is the number of its edges. The sum (symmetric difference) of sets is written $S \oplus T$. This applies in particular to circles, regarded as edge sets. Two paths are *internally disjoint* if they have no nodes or edges in common except possibly at their endpoints. A *theta graph* consists of three pairwise internally disjoint paths that have the same two endpoints; these three paths are *constituent paths* of the theta graph. A graph is 2-*connected* or *inseparable* if it is connected and any two edges lie in a common circle. A *block* of a graph is a maximal inseparable subgraph. A graph is 3-*connected* if it is inseparable and has at least 4 nodes, and there is no set of 2 or fewer nodes whose deletion leaves a disconnected graph.

Suppose Γ_1 and Γ_2 are two graphs that have in common a link e. An (*edge*) *amalgamation* of Γ_1 and Γ_2 along e, written $\Gamma_1 \cup_e \Gamma_2$, is a graph obtained by identifying the two copies of e. (It is not unique, since the copies can be identified in two ways.) An *edge sum* (or *2-sum*), written $\Gamma_1 \oplus_e \Gamma_2$, is $(\Gamma_1 \cup_e \Gamma_2) \setminus e$.

Suppose Γ is a graph and Ξ a subgraph. A *bridge of* Ξ in Γ is a maximal subgraph B of Γ with the properties that $E(B) \cap E(\Xi) = \varnothing$, $B \not\subseteq \Xi$, and any node common to B and an edge not in B lies in Ξ (see Tutte [35, Section I.8]).

Suppose Δ is a graph and γ is a nonzero cardinal number, possibly infinite: then, by $\gamma\Delta$ we mean Δ with every edge replaced by γ copies of itself. Thus the underlying graph of a regular biased expansion $\gamma \cdot \Delta$ is $\gamma\Delta$; note the importance of the dot in the notation.

Biased graphs were defined in the introduction. Some additional notions: A subgraph or edge set in a biased graph Ω is called *balanced* if every circle in it is balanced. A balanced biased or gain graph should be thought of as like an ordinary graph and the bias, i.e., the choice of balanced circles, as a kind of skewing; so the less balanced, the more biased.

A *gain graph* $\Phi = (\|\Phi\|, \varphi)$ with *gain group* $\mathfrak{G}$ is a graph $\|\Phi\|$ together with an orientable *gain function* $\varphi : E(\Phi) \to \mathfrak{G}$; that is, φ is defined on oriented edges and, letting e^{-1} denote e with the opposite orientation, $\varphi(e^{-1}) = \varphi(e)^{-1}$. A group expansion, obviously, is a gain graph. A circle in Φ is called *balanced* if the product of the gains of its edges is 1, the group identity; thus Φ produces a biased graph $\langle\Phi\rangle$. *Switching* a gain graph Φ by a *switching function* $\eta : N \to \mathfrak{G}$ means replacing φ by a new gain map, φ^η, defined by $\varphi^\eta(e) = \eta(v)^{-1}\varphi(e)\eta(w)$ if e is oriented from endpoint v to endpoint w. Switching gains does not change bias: $\langle\Phi\rangle = \langle\Phi^\eta\rangle$. Not all biased graphs are obtainable from gains. An expansion of C_{k+1} by a multiary quasigroup that is not isotopic to an iterated group is one example; we shall see others in Section 5. Biased graphs and gain graphs are from [37, Part I].

A *minor* of a graph, biased graph, or gain graph is a subgraph or contraction of a subgraph. Since contraction of biased and gain graphs is complicated and plays a minor role in this article, we omit the definitions, referring the reader to [37, Sections I.2 and I.5].

We shall have use for a property of chains of paths. If $A, B \subseteq \Delta$, an AB*-path* is a path with one endpoint in A, the other in B, and otherwise disjoint from $A \cup B$.

Lemma 2.1 (Path Lemma)**.** *In a 2-connected graph Δ let A and B be disjoint paths. If P_0 and P are two AB-paths, then there exist AB-paths $P_1, \ldots, P_k = P$ such that each $P_{i-1} \cup P_i \cup A \cup B$ contains exactly one circle.*

Proof. This lemma can be deduced from Tutte's Path Theorem [33, Theorem 4.34], but we give a direct proof suggested by Marcin Mazur [26]. The proof assumes $v_0v_1, v_{l-1}v_l \notin P$ but it can easily be modified to cover the other possibilities. The result is trivial if P_0 and P are internally disjoint. Otherwise, let the nodes of P_0, in order from A to B, be $v_0, v_1, \ldots, v_l$; define $P_0(v_j)$ to be that portion of P_0 from v_{j+1} to v_{l-1}; and make a similar definition for P. Let x_0 be the first node of P when traced from A to B. Let x_1 be the first node of $P(x_0)$ that lies in $P_0(v_0)$, x_2 the first node of $P(x_1)$ that lies in $P_0(x_1)$, and in general x_i the first node of $P(x_{i-1})$ that lies in $P_0(x_{i-1})$. Define $k-1$ as the last value of i for which an x_i exists. For $0 < i < k$, P_i is obtained by tracing P from x_0 to x_i and then P_0 from x_i to v_l; and P_k is P. Then $P_{i-1} \cup P_i$ contains the circle formed by the segments of P_0 and P from x_{i-1} to x_i, and no other circle; except that the unique circle in $P_0 \cup P_1$ consists of P_0 and P_1 up to x_1 along with A from v_0 to x_0, and the unique circle in $P_{k-1} \cup P_k$ consists of P_0 and P from x_{k-1} to their endpoints in B together with B between those endpoints. □

A *homomorphism* (synonym: *mapping*) of graphs is an incidence-preserving mapping of node and edge sets. A *homomorphism of biased graphs* is a homomorphism of the underlying graphs that preserves balance, but not necessarily imbalance, of edge sets.

In a biased graph there is a kind of closure called the *balance-closure* (not "balanced closure"), defined for any edge set by

$$\operatorname{bcl} S = S \cup \{e \notin S : \text{ there is a balanced circle } C \ni e \text{ such that } C \setminus e \subseteq S\}.$$

This is not an abstract closure operator, nor is it true that $\operatorname{bcl} S$ must be balanced; the essential property of balance-closure is

Lemma 2.2 ([37, Proposition I.3.1])**.** *For $S \subseteq E(\Omega)$, $\operatorname{bcl} S$ is balanced if and only if S is balanced.*

2.2. **Basics of expansions.** Elementary facts about expansions let us confine our attention to simple, inseparable base graphs. First, it is clear that a biased expansion of a graph Δ is the union of arbitrary biased expansions of the blocks of Δ. Second, biased expansion of a loop is uninteresting. Third, suppose e and f are parallel links in Δ. In a biased expansion Ω of Δ, there is a unique bijection between $p^{-1}(e)$ and $p^{-1}(f)$ such that, if $\tilde{e}$ and $\tilde{f}$ correspond, then $\tilde{e}\tilde{f}$ is balanced and, for any set $\tilde{P} \subseteq E(\Omega)$ that contains neither $\tilde{e}$ nor $\tilde{f}$, $\tilde{P} \cup \{\tilde{e}\}$ is a balanced circle if and only if $\tilde{P} \cup \{\tilde{f}\}$ is. Thus, Ω is completely determined by $\Omega \setminus p^{-1}(f)$. Moreover, any gains φ for Ω are completely determined by the gains on $\Omega \setminus p^{-1}(f)$ by the equation $\varphi(\tilde{f}) = \varphi(\tilde{e})$. (See Example 3.1.)

A *homomorphism* (or *mapping*) $\Omega \to \Omega'$ of biased expansions is a biased-graph homomorphism along with a homomorphism of base graphs such that the two mappings commute with projection. We shall have occasion to use only homomorphisms that are injective.

The *restriction of* Ω *to* $\Delta' \subseteq \Delta$, written $\Omega\big|_{\Delta'}$, is the subgraph $p^{-1}(\Delta')$ with the bias and projection mapping inherited from Ω. (This should not be confused with restricting Ω to an arbitrary subgraph of itself; $\Omega\big|_{\Delta'}$ is one such restriction, but not all restrictions are of that form.)

A basic property of expansions is the existence of balanced copies of Δ or any subgraph, extending any balanced subgraph of the expansion. A *lift* of an edge set $S \subseteq E(\Delta)$ is a subset $\tilde{S} \subseteq E(\Omega)$ for which $p\big|_{\tilde{S}}$ is a bijection onto S. We shall always mean by $\tilde{S}$ a lift of S.

Lemma 2.3. *Let Ω be a biased expansion of a graph Δ. Given any subsets $A \subseteq B \subseteq E(\Delta)$ and a balanced lift $\tilde{A}$, there is a balanced lift $\tilde{B}$ that contains $\tilde{A}$.*

Proof. Extend A to a maximal subgraph S of B that has no additional circles besides those in A. Take any lift $\tilde{S} \supseteq \tilde{A}$; it is balanced because $\tilde{A}$ is balanced. Then $\operatorname{bcl} \tilde{S}$ projects to $\operatorname{clos} S = \operatorname{clos} B$, where clos is the ordinary graphic matroid closure

$$\operatorname{clos} S = S \cup \{e \notin S : \text{ there is a circle } C \ni e \text{ such that } C \setminus e \subseteq S\}.$$

Thus, $\operatorname{bcl} \tilde{S}$ is balanced by Lemma 2.2, and it contains a lift of B. Take $\tilde{B} = p^{-1}(B) \cap \operatorname{bcl} \tilde{S}$. ∎

One can apply the lemma, for example, when A is a forest, since any lift of a forest is balanced. It is also the basis for an alternative definition of biased expansions; see [37, Part V].

The nongroup biased expansions are the same as the nongainable biased expansions, because if a biased expansion graph Ω has gains in a group $\mathfrak{H}$, then it is a group expansion by a subgroup $\mathfrak{G}$ of $\mathfrak{H}$ [37, Theorem V.2.1(a)]. Moreover, $\mathfrak{G}$ is unique up to isomorphism [37, Theorem V.2.1]. Furthermore, if $\Omega \downarrow \Delta$ is a group expansion by $\mathfrak{G}$, one can choose the gain mapping $\varphi : E(\Omega) \to \mathfrak{G}$ so that $\varphi^{-1}(1)$ is any desired balanced lift of $E(\Delta)$ (a consequence of [37, Lemma I.5.3]), and then φ is determined up to automorphisms of $\mathfrak{G}$ [37, Theorem V.2.1]. One can interpret the choosability of $\varphi^{-1}(1)$ to mean that selecting a balanced lift of Δ is the expansion-graph analog of isotoping a quasigroup to a loop, where one can choose arbitrarily the element that, after isotopy, becomes the loop identity.

2.3. **Expansion minors.** Certain minors of a biased expansion are themselves expansions. An instance is a restriction of a biased expansion to a subset of the fibers, that is, $\Omega\big|_{\Delta'}$ where Δ' is any subgraph of Δ; analogously, the restriction of $\mathfrak{G}\Delta$ is an expansion $\mathfrak{G}\Delta'$. Similarly, a contraction of a group or biased expansion by a balanced edge set is again a group or biased expansion save for possibly having extra balanced or unbalanced loops; for instance, if $\tilde{S}$ is a balanced edge set in a biased expansion Ω of Δ, then $\Omega/\tilde{S}$ without loops is a biased expansion of $\Delta/p(\tilde{S})$ without loops. We want a notion that combines both of these kinds of minors, that of an 'expansion minor'.

Let Ω be a biased expansion of a graph Δ. An *expansion minor* of Ω is any minor Ω' of Ω (without loose or half edges) whose edge set is a union of fibers $p^{-1}(e)$ of Ω; that is, $E(\Omega') = p^{-1}(S)$ for some $S \subseteq E$. An expansion minor of a group expansion is similar. As an example, any restriction $\Omega|_S$ for $S \subseteq E(\Delta)$, such as an induced subgraph of Ω, is an expansion minor.

Proposition 2.4. *Let Δ be a graph.*

(a) *An expansion minor Ω' of a biased expansion Ω of Δ is a biased expansion of a minor Δ' of Δ. If Ω is regular of multiplicity γ, then so is Ω'.*
(b) *An expansion minor of a group expansion $\mathfrak{G}\Delta$ is a group expansion $\mathfrak{G}\Delta'$ of a minor Δ' of Δ, and conversely.*
(c) *An expansion minor of an expansion minor of Ω or $\mathfrak{G}\Delta$ is an expansion minor of Ω or $\mathfrak{G}\Delta$, respectively.*

Part (b) is especially significant. It says that we can tell something about the gainability of a biased expansion from its *triangular expansion minors*, that is, expansion minors that are expansions of K_3. We apply this idea in Section 5.

The proof is in the more precise description of expansion minors contained in two lemmas. First we define a construction method for expansion minors.

> *Construction XM.* Given a biased expansion Ω of Δ, take $S \subseteq E(\Delta)$, a weak partition $E(\Delta) = S \cup T \cup D$ (that is, S, T, and D

are pairwise disjoint sets whose union is $E(\Delta)$; some of them may be void), and a balanced lift $\tilde{T}$ of T into Ω. From $(\Omega \setminus p^{-1}(D))/\tilde{T}$ delete $p^{-1}(T)$, and delete an arbitrary subset of the isolated nodes (if there are any). Call this Ω'.

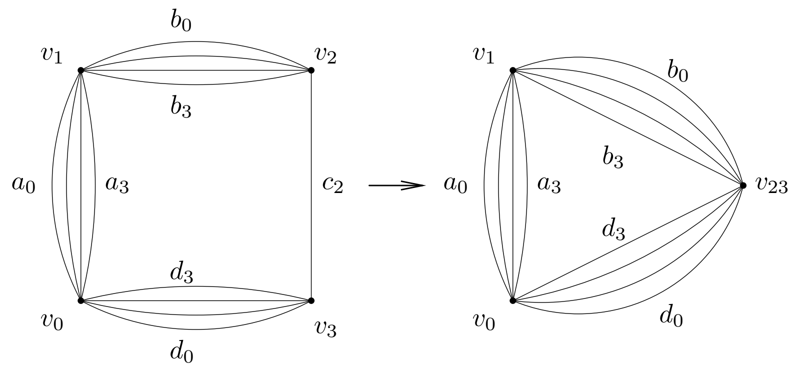


FIGURE 5. The right-hand graph is an expansion minor of order 3 of $\mathfrak{Q}_{3,4}C_4$ (Figure 4), obtained by taking $S = \{a, b, d\}$, $T = \{c\}$, $\tilde{T} = \{c_2\}$, and $D = \varnothing$ (this is the left-hand graph), and then contracting $\tilde{T}$. The balanced circles in the expansion minor are the circles $a_i b_j d_l$ such that $(a_i b_j c_2) = d_l$.

Lemma 2.5. (a) *The biased graph Ω' of Construction XM is an expansion minor of Ω, and every expansion minor of Ω is formed in this way.*

(b) *Ω' is a biased expansion of a graph Δ' which is a minor of Δ formed from $\Delta \setminus D/T$ by deleting some subset of its isolated nodes (if any).*

(c) *$E(\Omega') = p^{-1}(S)$; the projection mapping $p' = p\big|_{p^{-1}(S)}$; and $(p')^{-1}(e) = p^{-1}(e)$ for each $e \in S = E(\Delta')$.*

(d) *If Ω is regular, then Ω' is regular with the same multiplicity.*

Proof. We assume that the reader is acquainted with the definitions and notation of contraction and minors in [37, Sections I.2 and I.5].

(a) It is clear that Ω' is an expansion minor; the task is to prove the converse. A minor of Ω is formed by contracting an edge set A, then deleting a subset of A^c. (We ignore isolated nodes as a triviality.) Let $A_0 = A{:}N_0(A)$ and $\tilde{T} = A \setminus A_0$. Some of the edges after contraction may be half or loose edges if $A_0 \neq \varnothing$. The half edges come in entire fibers $p^{-1}(e)$, where e joins $N_0(A)$ to its complement. The loose edges come in fibers $p^{-1}(e)$ where $e \in E(\Delta){:}N_0(A)$ but $e \notin p(A_0)$, or in partial fibers $p^{-1}(e) \setminus A_0$ where $e \in p(A_0)$. In either case we may simply delete the entire fiber; at worst this leaves extra isolated nodes. Thus we delete $p^{-1}(D_1)$ where $D_1 = E(\Delta) \setminus E(\Delta{:}N_0(A)^c)$.

This leaves us contracting only the balanced part $\tilde{T}$ of A; a process that results in no half or loose edges. To get Ω' we must delete all the remaining edges in $p^{-1}(T)$, where $T = p(\tilde{T})$; all of these are loops. The remaining graph Ω'' now meets the definition of an expansion minor of Ω; it differs from Ω' only in that the latter may require deleting more edges, which must be whole fibers $p^{-1}(e)$ for $e \in D_2 \subseteq E(\Delta)$. Thus $D = D_1 \cup D_2$ and $S = E(\Delta) \setminus (T \cup D)$ in Construction XM.

(b) We have to prove that, for any circle C in $\Delta' = (\Delta \setminus D)/T$, edge $e \in C$, and lift $\tilde{P}$ of $C \setminus e$ into Ω', there is a unique edge $\tilde{e} \in (p')^{-1}(e)$ such that $\tilde{P} \cup \{\tilde{e}\}$ is balanced. C has the form $C_1 \cap S$ where C_1 is a circle in $\Delta \setminus D$. Lift $C_1 \setminus S$ to $\tilde{Q} \subseteq \tilde{T}$. Then $\tilde{P} \cup \tilde{Q}$ is a lift of $C_1 \setminus e$ into Ω, for which there is a unique $\tilde{e} \in p^{-1}(e)$ that makes $\tilde{P} \cup \tilde{Q} \cup \{\tilde{e}\}$ balanced. By the definition of contraction, for $\tilde{e} \in p^{-1}(e) = (p')^{-1}(e)$, $\tilde{P} \cup \tilde{Q} \cup \{\tilde{e}\}$ is balanced in Ω if and only if $\tilde{P} \cup \{\tilde{e}\}$ is balanced in Ω'. This concludes the proof of (b).

Parts (c) and (d) are obvious. □

There are an analogous construction and lemma for group expansions.

> *Construction GXM.* Given $\mathfrak{G}\Delta$, take S, T, D, and $\tilde{T}$ as in Construction XM and modify $(\mathfrak{G}\Delta \setminus p^{-1}(D))/\tilde{T}$ as in that construction to form Φ'.

Lemma 2.6. (a) *The gain graph Φ' of Construction GXM is an expansion minor of $\mathfrak{G}\Delta$, and every expansion minor of $\mathfrak{G}\Delta$ is formed in this way.*

(b) *$\Phi' \cong \mathfrak{G}\Delta'$, where Δ' is as in Lemma 2.5(b).*

(c) *For every minor Δ' of Δ, $\mathfrak{G}\Delta'$ is an expansion minor of $\mathfrak{G}\Delta$.*

(d) *$\langle \Phi' \rangle$ is an expansion minor of $\langle \mathfrak{G}\Delta \rangle$, and every expansion minor of $\langle \mathfrak{G}\Delta \rangle$ equals $\langle \mathfrak{G}\Delta' \rangle$ for a minor Δ' of Δ.*

(e) *Construction XM applied to $\langle \mathfrak{G}\Delta \rangle$ yields $\langle \Phi' \rangle$.*

Proof. Part (a) is proved as in Lemma 2.5. Part (c) follows from (a) by taking $\tilde{T} = p^{-1}(T) \cap E(\{1\}\Delta)$ in the construction. Part (e) is obvious from the constructions. Part (d) follows from (e) and (b).

(b) As a minor of $\mathfrak{G}\Delta$, Φ' has gains in $\mathfrak{G}$ [37, Theorem I.5.4]. We may assume by prior switching of $\Phi = \mathfrak{G}\Delta$ that $\varphi|_{\tilde{T}} \equiv 1$. Thus $\varphi' = \varphi|_{E(\Phi')}$, so $\varphi'|_{(p')^{-1}(e)}$ is a bijection onto $\mathfrak{G}$. It follows easily that $\Phi' \cong \mathfrak{G}\Delta'$. (We do not say $\Phi' = \mathfrak{G}\Delta'$ because the prior switching means that the edge ge, in $E(\Phi')$ as a subset of $E(\mathfrak{G}\Delta)$, may not have gain g in Φ'.) □

Example 2.1. To generate examples of Constructions XM and GXM we take the group expansion $\Phi = \mathfrak{G}K_4$ where $\mathfrak{G}$ is any nontrivial group and we let $\Omega = \langle \mathfrak{G}K_4 \rangle$. The edges of K_4 are e_{ij}, with the implied orientation where appropriate. The edge set of Ω is $\mathfrak{G} \times E$. The edge fibers are the sets $p^{-1}(e_{ij}) = \{ge_{ij} : g \in \mathfrak{G}\}$.

(a) For the first example we let $T = \{e_{12}, e_{34}\}$ and $D = \varnothing$, so $S = \{e_{13}, e_{14}, e_{23}, e_{24}\}$. Then we choose the balanced lift $\tilde{T} = \{1e_{12}, 1e_{34}\}$. The expansion minor Ω' has node set $\{v_{12}, v_{34}\}$, resulting from contraction of $\tilde{T}$ so that v_1 is identified with v_2

and v_3 with v_4. The edge set is $E(\Omega) \setminus (p^{-1}(e_{12}) \cup p^{-1}(e_{34})) = \{ge_{ij} : g \in \mathfrak{G},\ i = 1, 2,\ j = 3, 4\}$, because first $\tilde{T}$ is contracted, which makes the other edges of $p^{-1}(T)$ into loops (they are the edges ge_{12} and ge_{34} for $g \neq 1$), and then these loops are deleted. This defines the underlying graph $\Gamma' = \|\Omega'\|$. Now we have to find out which circles are balanced.

All the edges of Γ' are parallel, so the circles are digons. A digon is supposed to be balanced in Ω' if it arises by contracting a balanced circle in Ω; or since this example is a group expansion, it is balanced if it arises by contracting a circle in Φ whose gain is 1. For instance, a digon $\{ge_{13}, he_{14}\}$ is balanced in Ω' if and only if $\{ge_{13}, he_{14}, 1e_{34}\}$ is balanced in Ω, hence if and only if $g = h$. A digon $\{ge_{13}, he_{24}\}$ is balanced in Ω' if and only if $\{ge_{13}, he_{24}, 1e_{12}, 1e_{34}\}$ is balanced in Ω, hence again if and only if $g = h$. Note that, in accordance with the definition of a biased expansion, a digon $\{ge_{13}, he_{13}\}$ cannot be balanced, since $g \neq h$ because the digon has two edges. So the balanced digons of Ω' are those of the form $\{ge_{ij}, ge_{kl}\}$ where $i, k \in \{1, 2\}$, $j, l \in \{3, 4\}$, and $e_{ij} \neq e_{kl}$.

This defines Ω', which is clearly a biased expansion of K_4/T, the graph with four parallel edges $\{e_{ij} : i = 1, 2,\ j = 3, 4\}$ joining the two nodes v_{12} and v_{34}. If we keep the original gains on the edges in Ω' we have Φ', the expansion minor of Φ.

(b) For a second example we lift the same T to a different pair, $\tilde{T} = \{1e_{12}, g_0e_{34}\}$ where $g_0 \neq 1$. The contracted expansion graph and base graph are the same but the balanced circles are different because the criterion for balance of a digon gives a different equation. Indeed, a digon $\{ge_{13}, he_{14}\}$ is balanced in Ω' if and only if $\{ge_{13}, he_{14}, g_0e_{34}\}$ is balanced in Ω, hence if and only if $gg_0 = h$. This tells us the biased graph Ω', but it does not tell us Φ', for in contracting Φ by $\tilde{T}$ there is a switching step that was not necessary when $\tilde{T}$ had all identity gains.

For Φ' we have to switch Φ to Φ^η in which $\tilde{T}$ has identity gains. We may choose $\eta(v) = 1$ excepting that $\eta(v_3) = g_0$. Then $\varphi^\eta(he_{i3}) = hg_0$, where $i = 1, 2$. Thus, the gains in Φ' are the original gains for edges ge_{i4} but are right-multiplied by g_0 for edges he_{i3}. This gives a complete description of Φ'.

(c) We contract edges that contain a circle. Let $T = \{e_{12}, e_{13}, e_{23}\}$ and $D = \varnothing$ and choose the lift $\tilde{T} = \{1e_{12}, 1e_{13}, 1e_{23}\}$. (The first two gains force the third since $\tilde{T}$ must be balanced.) The rest of the construction is similar to that of (a); the base graph has three parallel edges and the balanced digons have the form $\{ge_{i4}, ge_{j4}\}$.

3. Extension of biased expansions

An *extension* of a biased expansion $\Omega \downarrow_p \Delta$ is a biased expansion $\Omega' \downarrow_{p'} \Delta'$ such that

(a) Δ is a spanning subgraph of Δ', and
(b) $\Omega'\big|_\Delta = \Omega$ (so that $p'\big|_\Delta = p$).

We may say Ω' is an extension of Ω *to* Δ', or *to* $E(\Delta') \setminus E(\Delta)$. The extension is *simple* if Δ' is a simple graph. It is a *maximal extension* if it has no simple proper extension.

We are interested in two types of extension. The first is extension to a link that is parallel to an existing edge of Δ.

Example 3.1 (Parallel Extension). Suppose Δ is any graph, f is a link in Δ, and e is an edge parallel to f but not in Δ. Ω always extends to e. Take $(p')^{-1}(e)$ to be a set in one-to-one correspondence with $p^{-1}(f)$; form balanced digons $\{\tilde{e}, \tilde{f}\}$ when $\tilde{e}$ and $\tilde{f}$ correspond; and for a circle $P \cup f$ in Δ, a lift $\tilde{P} \cup \tilde{e}$ is balanced in Ω' if and only if $\tilde{P} \cup \tilde{f}$ is balanced in Ω. Any selection of edges of Δ can be reduplicated in this way, as many times as desired.

This kind of extension can be technically useful, but the other kind is the more important one: that is extension by an edge e_{vw} joining nonadjacent nodes of Δ. The possibility or impossibility of such extension is crucial data about the structure of a biased expansion.

There are four principal extension theorems. First is uniqueness (Theorem 3.1). If a biased expansion of a 2-connected graph Δ extends to one of Δ', that extension is *unique*, by which we mean unique up to an isomorphism that is the identity on $\Omega \downarrow \Delta$. We can express this by the existence of a commutative diagram of extensions:

$$\begin{tikzcd}
 & \Omega \arrow[dl, hook] \arrow[dr, hook] \arrow[dd] & \\
\Omega' \arrow[rr, dashed, "\rho"'] \arrow[dd] & & \Omega'' \arrow[dd] \\
 & \Delta \arrow[dl, hook] \arrow[dr, hook] & \\
\Delta' \arrow[rr, dashed, "\rho"'] & & \Delta''
\end{tikzcd}$$

where the maps from Ω are embeddings and ρ is an isomorphism. In fact, ρ itself is unique. (Recall that a mapping of biased expansions includes a mapping of their base graphs that commutes with projection.)

The second result is the existence of a unique maximal (simple) extension (Theorem 3.2). The third result says that, if e joins the trivalent nodes of a theta graph in Δ, then Ω extends to e (Proposition 3.8). Last is the theorem that, if e is a chord of a circle $C \subseteq \Delta$, and $\Omega\big|_C$ extends to e, then Ω extends to e (Proposition 3.9).

Theorem 3.1 (Uniqueness of Extension)**.** *Let $\Omega \downarrow \Delta$ be a biased expansion of a 2-connected graph Δ. If $\Omega' \downarrow \Delta'$ and $\Omega'' \downarrow \Delta'$ are two extensions of Ω to Δ', then* *there is a unique biased-expansion isomorphism* $\rho : \Omega' \to \Omega''$ *such that* $\rho\big|_\Omega$ *is the identity, provided that* Δ' *is simple or, more generally, that* ρ*, the projections, and* $\mathrm{id}_{\Delta'}$ *commute.*

Proof. Let $e \in E(\Delta') \setminus E(\Delta)$ with endpoints v and w. These nodes lie in a common circle in Δ; let P_0 and P be the two vw-paths constituting the circle. To define $\rho(\tilde{e}')$ for $\tilde{e}' \in (p')^{-1}(e)$ we choose $\tilde{P}_0$ so that $\tilde{P}_0 \cup \tilde{e}'$ is balanced in Ω', then $\tilde{e}'' \in (p'')^{-1}(e)$ so that $\tilde{P}_0 \cup \tilde{e}''$ is balanced, and set $\rho_e(\tilde{e}') = \tilde{e}''$. It is clear that ρ_e is a bijection $(p')^{-1}(e) \to (p'')^{-1}(e)$ because the roles of Ω' and Ω'' are reversible.

We have to prove that $\tilde{e}''$ is independent of the choice of $\tilde{P}_0$. Take $\tilde{P}$ so that $\tilde{P} \cup \tilde{e}'$ is balanced; then $\tilde{P}_0 \cup \tilde{P}$ is balanced. Since $\tilde{P}_0 \cup \tilde{e}''$ and $\tilde{P}_0 \cup \tilde{P}$ are balanced, so is $\tilde{P} \cup \tilde{e}''$. Now suppose we change $\tilde{P}_0$ to $\tilde{P}_0^1$ so that $\tilde{P}_0^1 \cup \tilde{e}'$ is balanced. Then $\tilde{P}_0^1 \cup \tilde{P}$ is balanced (because $\tilde{P} \cup \tilde{e}'$ is), and since $\tilde{P} \cup \tilde{e}''$ is balanced, so is $\tilde{P}_0^1 \cup \tilde{e}''$. Therefore, $\rho_e(\tilde{e}')$ is independent of the choice of lift of P_0.

Still, we ought to prove $\rho_e(\tilde{e}')$ is independent of the choice of vw-path P_0. Obviously, P_0 could be any vw-path. Then suppose P is a vw-path such that $P_0 \cup P \cup e$ forms a theta graph. Let R_0, R, and R_e be the constituent paths of this theta graph that, respectively, lie in P_0, lie in P, and contain e. Fixing a lift of R, one can imitate the previous proof to show that any lifts of P_0 and P imply the same bijection ρ_e.

Now take the original P_0 and any other vw-path P. By the Path Theorem (see Corollary 2.1) and the preceding argument, all of $P_0, P_1, \ldots, P_k = P$ induce the same bijection ρ_e.

We now define $\rho(\tilde{f})$, for $\tilde{f} \in E(\Omega')$, to be $\tilde{f}$ if $f \in E(\Delta)$ and $\rho_f(\tilde{f})$ if $f \notin E(\Delta)$. It remains to prove that $\rho : E(\Omega') \to E(\Omega'')$ is an isomorphism of biased graphs.

For that it suffices to show that, if $\tilde{C}$ is a balanced circle in Ω', then $f(\tilde{C})$ is balanced in Ω'', and conversely. Choose a spanning tree T of Δ and a lift $\tilde{T}$ such that $\tilde{T} \cup \tilde{C}$ is balanced (possible by Lemma 2.3). Then $\tilde{C} \subseteq \mathrm{bcl}_{\Omega'} \tilde{T}$. By the definition of ρ, $\rho(\tilde{C}) \subseteq \mathrm{bcl}_{\Omega''} \tilde{T}$. Since the latter is balanced, $\rho(\tilde{C})$ is balanced. This reasoning works in both directions: if $\tilde{C}'' \in \mathcal{B}(\Omega'')$, then $\rho^{-1}(\tilde{C}'')$ is balanced. That concludes the proof. ∎

Theorem 3.2 (Maximal Extension)**.** *Given any biased expansion* Ω *of a* 2*-connected simple graph* Δ*, there is a unique maximal extension of* Ω*; its base graph is* $\Delta \cup X$ *where*

$$X = \{e \notin E(\Delta) : \Omega \text{ extends to } e\}.$$

Remember that "uniqueness" is up to isomorphisms that are the identity on Ω.

Proof. No extension $\Omega'' \downarrow \Delta''$ can possibly have $\Delta'' \not\subseteq \Delta \cup X$, so we need only produce an extension of Ω to $\Delta \cup X$ and call upon the Uniqueness Theorem.

For each $e \in X$, let Ω_e be an extension to e. The major part of the proof is to show that Ω_{e_1} and Ω_{e_2} are compatible.

Proposition 3.3 (Common Extension)**.** *If $e_1, e_2 \in X$, then Ω extends to $\Delta \cup \{e_1, e_2\}$.*

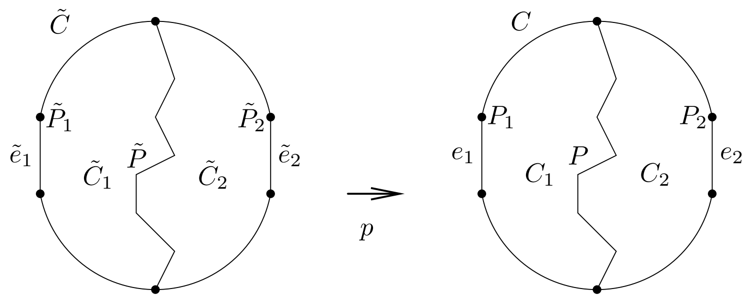


FIGURE 6. The parts of $C = P_1 \cup P_2$ (right) with $e_i \in P_i$, the chordal path P, and the circles $C_i := P_i \cup P$, for defining balance of a lift $\tilde{C}$ (left) in Proposition 3.3.

Proof. The core of the proposition is the definition of balance in the common extension Ω_{12} of Ω_1 and Ω_2 (meaning Ω_{e_1} and Ω_{e_2}). The graph $\|\Omega_{12}\|$ is simply $\|\Omega_1\| \cup \|\Omega_2\|$. Balance of a circle $\tilde{C}$ that covers a circle C in $\Delta_{12} = \Delta \cup \{e_1, e_2\}$ is as in Ω_1 or Ω_2, except when C contains both e_1 and e_2. Then we define $\tilde{C}$ to be balanced if and only if there is a path P that forms with C a theta graph whose three constituent paths are P and two paths, P_1 and P_2, of which P_1 contains e_1 and P_2 contains e_2 (we call P a *connecting chordal path* of C because it connects the two components of $C \setminus \{e_1, e_2\}$), and P has a lift $\tilde{P}$ such that $\tilde{C}_1$ and $\tilde{C}_2$ are both balanced. (The notation is that $C_i = P_i \cup P$, $\tilde{P}_i$ is the lift of P_i that is contained in $\tilde{C}$, and $\tilde{C}_i = \tilde{P}_i \cup \tilde{P}$.) Then Ω_{12} is $\|\Omega_{12}\|$ with balanced circles as just defined.

It is important to know that the definition of balance is independent of the various choices implicit in it. We need a bit more notation. For a connecting chordal path path P and edge $e \in P$, let $R = P \setminus e$. For a different connecting chordal path path P' and $e' \in P'$, we define R', C_1', C_2' analogously to R, C_1, C_2. We begin with a little lemma.

Lemma 3.4. *Let P and P' be two connecting chordal paths of C such that $(C \setminus \{e_1, e_2\}) \cup P \cup P'$ contains a unique circle, D. Let $e \in P \setminus P'$ and $e' \in P' \setminus P$, or let $e = e' \in P \cap P'$. Let $\tilde{C}$ be a lift of C and choose arbitrary lifts $\tilde{R}$ and $\tilde{R}'$ that agree on $R \cap R'$ and such that $\tilde{D}$ is balanced if $e = e' \in P \cap P'$. Then, for each lift $\tilde{e}$ such that $\tilde{C}_1$ and $\tilde{C}_2$ are balanced (in Ω_1 and Ω_2, respectively), there is a unique lift $\tilde{e}'$ such that $\tilde{C}_1'$ and $\tilde{C}_2'$ are balanced (in Ω_1 and Ω_2, respectively); and $\tilde{e} = \tilde{e}'$ if $e = e'$* (Fig. 7)*.*

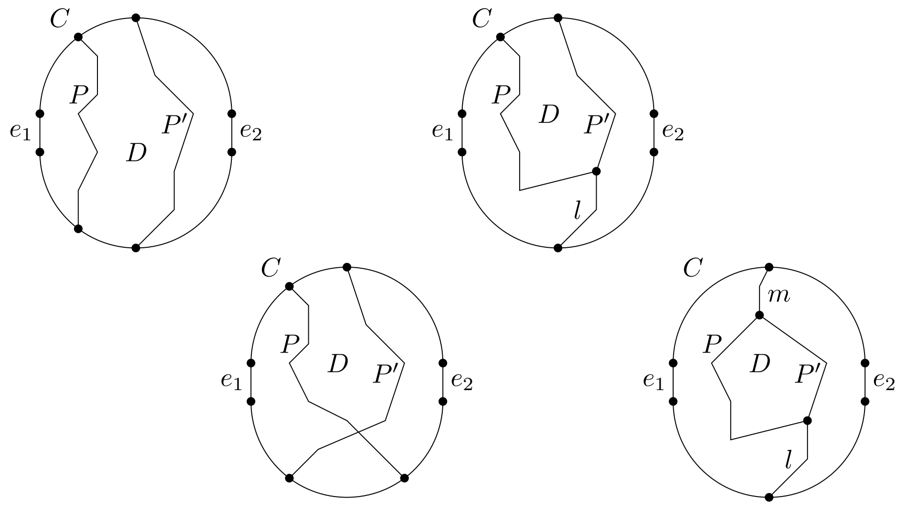


FIGURE 7. The four ways P and P' can appear in Lemma 3.4. l and m are lengths ≥ 0.

Proof. Let $A = C \setminus \{e_1, e_2\}$. Note that $D \subseteq A \cup R \cup R'$ if $e = e'$, but $e, e' \in D$ if $e \neq e'$.

Suppose $e = e'$. In Ω_1, $\tilde{C}_1 \cup \tilde{D}$ is a theta graph, $\tilde{C}_1' = \tilde{C}_1 \oplus \tilde{D}$, and $\tilde{D}$ is balanced, so $\tilde{C}_1$ is balanced if and only if $\tilde{C}_1'$ is. Similarly, $\tilde{C}_2$ is balanced if and only if $\tilde{C}_2'$ is. Also, the $\tilde{e}$ that makes $\tilde{C}_i$ balanced is unique, by the circle lifting property in Ω_i. It follows that $\tilde{e} = \tilde{e}'$.

If $e \neq e'$, then for each lift $\tilde{e}$ there is a unique $\tilde{e}' = \theta(\tilde{e})$ for which $\tilde{D}$ is balanced (in Ω), and θ is a bijection from $p^{-1}(e)$ to $p^{-1}(e')$. Suppose we lift e to $\tilde{e}$ such that $\tilde{C}_1$ is balanced. Then lifting e' to $\tilde{e}'$, $\tilde{C}_1'$ is balanced $\iff$ $\tilde{D}$ is balanced $\iff$ $\tilde{e}' = \theta(\tilde{e})$. A similar argument applies to $\tilde{C}_2$ and $\tilde{C}_2'$. $\square$

The next lemma shows, in particular, that the definition of balance of a lift of C is independent of the choice of connecting chordal path.

Lemma 3.5. *Given C containing e_1 and e_2, any connecting chordal path P, any edge $e \in P$, and any lift $\tilde{R}$ of $R = P \setminus e$, then a lift $\tilde{C}$ is balanced if and only if there exists $\tilde{e}$ such that $\tilde{C}_1$ and $\tilde{C}_2$ are balanced, and this $\tilde{e}$ is unique.*

Proof. By definition, $\tilde{C}$ is balanced if there is $\tilde{e}$ such that $\tilde{C}_1$ and $\tilde{C}_2$ are balanced.

Suppose, conversely, that $\tilde{C}$ is balanced; thus, there exist a connecting chordal path P' and a lift $\tilde{P}'$ such that $\tilde{C}_1'$ and $\tilde{C}_2'$ are balanced. Choose $e' \in P'$.

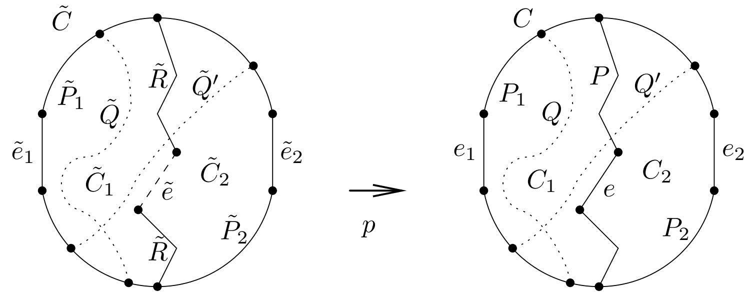


Figure 8. Example of chordal paths Q, Q' in Lemma 3.5 (right) and the process (left) of balancing $\tilde{C}_1$ and $\tilde{C}_2$, by choosing $\tilde{e}$, to get the lift $\tilde{P} = \tilde{R} \cup \tilde{e}$ of P that implies balance of $\tilde{C}$.

Since Δ is inseparable, there exist two connecting chordal paths of C, Q and Q', that are internally disjoint (Fig. 8). By the Path Lemma 2.1 there is a chain $P' = Q_0, Q_1, \ldots, Q_k = P$ of connecting chordal paths that includes $Q = Q_{m-1}$ and $Q' = Q_m$, such that $(C \setminus \{e_1, e_2\}) \cup Q_{i-1} \cup Q_i$ contains a unique circle for all i. Choose edges $f_i \in Q_i$ so that $f_0 = e'$, $f_k = e$, and $f_{i-1} \in Q_i \Rightarrow f_i = f_{i-1}$. This is possible because Q_{m-1} and Q_m are edge disjoint. Thus, at worst we may be forced to take $f_{m-1} = f_0$ and $f_m = f_k$, but there is no necessary relation between f_0 and f_k.

Let $R_i = Q_i \setminus f_i$ and let C_{i1} and C_{i2} be the circles in $C \cup Q_i$ that contain, respectively, e_1 and e_2 but not both. We may apply Lemma 3.4 k times to conclude that, for any lifts $\tilde{R}_0$ and $\tilde{R}_k$, in particular, $\tilde{R}_0 \subseteq \tilde{P}'$ and $\tilde{R}_k = \tilde{R}$, and for any $\tilde{e}'$ such that $\tilde{C}_{01}$ and $\tilde{C}_{02}$ are balanced, there is a unique lift $\tilde{e}$ such that $\tilde{C}_{k1}$ and $\tilde{C}_{k2}$ are balanced. (If $\theta_i : p^{-1}(f_i) \to p^{-1}(f_{i-1})$ is as in the proof of Lemma 3.4, then $\tilde{e} = (\theta_1\theta_2 \cdots \theta_k)^{-1}(\tilde{e})$.) $\square$

To prove Proposition 3.3 we need just two more steps: to prove, first, the circle lifting property in Ω_{12}, and second, that $\mathcal{B}(\Omega_{12})$ is a linear class.

Step 1. Circle lifting. We need to consider a circle $C \ni e_1, e_2$ and an edge $f \in C$. Letting $S = C \setminus f$, we assume $\tilde{S}$ given and must prove there is a unique $\tilde{f}$ such that $\tilde{S} \cup \tilde{f}$ is balanced. We take P, e, and R as in Lemma 3.5, and fix $\tilde{R}$. We may assume $f \in P_2$ (Fig. 9).

Choose $\tilde{e}$ so that $\tilde{C}_1 \subseteq \tilde{S} \cup \tilde{P}$ is balanced in Ω_1, then $\tilde{f}$ so that $\tilde{C}_2 \subseteq \tilde{S} \cup \tilde{P} \cup \tilde{f}$ is balanced in Ω_2. By definition, $\tilde{C} = \tilde{S} \cup \tilde{f}$ is then balanced. Suppose both $\tilde{f}^1$ and $\tilde{f}^2$ make $\tilde{S} \cup \tilde{f}^i = \tilde{C}^i$ balanced. By Lemma 3.5, for $i = 1$ and 2,

$$(\exists \tilde{e}^i)\ \tilde{P}_1 \cup \tilde{R} \cup \tilde{e}^i \quad \text{and} \quad \tilde{P}_2^i \cup \tilde{R} \cup \tilde{e}^i \quad \text{are balanced,}$$

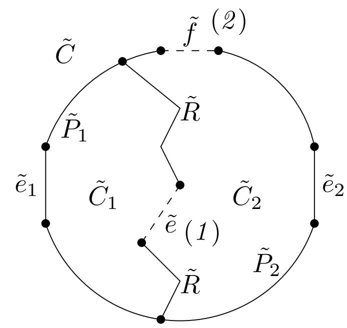


FIGURE 9. Adding (*1*) $\tilde{e}$ to make $\tilde{C}_1$ balanced and (*2*) $\tilde{f}$ to make $\tilde{C}_2$ balanced, in Step 1 of proving Proposition 3.3.

where $\tilde{P}_2^i$ is the lift of P_2 contained in $\tilde{S} \cup \tilde{f}^i$. Comparing $\tilde{C}_1^1$ with $\tilde{C}_1^2$ in Ω_1, $\tilde{e}^1 = \tilde{e}^2$. Then, comparing $\tilde{C}_2^1$ with $\tilde{C}_2^2$ in Ω_2, $\tilde{f}^1 = \tilde{f}^2$. Thus, $\tilde{f}$ is unique.

Step 2. Linearity. We must examine lifts of a theta graph Θ that contains both e_1 and e_2. There are two cases, according as e_1 and e_2 are in the same or different paths of Θ.

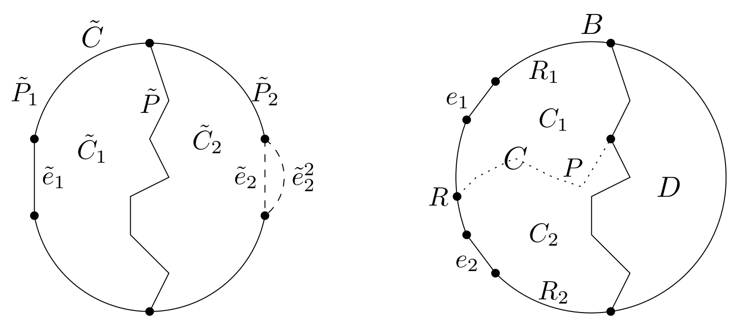


FIGURE 10. Case 1 (left) and Case 2 (right) in Step 2 of the proof of Proposition 3.3.

Case 1. If e_1 and e_2 are in different paths, we can use the notation of Lemma 3.5. Suppose a lift such that $\tilde{C}_1$ and $\tilde{C}_2$ are balanced: then $\tilde{C}$ is balanced by definition. On the other hand, suppose $\tilde{C}_1$ and $\tilde{C}$ are balanced while $\tilde{C}_2$ is unbalanced. By changing $\tilde{e}_2$ to $\tilde{e}_2^2$ we get a balanced lift of C_2, namely, $\tilde{C}_2^2 = (\tilde{C}_2 \setminus \tilde{e}_2) \cup \tilde{e}_2^2$. Then $\tilde{C}^2 = (\tilde{C} \setminus \tilde{e}_2) \cup \tilde{e}_2^2$ is balanced. However, in Step 1 we showed that $\tilde{C}$ and $\tilde{C}^2$ cannot both be balanced. Therefore, $\tilde{C}_2$ must have been balanced after all.

Case 2. If e_1 and e_2 lie in the same path of Θ, we need new notation. $\Theta \setminus \{e_1, e_2\}$ has two components and contains a unique circle, call it D. Let B and C be the other circles in Θ, and let P be a minimal path in Δ connecting the two components of $\Theta \setminus \{e_1, e_2\}$. We may assume that P has both endpoints in $N(C)$, so that $C \cup P$ is a theta graph with circles $C_1 \ni e_1$ and $C_2 \ni e_2$. If we write the path $B \cap C$ as a concatenation of paths, $R_1 e_1 R e_2 R_2$, we may also assume that P, which has one endpoint in R, has the other end not in R_2. Therefore, $B \cup C_1$ and $D \cup C_2$ are theta graphs. In addition, $B \oplus C_1 = D \oplus C_2$.

Now we prove that, if two of the circles $\tilde{B}, \tilde{C}, \tilde{D}$ in a lift $\tilde{\Theta}$ are balanced, then the third one is balanced.

Suppose $\tilde{B}$ and $\tilde{C}$ are balanced. By Lemma 2.3, there is a lift $\tilde{P}$ such that $\tilde{C}_1$ and $\tilde{C}_2$ are balanced. Therefore $\tilde{B} \oplus \tilde{C}_1$ is balanced, and as this equals $\tilde{D} \oplus \tilde{C}_2$ and $\tilde{C}_2$ is balanced, $\tilde{D}$ is balanced.

If, however, it is $\tilde{C}$ and $\tilde{D}$ that are balanced, then $\tilde{D} \oplus \tilde{C}_2 = \tilde{B} \oplus \tilde{C}_1$ is balanced, whence $\tilde{B}$ is balanced.

Supposing finally that $\tilde{B}$ and $\tilde{D}$ are balanced, we choose $\tilde{P}$ so that $\tilde{C}_1$ is balanced. Consequently, $\tilde{B} \oplus \tilde{C}_1$ is balanced. This being $\tilde{D} \oplus \tilde{C}_2$, we conclude that $\tilde{C}_2$ is balanced, whence $\tilde{C}$ is balanced.

Thus in every case linearity is satisfied, and therefore, Ω_{12} is a biased graph. $\square$

Lemma 3.6. *Suppose Ω' extends Ω to Δ' and $e \in X \setminus E(\Delta')$; then there is an extension of Ω to $\Delta' \cup e$.*

Proof. Let

$$\mathcal{F} = \{\Delta'' \subseteq \Delta' : \Omega'\big|_{\Delta''} \text{ extends to } e\}.$$

If $\Delta'' \in \mathcal{F}$ and $f \in E(\Delta') \setminus E(\Delta'')$, then $\Omega'\big|_{\Delta''}$ extends both to e and to f; by Proposition 3.3 it extends to $\{e, f\}$, so $\Delta'' \cup f \in \mathcal{F}$. This suffices to prove the lemma when X is finite.

Otherwise, we apply Zorn's Lemma in the usual way. Take a maximal chain $\{\Delta_i\}$ in $\mathcal{F}$; let Δ'' be its union. Write Ω_i for the extension of $\Omega'\big|_{\Delta_i}$ to e. By Unique Extension we can regard each Ω_i for $i < j$ as the restriction $\Omega_j\big|_{\Delta_i \cup e}$. Therefore $\Omega'' = \bigcup_i \{\Omega_i\}$ is a well defined graph. It is a biased expansion of $\Delta'' \cup e$ because any circle in $\Delta'' \cup e$ or theta graph in Ω'' is contained in some $\Delta_i \cup e$ or Ω_i. It extends $\Omega'\big|_{\Delta''}$ because $\Omega'\big|_{\Delta''} = \bigcup_i \{\Omega'\big|_{\Delta_i}\}$. Therefore, $\Delta'' \in \mathcal{F}$. If $\Delta'' \subset \Delta'$, there is an $f \in E(\Delta') \setminus E(\Delta'')$ and, by the first part of the proof, $\Delta'' \cup f \in \mathcal{F}$. As that contradicts the maximality of the original chain, Δ'' must be Δ', so Ω' extends to e. $\square$

Lemma 3.7. *Suppose Ω' extends Ω to $\Delta' \supset \Delta$; then Ω' extends to $\Delta \cup X$.*

Proof. Here let

$$\mathcal{F} = \{\Delta'' \subseteq \Delta \cup X \mid \Delta'' \supseteq \Delta' \text{ and } \Omega' \text{ extends to } \Delta''\}.$$

There can be only one maximal member of $\mathcal{F}$, namely, $\Delta \cup X$, since for any other Δ'', taking $e \in X \setminus E(\Delta'')$ we know that Ω'', an extension of Ω' to Δ'', extends to e. This proves the lemma when $X \setminus E(\Delta')$ is finite.

In the infinite case, again we apply Zorn's Lemma. The union of a maximal chain of graphs in $\mathcal{F}$ is itself in $\mathcal{F}$, and this union must be $\Delta \cup X$ or the chain could not have been maximal. □

To complete the proof of the Maximal Extension Theorem we need only appeal to the Unique Extension Theorem. □

Proposition 3.8 (Theta Extension)**.** *Any biased expansion of a theta graph with trivalent nodes v and w extends to the edge e_{vw}.*

Proof. Let the theta graph Δ have constituent paths P_1, P_2, and P_3 and write e for e_{vw}. By Example 3.1 we may assume v and w are nonadjacent in Δ. Define a set E_e in one-to-one correspondence with some fiber $p^{-1}(f)$ for $f \in E(\Delta)$. Letting each $\tilde{e} \in E_e$ have endpoints v and w defines a graph $\|\Omega'\|$ that covers $\Delta \cup e$. The task is to define balance and show it results in a biased expansion $\Omega' \downarrow \Delta \cup e$ extending the original biased expansion $\Omega \downarrow \Delta$.

Choose a fixed edge $f_1 \in P_1$, let $Q_1 = P_1 \setminus f_1$, and fix a lift $\tilde{Q}_1^0$. Choose a bijection $\psi : p^{-1}(f_1) \to E_e$ and, for $\tilde{e} \in E_e$, define

$$\tilde{Q}_1^0 \cup \tilde{f}_1 \cup \tilde{e} \text{ balanced} \iff \tilde{e} = \psi(\tilde{f}_1).$$

For any lift $\tilde{P}_2$ and any $\tilde{e} \in E_e$, we define

$$\tilde{P}_2 \cup \tilde{e} \text{ balanced} \iff \tilde{P}_2 \cup \tilde{Q}_1^0 \cup \psi^{-1}(\tilde{e}) \text{ is balanced.}$$

For $\tilde{P}_3 \cup \tilde{e}$ the definition is similar. (This leaves balance of $\tilde{P}_1 \cup \tilde{e}$ undefined as yet, in general.) We need to show consistency between the states of balance of $\tilde{P}_2 \cup \tilde{e}$ and of $\tilde{P}_3 \cup \tilde{e}$. If both are balanced, $\tilde{P}_2 \cup \tilde{Q}_1^0 \cup \psi^{-1}(\tilde{e})$ and $\tilde{P}_3 \cup \tilde{Q}_1^0 \cup \psi^{-1}(\tilde{e})$ are balanced, so $\tilde{P}_2 \cup \tilde{P}_3$ is balanced. Similarly, if only one of $\tilde{P}_2 \cup \tilde{e}$ and $\tilde{P}_3 \cup \tilde{e}$ is balanced, $\tilde{P}_2 \cup \tilde{P}_3$ cannot be balanced. Thus, linearity is satisfied for lifts of $P_2 \cup P_3 \cup e$. We call this *23-consistency*.

Now, for a lift $\tilde{P}_1$ we define $\tilde{P}_1 \cup \tilde{e}$ to be balanced if $\tilde{P}_2 \cup \tilde{e}$ is balanced for some $\tilde{P}_2$ such that $\tilde{P}_1 \cup \tilde{P}_2$ is balanced. Suppose we took two lifts $\tilde{P}_2^1$ and $\tilde{P}_2^2$ such that both $\tilde{P}_1 \cup \tilde{P}_2^j$ are balanced, and say $\tilde{P}_2^j \cup \tilde{e}^j$ is balanced. Pick $\tilde{P}_3$ so that $\tilde{P}_1 \cup \tilde{P}_3$ is balanced. Then each $\tilde{P}_2^j \cup \tilde{P}_3$ is balanced. By 23-consistency, $\tilde{P}_3 \cup \tilde{e}^j$ is balanced for $j = 1, 2$; thus $\tilde{P}_3 \cup \tilde{Q}_1^0 \cup \tilde{f}_1^j$ is balanced for $\tilde{f}_1^j = \psi^{-1}(\tilde{e}^j)$, but since $\tilde{f}_1^1 = \tilde{f}_1^2$, we see $\tilde{e}^1 = \tilde{e}^2$. Therefore, balance of $\tilde{P}_1 \cup \tilde{e}$ is independent of the choice of $\tilde{P}_2$. We call this *12-consistency*.

We show that, if $\tilde{P}_1 \cup \tilde{P}_3$ is balanced, then $\tilde{P}_1 \cup \tilde{e}$ is balanced if and only if $\tilde{P}_3 \cup \tilde{e}$ is balanced. Take $\tilde{P}_2$ so that $\tilde{P}_1 \cup \tilde{P}_2 \cup \tilde{P}_3$ is balanced. Then $\tilde{P}_1 \cup \tilde{e}$ is balanced $\iff$ (by 12-consistency) $\tilde{P}_2 \cup \tilde{e}$ is balanced $\iff$ (by 23-consistency) $\tilde{P}_3 \cup \tilde{e}$ is balanced.

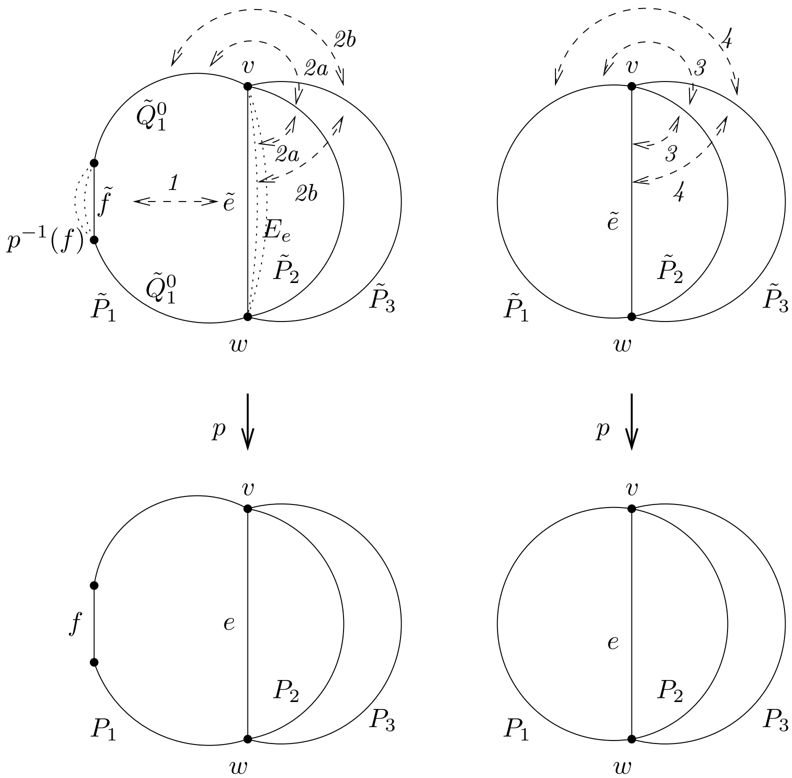


Figure 11. Illustrating the proof of Proposition 3.8. *Step 1*: Use a lift $\tilde{Q}_1^0$ of $P \setminus f_1$ to form balanced circles which imply a correspondence between $\tilde{f}_1 \in p^{-1}(f_1)$ and $\tilde{e} \in E_e$. *Step 2*: Define balance of $\tilde{P}_2 \cup \tilde{e}$ and $\tilde{P}_3 \cup \tilde{e}$. *Step 3*: Use balance of $\tilde{P}_1 \cup \tilde{P}_2$ and $\tilde{P}_2 \cup \tilde{e}$ to define balance of $\tilde{P}_1 \cup \tilde{e}$. *Step 4*: Use balance of $\tilde{P}_1 \cup \tilde{P}_3$ and $\tilde{P}_3 \cup \tilde{e}$ to finish proving consistency of balance. *Step 5* (not illustrated): Prove uniqueness in the circle lifting property.

We still have to prove uniqueness in the circle lifting property. First, we treat lifts of e. Any $\tilde{P}_i$ has a balanced completion $\tilde{P}_i \cup \tilde{e}$, as we have seen. Suppose $\tilde{P}_i \cup \tilde{e}^1$ and $\tilde{P}_i \cup \tilde{e}^2$ are balanced. If $i = 2, 3$, just take $\tilde{Q}_1^0 \cup \tilde{f}_1$ such that $\tilde{P}_i \cup \tilde{Q}_1^0 \cup \tilde{f}_1$ is balanced. Then $\tilde{f}_1 = \psi^{-1}(\tilde{e}^j)$ for $j = 1, 2$, whence $\tilde{e}^1 = \tilde{e}^2$. If $i = 1$, take $\tilde{P}_2$ so that $\tilde{P}_1 \cup \tilde{P}_2$ is balanced: then $\tilde{P}_2 \cup \tilde{e}^1$ and $\tilde{P}_2 \cup \tilde{e}^2$ are balanced, so $\tilde{e}^1 = \tilde{e}^2$.

Now we treat lifts of $f \in P_i$. Let $R = P_i \setminus f$ and take any $\tilde{R}$ and $\tilde{e}$. If $i = 2, 3$, we know that $\tilde{Q}_1^0 \cup \psi^{-1}(\tilde{e}) \cup \tilde{e}$ is balanced, and there exists $\tilde{f}$ for which $\tilde{Q}_1^0 \cup \psi^{-1}(\tilde{e}) \cup \tilde{R} \cup \tilde{f}$ is balanced (and it is unique); by definition, for this $\tilde{f}$ and no other, $(\tilde{R} \cup \tilde{f}) \cup \tilde{e}$

is balanced. If $i = 1$, we choose any $\tilde{P}_2$ such that $\tilde{P}_2 \cup \tilde{e}$ is balanced. Then there is a unique $\tilde{f}$ making $\tilde{P}_2 \cup \tilde{R} \cup \tilde{f}$ balanced, and by definition that is the only $\tilde{f}$ for which $(\tilde{R} \cup \tilde{f}) \cup \tilde{e}$ can be balanced. Thus we have a biased expansion of $\Delta \cup e$. $\square$

Proposition 3.9 (Chordal Extension)**.** *Suppose Ω is a biased expansion of a 2-connected graph Δ and $e \notin E(\Delta)$. For any circle $C \subseteq \Delta$ of which e is a chord, Ω extends to e if and only if $\Omega\big|_C$ extends to e.*

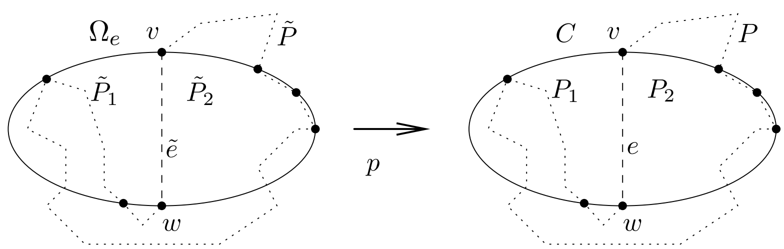


FIGURE 12. The configuration in the proof of Proposition 3.9, proving consistency of $\tilde{e}$. P (dotted), a path from v to w in Ω, may intersect C (solid curve).

Proof. We need only prove sufficiency. Let $e = vw$. Take C, of which e is a chord, such that $\Omega\big|_C$ extends to e. Let P_1 and P_2 be the paths into which e divides C. Let Ω_e be the extension to e of $\Omega\big|_C$. To define Ω', the extension of Ω, we set $E(\Omega') = E(\Omega) \cup p_e^{-1}(e)$ and define a circle $\tilde{P} \cup \tilde{e}$ in Ω', lifting a circle $P \cup e$ in $\Delta \cup e$, to be balanced if and only if there is a lift $\tilde{P}_1$ such that both $\tilde{P} \cup \tilde{P}_1$ and $\tilde{P}_1 \cup \tilde{e}$ are balanced. It remains to prove that Ω' is a biased graph and a biased expansion of Δ'. First we show that P_2 works as well as P_1 in defining balance of $\tilde{P} \cup \tilde{e}$.

Lemma 3.10. *$\tilde{P} \cup \tilde{e}$ is balanced if and only if there is a choice of $\tilde{P}_2$ so that $\tilde{P} \cup \tilde{P}_2$ and $\tilde{P}_2 \cup \tilde{e}$ are balanced.*

Proof. First, suppose $\tilde{P} \cup \tilde{e}$ is balanced: then there is a $\tilde{P}_1$ such that $\tilde{P} \cup \tilde{P}_1$ and $\tilde{P}_1 \cup \tilde{e}$ are balanced. Choose $\tilde{P}_2$ so that $\tilde{P} \cup \tilde{P}_1 \cup \tilde{P}_2$ is balanced. (That is possible by Lemma 2.3.) Then $\tilde{P}_1 \cup \tilde{P}_2 \cup \tilde{e}$ is a theta graph in Ω_e, so $\tilde{P}_2 \cup \tilde{e}$ is balanced. Thus, $\tilde{P}_2$ exists as desired.

The converse is similar. $\square$

We should prove that different choices of $\tilde{P}_1$ give consistent definitions of balance of $\tilde{P} \cup \tilde{e}$. Assume $P \neq P_1, P_2$.

Lemma 3.11. *Suppose $\tilde{P}_1^1 \cup \tilde{P}$ and $\tilde{P}_1^2 \cup \tilde{P}$ are balanced, and $\tilde{P}_1^1 \cup \tilde{e}^1$ and $\tilde{P}_1^2 \cup \tilde{e}^2$ are balanced. Then $\tilde{e}^1 = \tilde{e}^2$.*

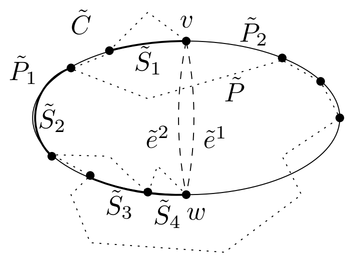


Figure 13. The configuration in the proof of Lemma 3.11, using $\tilde{P}$ (*dotted*) to prove consistency of $\tilde{e}$ with regard to $\tilde{C}$. P_1 may contain nodes or edges of P, forming segments $S_1, \ldots, S_k$ (whose lifts $\tilde{S}_i$ are heavy curves; $k = 4$ in the example) in $P_1 \setminus P$.

Proof. Choose $\tilde{P}_2$ so $\tilde{P} \cup \tilde{P}_1^1 \cup \tilde{P}_2$ is balanced (by Lemma 2.3). Then $\tilde{P}_2 \cup \tilde{e}^1$ is balanced in Ω_e, because of the theta graph $\tilde{P}_1^1 \cup \tilde{e}^1 \cup \tilde{P}_2$.

The maximal subpaths of P_1 that are internally disjoint from P form $k \geq 1$ segments $S_1, \ldots, S_k$ of positive length. Choose $e_i \in S_i$ and let $R = P_1 \setminus \{e_1, \ldots, e_k\}$. Then

(1) $R \cup P$ is connected, so $R \cup P \cup P_2$ is connected, and
(2) no edge of any S_i is contained in any circle of $R \cup P$, nor of $R \cup P \cup P_2$.

Consequently, writing $\tilde{R}^2$ for the lift of R contained in $\tilde{P}_1^2$,

(3) $\tilde{R}^2 \cup \tilde{P}$ and $\tilde{R}^2 \cup \tilde{P} \cup \tilde{P}_2$ are connected, and
(4) $\tilde{R}^2 \cup \tilde{P} \cup \tilde{P}_2$ is balanced, because any circle in it lies in $\tilde{P} \cup \tilde{P}_2$, which is balanced.

Now, $\tilde{e}_i^2$ lies in a circle in $\tilde{R}^2 \cup \tilde{P} \cup \tilde{e}_i^2$ by (3), which is balanced because it is in $\tilde{P}_1^2 \cup \tilde{P}$. Therefore $\tilde{e}_i^2 \in \operatorname{bcl}(\tilde{R}^2 \cup \tilde{P})$. So

$$\tilde{P}_1^2 \cup \tilde{P} \cup \tilde{P}_2 \subseteq \operatorname{bcl}(\tilde{R}^2 \cup \tilde{P} \cup \tilde{P}_2),$$

which is balanced (Lemma 2.2). Thus $\tilde{P}_1^2 \cup \tilde{P}_2$ is balanced, so $\tilde{P}_2 \cup \tilde{e}^2$ is balanced by the theta graph $\tilde{P}_1^2 \cup \tilde{e}^2 \cup \tilde{P}_2$ in Ω_e. As Ω_e is a biased expansion and both $\tilde{e}^1$ and $\tilde{P}_2 \cup \tilde{e}^2$ are balanced, $\tilde{e}^1 = \tilde{e}^2$. □

Thus, we have a well defined notion of balance in $\|\Omega'\| = \|\Omega\| \cup \|\Omega_e\|$.

The lemma applies as well to P_2 as to P_1, of course, due to Lemma 3.10.

Lemma 3.12. *Suppose $\tilde{P}_1$ chosen so that $\tilde{P} \cup \tilde{P}_1$ is balanced. Then $\tilde{P} \cup \tilde{e}$ is balanced if and only if $\tilde{P}_1 \cup \tilde{e}$ is balanced.*

Proof. There is a unique $\tilde{e}^0$ for which $\tilde{P}_1 \cup \tilde{e}^0$ is balanced, because Ω_e is a biased expansion. Then $\tilde{P} \cup \tilde{e}^0$ is balanced, but by Lemma 3.11 no other $\tilde{P} \cup \tilde{e}$ can be balanced. □

Lemma 3.13. *Suppose $\tilde{C}$ chosen so that $\tilde{P} \cup \tilde{C}$ is balanced. Then $\tilde{P} \cup \tilde{e}$ is balanced in Ω' if and only if $\tilde{C} \cup \tilde{e}$ is balanced in Ω_e.*

Proof. Apply Lemma 3.12 to P_1 and P_2, the latter requiring Lemma 3.10. □

The rest of the proof shows that Ω' is a biased expansion. First, the uniqueness of circle lifting.

Lemma 3.14. *If C' is a circle in Δ' and $f \in C'$, and if $\tilde{P}'$ is any lift of $C' \setminus f$, then there is exactly one lift $\tilde{f}$ that makes $\tilde{P}' \cup \tilde{f}$ balanced.*

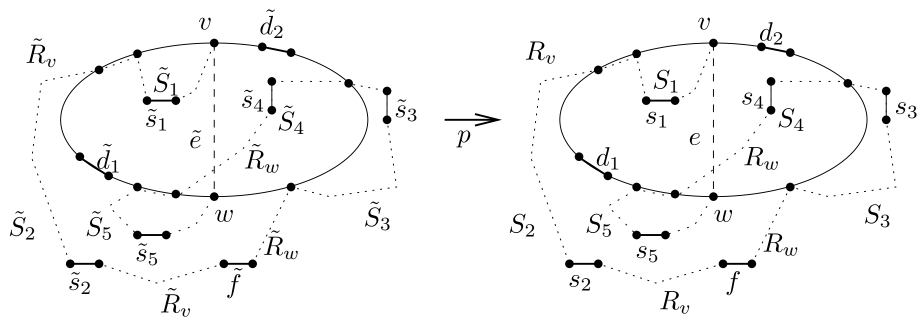


FIGURE 14. The configuration in the proof of Lemma 3.14, proving general consistency of $\tilde{e}$. C is the solid ellipse; P is dotted. There are $m = 5$ bridges of C in $C \cup P$ and $k = 2$ segments of C that connect R_v to R_w. $t = 2$ so $S_t = S_2$.

Proof. We may assume $e \in C'$. When $f = e$, this is a consequence of Lemma 3.12. Otherwise, let $P = C' \setminus e$, so $f \in P$. (We may assume $P \neq P_1, P_2$.)

We shall have need of the graph of P, which is $(N(P), P)$, and that of C. Removing f, $(N(P), P)$ falls into two connected halves, one containing v and the other w; we write $R = (N(P), P) \setminus f$ and R_v, R_w for the two halves. We shall be careless with notation, using P, R, etc., to denote both the graph and the edge set, trusting that all will be clear.

A *bridge of* C in $C \cup P$ is a maximal subpath of P whose internal nodes are in $P \setminus C$, and a *bridge of* P in $C \cup P$ is a maximal subpath of C whose internal nodes lie in $C \setminus P$ (excluding edgeless subpaths in both cases). Call the bridges of C (which are subpaths of P) $S_1, S_2, \ldots, S_m$ and choose an edge $s_i \in S_i$ for each bridge. Let $S = \{s_1, s_2, \ldots, s_m\}$. Amongst the bridges of P (which are subpaths of C), we are interested only in those that connect R_v to R_w. For each such bridge choose an edge d_i in it, and let $D = \{d_1, \ldots, d_k\}$, there being k such bridges. D depends on f. Let $D' = C \setminus D$ (as an edge set) (Fig. 14).

So far, we have two biased expansions: $\Omega \downarrow \Delta$ and $\Omega_e \downarrow C \cup e$. The Theta Extension Lemma generates others, which we employ as auxiliary graphs. Since $C \cup S_i$ is a theta subgraph of Δ, $\Omega\big|_{C\cup S_i}$ extends to a chord e_i of C that joins the endpoints of S_i. Call Ω_i the resulting biased expansion of $C \cup S_i \cup e_i$, and let $H = \{e_1, e_2, \ldots, e_m\}$. Taking every Ω_i separately, we get extensions $\Omega_i\big|_{C\cup e_i}$ of $\Omega\big|_C$. By the Maximal Extension Theorem, all the extensions of $\Omega\big|_C$, including Ω_e, are compatible; that is, there is an extension Ω_P of $\Omega\big|_C$ to $H \cup e$, determined by Ω_e and the subpaths S_i. We now have three groups of biased expansions: Ω, Ω_P extending Ω_e, and Ω_i extending $\Omega\big|_{C\cup S_i}$ to e_i (Fig. 15). All this is independent of f.

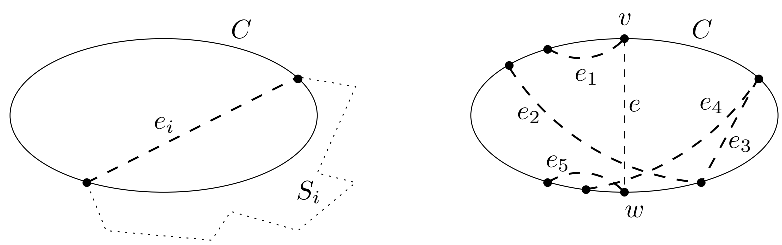


FIGURE 15. The two kinds of extension of $\Omega|_C$: Extension to $C \cup S_i \cup e_i$ (left) and to Ω_P (right).

We wish to prove that, given $\tilde{R}$ and $\tilde{e}$, there is a unique $\tilde{f}$ such that $\tilde{R} \cup \{\tilde{e}, \tilde{f}\}$ is balanced. First we establish a tool.

Lemma 3.15. *Let $\tilde{P}$ and $\tilde{C}$ be arbitrary lifts of P and C, let $\tilde{e}_i$ be the lift that makes $\tilde{S}_i \cup \tilde{C}$ balanced in Ω_i, and let $\tilde{H} = \{\tilde{e}_1, \ldots, \tilde{e}_m\}$. Then $\tilde{P} \cup \tilde{C}$ is balanced in $\Omega \iff \tilde{C} \cup \tilde{H}$ is balanced in Ω_P.*

Furthermore, let $\tilde{e}$ be any lift of e such that $\tilde{C} \cup \tilde{e}$ is balanced. Then $\tilde{P} \cup \tilde{e}$ is balanced (in our definition given previously) $\iff \tilde{C} \cup \tilde{H} \cup \tilde{e}$ is balanced in Ω_P.

Proof. For the first part, when $\tilde{P} \cup \tilde{C}$ is balanced, from Ω_i we know every $\tilde{C} \cup \tilde{e}_i$ is balanced. Thus, $\tilde{e}_i \in \mathrm{bcl}_{\Omega_P} \tilde{C}$. By Lemma 2.2, $\tilde{C} \cup \tilde{H}$ is balanced.

Conversely, if $\tilde{C} \cup \tilde{H}$ is balanced, then each $\tilde{C} \cup \tilde{S}_i$ is balanced. Therefore, $\tilde{s}_i \in \mathrm{bcl}_\Omega(\tilde{C} \cup \tilde{P} \setminus \tilde{S})$. $\tilde{C} \cup \tilde{P} \setminus \tilde{S}$ is balanced because its only circle is $\tilde{C}$. It follows that $\tilde{C} \cup \tilde{P}$ is balanced.

For the second part, because we assume balance of $\tilde{C} \cup \tilde{e}$, $\tilde{P} \cup \tilde{e}$ is balanced $\iff$ $\tilde{P} \cup \tilde{C}$ is balanced in Ω. We can reformulate the statement as: $\tilde{P} \cup \tilde{C}$ is balanced (in Ω) $\iff$ $\tilde{C} \cup \tilde{H} \cup \tilde{e}$ is balanced (in Ω_P). The proof is like that of the first part. $\square$

Let $A = (P \cup D') \setminus f$. In case $f \in P \setminus C$, f lies in a subpath S_t corresponding to a chord e_t. If $f \in C$, we leave S_t and e_t undefined. Define $B = (D' \cup H) \setminus \{f, e_t\}$.

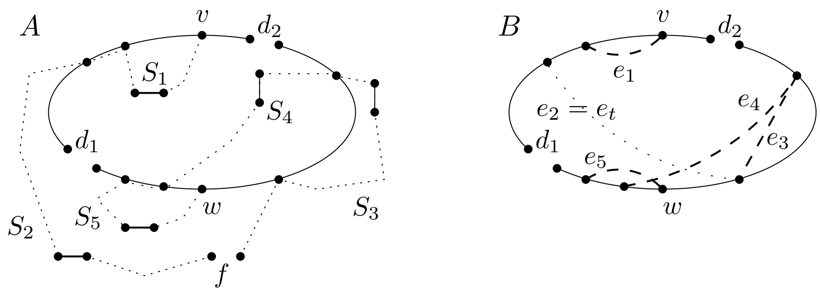


FIGURE 16. A and B for lifting f to balance $\tilde{R} \cup \tilde{e} \cup \tilde{f}$ (illustrating the case $f \notin C$). Edges $d_1, \ldots, d_k$ and f are absent from A, and edges $d_1, \ldots, d_k$ and e_t are absent from B, leaving each disconnected into one component that contains v and another that contains w.

Then each of A and B contains $R := P \setminus F$ but, due to the absence of D, f, and (when appropriate) e_t, remains disconnected into a v-component and a w-component. Adding in any one of e, d_i for $1 \leq i \leq k$, or f or e_t makes A and B connected (Fig. 16).

We were given $\tilde{R}$ and we can extend it (in Ω) to a balanced lift $\tilde{A}$. Each $\tilde{S}_i \subseteq \tilde{R}$, except when $i = t$, implies a unique $\tilde{e}_i$ for which $\tilde{S}_i \cup \tilde{e}_i$ is balanced (in Ω_i). Thus we have a balanced lift $\tilde{B}$ (in Ω_P) as well, uniquely defined. Now we add the given $\tilde{e}$ and take $\operatorname{bcl}_{\Omega_P}(\tilde{B} \cup \tilde{e})$. It contains exactly one lift $\tilde{d}_i$ for each i and one $\tilde{f}$ (if $f \in C$) or $\tilde{e}_t$ (if $f \notin C$), and it is balanced. Thus we have a balanced lift $\tilde{B} \cup \tilde{D} \cup \{\tilde{e}, \tilde{f}\}$ (if $f \in C$) or $\tilde{B} \cup \tilde{D} \cup \{\tilde{e}, \tilde{e}_t\}$ (if not), which in both cases is $\tilde{C} \cup \tilde{H} \cup \tilde{e}$. Moreover, the lifts $\tilde{D}$ and (if $f \in C$) $\tilde{f}$ are the only ones that give balance, by the circle lifting property in Ω_P. When $f \in C$, Lemma 3.15 shows that, not only is $\tilde{P} \cup \tilde{e}$ balanced, but $\tilde{f}$ is the only lift of f for which this is true. When $f \notin C$, we find $\tilde{f}$ as the unique edge in $p^{-1}(f) \cap \operatorname{bcl}_{\Omega_t}(\widetilde{S_t \setminus f} \cup \tilde{C} \cup \tilde{e}_t)$. Balance of $\tilde{P} \cup \tilde{e}$ follows from the second part of Lemma 3.15. In both cases, $\tilde{f}$ exists and is unique. $\square$

Lemma 3.16. *Ω' is a biased graph.*

Proof. We look at a theta graph that contains e. Let $P \cup e$ and $P' \cup e$ be its circles that contain e and $D = P \oplus P'$ the third circle.

Suppose, in a lift of $P \cup P'$, $\tilde{D}$ is balanced. Since $\tilde{P} \cup \tilde{P}'$ is balanced, we can choose $\tilde{P}_1$ so that $\tilde{P} \cup \tilde{P}' \cup \tilde{P}_1$ is balanced. Then for any $\tilde{e}$, $\tilde{P} \cup \tilde{e}$ is balanced $\iff$ $\tilde{P}_1 \cup \tilde{e}$ is balanced $\iff$ $\tilde{P}' \cup \tilde{e}$ is balanced. That is, one or three circles in $\tilde{P} \cup \tilde{P}' \cup \tilde{e}$ are balanced. Thus $\tilde{e}$ is unique due to Lemma 3.11.

Suppose, however, that $\tilde{D}$ is not balanced. Take $f \in P \setminus P'$ and replace $\tilde{f} \in \tilde{P}$ by $\tilde{f}^0$ such that $\tilde{P}^0 \cup \tilde{P}_1$ is balanced. (No other lift edges are altered.) By the first part, $\tilde{P}^0 \cup \tilde{e}$ is balanced $\iff$ $\tilde{P}' \cup \tilde{e}$ is balanced. If both are balanced, then $\tilde{P} \cup \tilde{e}$ is unbalanced by Lemma 3.14, so only one circle is balanced in $\tilde{P} \cup \tilde{P}' \cup \tilde{e}$. On the other hand, if neither is balanced, then $\tilde{D}$ and $\tilde{P}' \cup \tilde{e}$ are unbalanced, so at most one circle is balanced in $\tilde{P} \cup \tilde{P}' \cup \tilde{e}$. □

The combination of Lemmas 3.14 and 3.16 proves Proposition 3.9 . □

Call a graph *theta-complete* if the trivalent nodes of any theta subgraph are adjacent. The *theta completion* $\theta(\Delta)$ of a simple graph Δ is the smallest theta-complete simple graph that contains Δ. The results of this section imply:

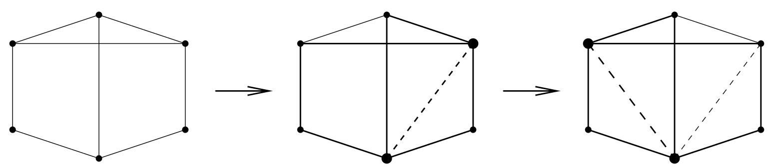


FIGURE 17. The theta completion $\theta(\Delta)$ (solid and dashed lines) of a graph Δ (solid lines). Heavy lines show a theta subgraph that leads to each added edge.

Theorem 3.17. *A biased expansion of a simple graph Δ extends uniquely to $\theta(\Delta)$. If Δ is the base graph of a maximal biased expansion, then Δ is theta-complete.* □

A theta-complete graph has a simple structure: see Proposition 6.4. We may conclude from its structure that, to construct $\theta(\Delta)$, it is sufficient to find all non-adjacent pairs $v, w \in V$ that are the trivalent nodes of a theta subgraph of Δ and adjoin the edges e_{vw}. The resulting graph Δ' is $\theta(\Delta)$; it is not necessary to look for new theta subgraphs in Δ' and repeat the adjunction process. Hence, when extending a biased expansion $\Omega \downarrow \Delta$, one gets the maximal extension by finding Δ' and extending Ω (as in Proposition 3.8) independently to every edge of $E(\Delta') \setminus E(\Delta)$; after that, no further step is needed.

4. INESCAPABLE GROUPS (3-CONNECTION)

Our extension results imply a strong characterization of biased expansions of well-connected graphs.

Theorem 4.1. *Every biased expansion of a 3-connected graph of order at least four is a group expansion. The group is unique.*

The graph being expanded may have finite or infinite order.

Lemma 4.2. *A biased expansion of a complete graph of any finite or infinite order not less than four is a group expansion by a unique group.*

Proof of Lemma. Let K be the complete graph and Ω its biased expansion.

In the finite case the lemma is a consequence of the theorem of "generalized associativity" stated by Belousov [3] and proved by Hosszú [22], Aczél, Belousov, and Hosszú [1, Theorem 1], and Belousov [4] (see [10, pp. 76–78]), and independently proved by Kahn and Kung [24, Section 7, pp. 490–492]. "Generalized associativity" states that, if a set has four quasigroup operations that satisfy $g_1(h_1(x_1, x_2), x_3) = g_2(x_1, h_2(x_2, x_3))$, all four operations are isotopic to the same associative quasigroup. (It follows that, if a finitary quasigroup factors into binary quasigroups in all possible ways, then the quasigroup is an iterated group isotope.) Aczél, Belousov, and Hosszú prove this by producing explicit isotopisms. Kahn and Kung construct four quasigroups that satisfy the same equation, from combinatorial data equivalent to a biased expansion of K_4, in such a way that they have identity elements; thus they are obviously equal, hence a group. Either way, it follows from generalized associativity that every restriction $\Omega\big|_{K'}$ to a K_4 subgraph $K' \subseteq K$ is a group expansion, say $\langle \mathfrak{G}_{K'} K' \rangle$. Since $\Omega\big|_{K''} \cong \langle \mathfrak{G}_{K'} K_3 \rangle$ for $K'' \subseteq K'$ of order three, and $\mathfrak{G}_{K'}$ is unique up to isomorphism (by a theorem of Bruck [9], or proved directly by Dowling [13, Theorem 8]), Kahn and Kung deduce that all $\mathfrak{G}_{K'}$ are isomorphic and $\Omega = \langle \mathfrak{G} K \rangle$ for a group $\mathfrak{G}$.

In essence, these approaches depend on interpreting $\Omega\big|_C$ for a spanning circle C of K' as encoding a quasigroup *multiplication*. Partly for completeness' sake and partly because it is such a natural way of deducing the group directly from the biased expansion, we give a new proof that depends on setting up the *division* operation of the group by means of a spanning star subgraph of K.

Let $v_0 \in N(K)$, and distinguish a balanced lift $\tilde{K}^0$ of K. Take a set $\mathfrak{Q}$ in one-to-one correspondence with each fiber $p^{-1}(e)$. Holding $\tilde{K}^0 \setminus v_0$ fixed, and letting v_1 be another node of K, each choice of edge $\tilde{e}_{01}$ implies by balance-closure one edge $\tilde{e}_{0j}$ for each $j \neq 0, 1$ such that the lift $\tilde{K}^0 \supseteq \tilde{K}^0 \setminus v_0$ is balanced. This determines bijections $\zeta_j : p^{-1}(e_{01}) \to p^{-1}(e_{0j})$. We define ψ_1 to be any one bijection $\psi_1 : p^{-1}(e_{01}) \to \mathfrak{Q}$ and $\psi_j : p^{-1}(e_{0j}) \to \mathfrak{Q}$ to be the bijection $\zeta_j^{-1} \circ \psi_1$. We also define, for each ordered pair of distinct $i, j \neq 0$,

$$\varepsilon = \psi_j(\tilde{e}^0_{0j}) \quad \text{for all} \quad j \neq 0.$$

This will serve as the group identity.

We have now labelled (from $\mathfrak{Q}$) all edges $\tilde{e}_{0i}$. The next task is to label all $\tilde{e}_{ij}$. We define

$$\psi_{ij}(\tilde{e}_{ij}) = \psi_i(\tilde{e}_{0i}) \;\text{ if }\; \tilde{e}_{0i}\tilde{e}^0_{0j}\tilde{e}_{ij} \text{ is balanced.} \tag{4.1}$$

In particular, $\psi(\tilde{e}^0_{ij}) = \psi_i(\tilde{e}^0_{0i})$ since $\tilde{e}_{0i}\tilde{e}^0_{0j}\tilde{e}^0_{ij}$ is balanced when $\tilde{e}_{0i} = \tilde{e}^0_{0i}$, so $\psi_{ij}(\tilde{e}^0_{ij}) = \varepsilon$. Finally, we define division. Actually, we define an operation $(\alpha/\beta)_{ij}$ for each ordered pair of distinct $i, j \neq 0$ by

$$(\alpha/\beta)_{ij} = \psi_{ij}(\tilde{e}_{ij}) \;\text{ if }\; \psi_i^{-1}(\alpha)\psi_j^{-1}(\beta)\tilde{e}_{ij} \;\text{ is balanced.}$$

These definitions are illustrated in Figure 18. Since $\tilde{e}_{0i}\tilde{e}_{0j}\tilde{e}^0_{ij}$ is balanced when $\psi_i(\tilde{e}_{0i}) = \psi_j(\tilde{e}_{0j})$, writing α for this latter value we find that

$$(\alpha/\alpha)_{ij} = \varepsilon \quad \text{for every } \alpha \in \mathfrak{Q}.$$

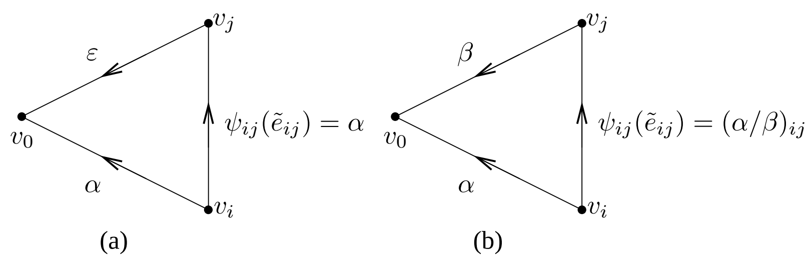


FIGURE 18. (a) The way an edge $\tilde{e}_{ij}$ is labelled. (b) The definition of $(\alpha/\beta)_{ij}$. All triangles are balanced.

The next step is to prove that division is independent of the first subscript:

$$(\alpha/\beta)_{ik} = (\alpha/\beta)_{jk}. \tag{4.2}$$

Look at Figure 19(a): If the K_4 is balanced with edges at v_0 labelled α, α, β, the labels on $\triangle v_i v_j v_k$ are as shown. Keeping this triangle, change the v_0 edges to those labelled as in Figure 19(b). The label ε on $\tilde{e}_{ij}$ implies that $\gamma' = \gamma$. The definition (4.1) implies that $\gamma = (\alpha/\beta)_{ik}$ and $\gamma' = (\alpha/\beta)_{jk}$. Thus (4.2) is proved.

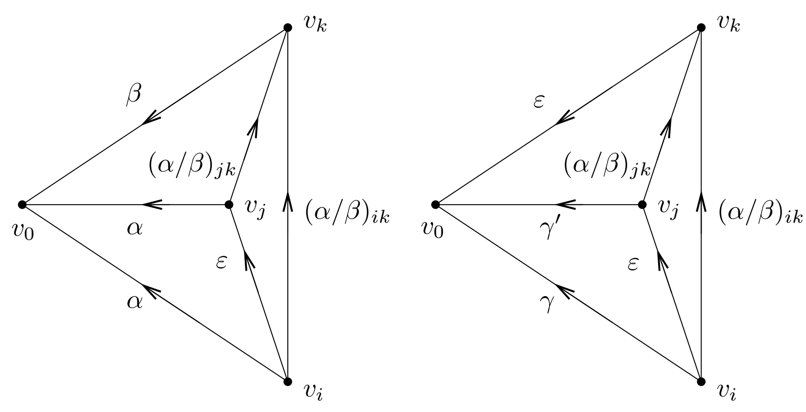


FIGURE 19. Illustrating the proof of (4.2). The graphs are balanced.

Another consequence of the definition of division is the reversal property $(\alpha/\beta)_{ij} = (\beta/\alpha)_{ji}$. Assuming there are at least four nodes and applying (4.2) thrice,

$$(\alpha/\beta)_{ij} = (\alpha/\beta)_{kj} = (\beta/\alpha)_{jk} = (\beta/\alpha)_{ik} = (\alpha/\beta)_{ki} = (\alpha/\beta)_{ji}.$$

Thus, all $(\alpha/\beta)_{ij}$ are equal: we have a single well-defined operation α/β.

From the value $(\alpha/\alpha)_{ij} = \varepsilon$ and the labelling rule for $\tilde{e}_{ij}$ we have

(L1) $\alpha/\alpha = \varepsilon$,
(L2) $\alpha/\varepsilon = \alpha$.

By the reversal property, $(\varepsilon/(\beta/\gamma))_{ij} = ((\beta/\gamma)/\varepsilon)_{ji} = (\beta/\gamma)_{ji} = (\gamma/\beta)_{ij}$, so

(L3) $\varepsilon/(\beta/\gamma) = \gamma/\beta$.

These are three of the four axioms for a group defined by division, given in [21, p. 6]. It remains to prove that

(L4) $(\alpha/\gamma)/(\beta/\gamma) = \alpha/\beta$.

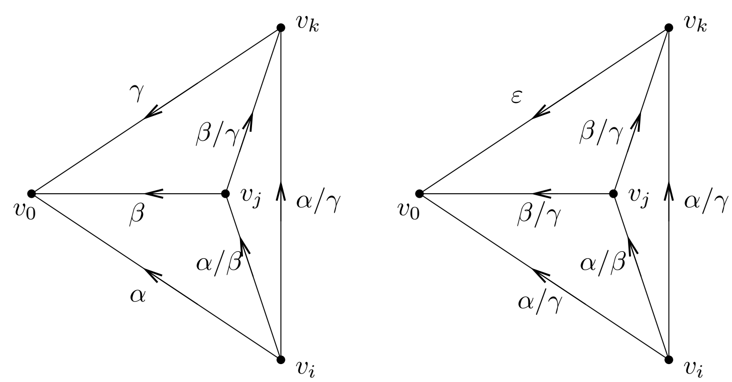


FIGURE 20. Diagrams for the proof of property (L4) expressing the product of quotients.

Again, we use two diagrams: see Figure 20. Diagram (a) is just definitions. Holding $\triangle v_i v_j v_k$ fixed, we change the edge labels at v_0 so that $\tilde{e}_{0k}$ has label ε. The labels on $\tilde{e}_{0i}$ and $\tilde{e}_{0j}$ are from the definition of division. Then $\tilde{e}_{ij}$ has label $(\alpha/\gamma)/(\beta/\gamma)$, but we already know its label is α/β. That proves (L4).

Therefore, $\mathfrak{Q}$ is a group, and it is easy to verify that $\Omega = \langle \mathfrak{Q} K \rangle$. □

Proof of Theorem 4.1. Let $\Omega \downarrow \Delta$ where Δ is 3-connected. By Example 3.1 we may assume Δ is simple. If v and w are nonadjacent nodes in Δ, they are the trivalent nodes of a theta subgraph of Δ. By Propositions 3.8 and 3.9 and Theorem 3.2, Ω extends to Ω', an expansion of the complete graph on $N(\Delta)$. By Lemma 4.2, then, Ω' is a group expansion; hence, so is Ω. □

5. Amalgamation (2-separation)

Biased expansions of the same multiplicity can be assembled by an analog of the ordinary graphical operation of edge amalgamation. This operation is essential to the structure theory of biased expansions. Besides that, it enables us to produce nongroup expansions out of group expansions, in two different ways. The easy way is to combine expansions by different (quasi)groups of the same order. For instance, in multiplicity 4 we can assemble a $\mathbb{Z}_4$-expansion and a $\mathfrak{V}_4$-expansion, $\mathfrak{V}_4$ being the Klein four-group. A more sophisticated kind of application combines expansions by the same group but with a nasty twist.

The first task is to define and justify the method of combination.

If a graph Δ is the union of two subgraphs, Δ_1 and Δ_2, that have in common only a link e and its endpoints, i.e. $\Delta_1 \cap \Delta_2 = (N(e), \{e\})$, we say Δ is the *edge amalgamation* (or *parallel connection*) of Δ_1 and Δ_2 *along* e, written $\Delta_1 \cup_e \Delta_2$, and we call $\Delta \setminus e$ the *edge sum* (or 2-*sum*) of Δ_1 and Δ_2 along e, written $\Delta_1 \oplus_e \Delta_2$. Another way to look at edge amalgamation or edge sum is as identification or cancellation of distinct links $e_1 \in E(\Delta_1)$ and $e_2 \in E(\Delta_2)$. We shall sometimes take this point of view.

These constructions can be modelled in biased expansions. Suppose Ω_1 and Ω_2 are biased expansions of Δ_1 and Δ_2. We construct an *expanded edge amalgamation* of Ω_1 and Ω_2 *along* e, written $\Omega_1 \cup_e \Omega_2$ or in full $\Omega_1 \cup_{e,\beta} \Omega_2$, by choosing a bijection $\beta : p_1^{-1}(e) \to p_2^{-1}(e)$ and using it to identify $p_1^{-1}(e)$ with $p_2^{-1}(e)$. The edge set of $\Omega_1 \cup_e \Omega_2$ is thus the disjoint union of $E(\Omega_1)$ and $E(\Omega_2)$ with $p_1^{-1}(e)$ and $p_2^{-1}(e)$ identified by β. A circle $\tilde{C}$ in $\Omega_1 \cup_e \Omega_2$ is balanced if it belongs to $\mathcal{B}(\Omega_1) \cup \mathcal{B}(\Omega_2)$ or it has the form $\tilde{C}_1 \cup \tilde{C}_2 \setminus [p_1^{-1}(e) \cup p_2^{-1}(e)]$ where $\tilde{C}_i \in \mathcal{B}(\Omega_i)$ and $e \in p(\tilde{C}_i)$ for $i = 1, 2$ and $\beta(\tilde{C}_1 \cap p_1^{-1}(e)) = \tilde{C}_2 \cap p_2^{-1}(e)$. (We may write $\tilde{C}$ more simply as $\tilde{C}_1 \oplus \tilde{C}_2$ if we bear in mind the identification of $p_1^{-1}(e)$ with $p_2^{-1}(e)$.) The *expanded edge sum* along e is $\Omega_1 \oplus_e \Omega_2 := \Omega_1 \oplus_{e,\beta} \Omega_2 := (\Omega_1 \cup_e \Omega_2) \setminus p^{-1}(e)$, p being the projection mapping of $\Omega_1 \cup_e \Omega_2$. Both constructions apply to group expansions $\mathfrak{G}_1\Delta_1$ and $\mathfrak{G}_2\Delta_2$ by taking $\Omega_i = \langle \mathfrak{G}_i\Delta_i \rangle$.

An example is any biased expansion Ω of $\Delta_1 \cup_e \Delta_2$. If $\Omega_i = p^{-1}(\Delta_i)$ and β is the identity map, then $\Omega_1 \cup_e \Omega_2 = \Omega$. On the other hand, a biased expansion of $\Delta_1 \oplus_e \Delta_2$ need not be an expanded edge sum $\Omega_1 \oplus_e \Omega_2$: see Example 5.4.

Figure 21 shows an expanded edge amalgamation of two group expansions $\mathbb{Z}_4C_4$ and $\mathbb{Z}_4C_3$, which is not itself a group expansion because the bijection β is not a pseudoisomorphism (see Theorem 5.3). The base graph of the amalgamation is C_5 with a chord e. Figure 22 shows the construction of an expanded edge amalgamation of two quasigroup expansions of a triangle when the two quasigroups happen to be the same.

Theorem 5.1. *Let $\Delta = \Delta_1 \cup_e \Delta_2$ or $\Delta_1 \oplus_e \Delta_2$, the amalgamation or sum along e of graphs Δ_1 and Δ_2, and let Ω_1 and Ω_2 be biased expansions of Δ_1 and Δ_2 such that $\#p_1^{-1}(e) = \#p_2^{-1}(e)$. Any expanded edge amalgamation $\Omega_1 \cup_e \Omega_2$ or expanded*

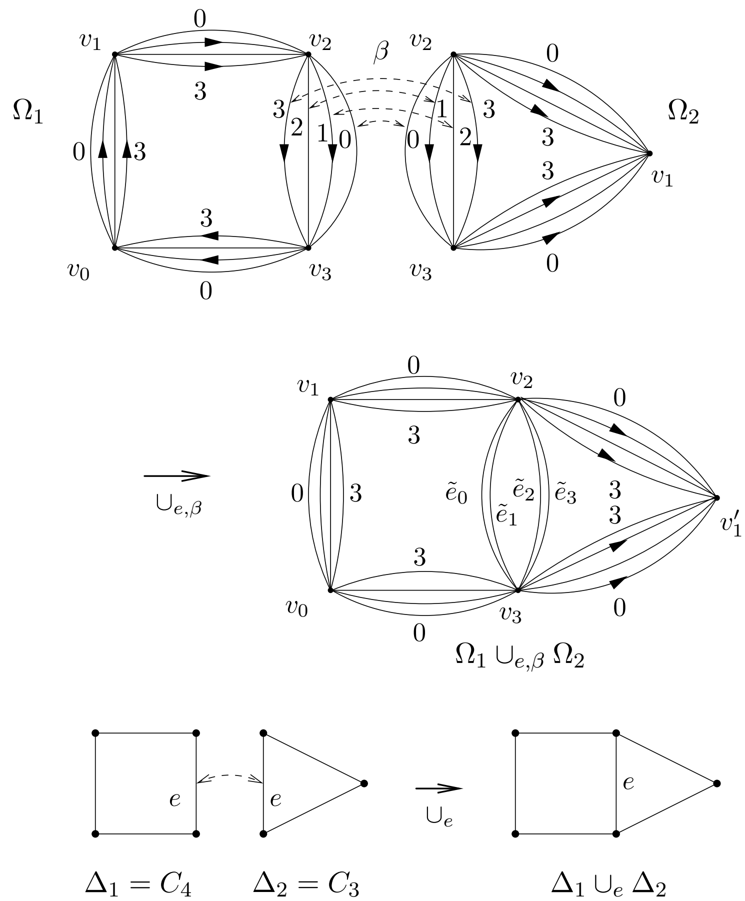


FIGURE 21. A non-group biased expansion Ω (upper graphs) of $C_5 \cup e$, where e is a chord, obtained by expanded edge amalgamation of two group expansions, $\Omega_1 = \langle \mathbb{Z}_4 C_4 \rangle$ and $\Omega_2 = \langle \mathbb{Z}_4 C_3 \rangle$. The expanded edge amalgamation is $\Omega := \Omega_1 \cup_{e,\beta} \Omega_2$; the base graph Δ is the edge amalgamation $C_4 \cup_e C_3 = C_5 \cup e$. The bijection β of the amalgamation is indicated by the dashed arrows. Ω is not gainable because β is not a pseudoisomorphism. The expanded 2-sum $\Omega_1 \oplus_{e,\beta} \Omega_2$ is Ω with the $\tilde{e}$ edges deleted; it is a non-group biased expansion of $C_4 \oplus_e C_3 = C_5$.

edge sum $\Omega_1 \oplus_e \Omega_2$ is a biased expansion of Δ. If Δ is the edge amalgamation, then Ω_1 and $\Omega_2 \subseteq \Omega_1 \cup_e \Omega_2$. If Δ is the edge sum and e is not an isthmus in either Δ_1 or Δ_2, then Ω_1 and Ω_2 are expansion minors of $\Omega_1 \oplus_e \Omega_2$.

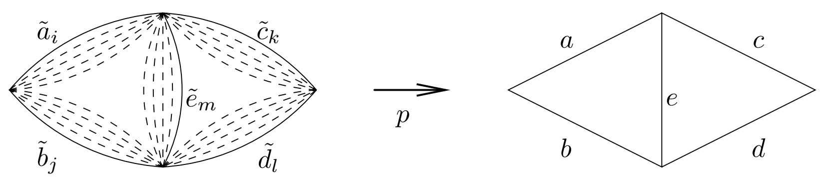


Figure 22. An expanded edge amalgamation $(\mathfrak{Q} \cdot abe) \cup_e (\mathfrak{Q} \cdot cde) \downarrow (abe) \cup_e (cde)$ of two expansions of K_3 by the same (binary) quasigroup $\mathfrak{Q}$. A triangle (solid lines) $\tilde a_i \tilde b_j \tilde e_m$ $(i, j, m \in \mathfrak{Q})$ in $\mathfrak{Q} \cdot abe$ (or, $\tilde c_k \tilde d_l \tilde e_m$ in $\mathfrak{Q} \cdot cde$) is balanced if and only if $i \cdot j = m$ (or, $k \cdot l = m$) in $\mathfrak{Q}$. A quadrilateral $\tilde a_i \tilde b_j \tilde c_k \tilde d_l$ is balanced if and only if there is an edge $\tilde e_m$ such that both $\tilde a_i \tilde b_j \tilde e_m$ and $\tilde c_k \tilde d_l \tilde e_m$ are balanced, which will happen if and only if $i \cdot j = k \cdot l$.

Proof. We show first that $\Omega = \Omega_1 \cup_e \Omega_2$ is a biased graph and a biased expansion of $\Delta_1 \cup_e \Delta_2$. For convenience of notation we assume that the identification prescribed by β has been carried out.

Suppose $\tilde C_1 \cup \tilde C_2$ is a theta graph in Ω and $\tilde C_1$ and $\tilde C_2$ are balanced in Ω. We want $\tilde C_1 \oplus \tilde C_2$ to be balanced. If $\tilde C_1 \cup \tilde C_2 \subseteq \Omega_i$, this will be so. There are two ways $\tilde C_1 \cup \tilde C_2$ may not be in Ω_1 or Ω_2: one of its three constituent paths may be an edge $\tilde e \in p^{-1}(e)$, or $\tilde C_1 \cup \tilde C_2$ may be disjoint from $p^{-1}(e)$. In the first case $\tilde C_i \setminus \tilde e$ is a path in Ω_i and $\tilde C_1 \oplus \tilde C_2$ is balanced by the definition of $\mathcal{B}(\Omega)$. In the second case, one circle is contained in an Ω_i, say $\tilde C_1 \subseteq \Omega_1$; then $\tilde C_2$ lies partly in Ω_1 and partly in Ω_2. Because $\tilde C_2$ is balanced, it must be the sum $\tilde C_1' \oplus \tilde C_2'$ of balanced circles $\tilde C_i' \subseteq \Omega_i$ that contain an edge $\tilde e \in p^{-1}(e)$. Then $\tilde C_1 \cup \tilde C_1'$ is a theta graph in Ω_1 and is the union of balanced circles; thus $\tilde C_1 \oplus \tilde C_1'$ is balanced. Hence $(\tilde C_1 \oplus \tilde C_1') \oplus \tilde C_2'$ is balanced, and this equals $\tilde C_1 \oplus \tilde C_2$. We have proved that Ω is a biased graph.

Given a circle C in Δ, $f \in C$, and a lift $\tilde P$ of $P = C \setminus f$ into Ω, we want to prove there is one and only one $\tilde f \in p^{-1}(f)$ that makes $\tilde P \cup \tilde f$ balanced. If $C \subseteq \Delta_i$ there is nothing to prove, so we assume $C = P_1 \cup P_2$ where P_i is a path in Δ_i with endpoints $N(e)$ and that $f \in P_2$. Let $C_i = P_i \cup e$. Then P_1 lifts to $\tilde P_1 \subseteq \tilde P$ and $P_2 \setminus f$ lifts to $\tilde Q_2 \subseteq \tilde P$. There is a unique $\tilde e$ for which $\tilde P_1 \cup \tilde e$ is balanced. Then there is just one $\tilde f$ for which $\tilde Q_2 \cup \{\tilde e, \tilde f\}$ is balanced. Now we have two balanced circles, $\tilde P_1 \cup \tilde e$ and $\tilde Q_2 \cup \{\tilde e, \tilde f\}$, whose union is a theta graph with $\tilde e$ as one constituent path; the other paths form a circle $\tilde P \cup \tilde f$, balanced by the definition of $\mathcal{B}(\Omega)$, that projects to C. Hence $\tilde f$ exists as desired. Its uniqueness is obvious.

The remaining part that is not obvious is that Ω_1 is a minor of $\Omega_1 \oplus_e \Omega_2$. Because e is not an isthmus in Δ_2, there is a circle C in Δ_2 that contains e and an arbitrary other link f_2. Let $C = eQ_2' f_2 Q_2''$, lift $Q_2' \cup Q_2''$ arbitrarily to $\tilde Q_2$, and form the subgraph $\Omega_{f_2} = \Omega_1 \cup p^{-1}(f_2) \cup \tilde Q_2$. We prove that $\Omega_1 \cong (\Omega_{f_2} \setminus p^{-1}(e))/\tilde Q_2$ by the isomorphism ε_1 that is the identity on $\Omega_1 \setminus p^{-1}(e)$ and is defined on $p^{-1}(e)$ by $\varepsilon_1(\tilde{e}) =$ that edge $\tilde{f}_2$ for which $\{\tilde{e}, \tilde{f}_2\} \cup \tilde{Q}_2$ is balanced. What has to be proved is that, for a circle $\tilde{C} \subseteq E(\Omega_1)$, $\tilde{C}$ is balanced if and only if $\varepsilon_1(\tilde{C})$ is balanced. Let $\tilde{C} \cap p^{-1}(e) = \{\tilde{e}\}$. Then $\tilde{C} \cup \tilde{Q}_2 \cup \{\varepsilon_1(\tilde{e})\}$ is a theta graph in which $\tilde{Q}_2 \cup \{\tilde{e}\} \cup \{\varepsilon_1(\tilde{e})\}$ is balanced. The conclusion follows. $\square$

Theorem 5.1 allows us to produce arbitrarily large biased expansions that are not group expansions, of any multiplicity $\gamma \geq 4$.

Example 5.1. Let $\Delta = \Delta_1 \oplus_e \Delta_2$, where Δ_1 and Δ_2 are 2-connected of order at least 3, and let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be different groups of the same order γ. Form $\Omega_i = \langle \mathfrak{G}_i \Delta_i \rangle$. Any bijection $\mathfrak{G}_1 \to \mathfrak{G}_2$ induces a bijection $\beta : p_1^{-1}(e) \to p_2^{-1}(e)$ by which we can form an expanded edge sum $\Omega_1 \oplus_e \Omega_2$. The sum has as minors both $\langle \mathfrak{G}_1 \Delta_1 \rangle$ and $\langle \mathfrak{G}_2 \Delta_2 \rangle$, and these in turn have minors $\langle \mathfrak{G}_1 K_3 \rangle$ and $\langle \mathfrak{G}_2 K_3 \rangle$. If $\Omega = \langle \mathfrak{G} \Delta \rangle$, then all triangular minors are isomorphic to $\langle \mathfrak{G} K_3 \rangle$, but this is impossible. Therefore Ω is a nongroup regular biased expansion of Δ. Note that this construction cannot be carried out for prime multiplicities γ.

Example 5.2. In the preceding construction take $\Delta_2 = K_3$ and let Ω_2 be any quasigroup expansion of K_3 having multiplicity γ but not isomorphic to $\langle \mathfrak{G}_1 K_3 \rangle$. Then $\Delta = \Delta_1 \oplus_e K_3$ has a regular biased expansion with nonisomorphic triangular minors $\langle \mathfrak{G}_1 \Delta_1 \rangle$ and Ω_2, so it is a nongroup regular biased expansion of Δ. This construction can be carried out for all multiplicities $\gamma \geq 4$.

The technique of summing with quasigroup expansions of a triangle yields highly nongainable biased expansions of series-parallel graphs, just to mention a sizeable class to which it applies. The reason is that every series-parallel graph Δ is constructed by doubling edges in parallel, an operation that is trivial to reproduce in a biased expansion of Δ (see Example 3.1), and by subdividing edges, which is equivalent to taking an edge sum with a triangle. On the other hand, the methods of Example 5.1 and 5.2 together still do not give non-group biased expansions with all multiplicities $\gamma \geq 4$ of all 2-separable inseparable graphs. For that see Corollary 6.7.

Example 5.3. In Example 5.1, take $\mathfrak{G}_1 = \mathfrak{G}_2 = \mathfrak{G}$. Then a bijection $\beta : \mathfrak{G} \to \mathfrak{G}$ produces a $\mathfrak{G}$-expansion of the base graph if β is a pseudoisomorphism, or a nongroup biased expansion if β is not a pseudoisomorphism. Figure 21 shows an example of the latter type.

Example 5.4. Here is an example of a biased expansion of $\Delta_1 \oplus_e \Delta_2$ that is not an expanded edge sum of expansions of Δ_1 and Δ_2. In the example, $\Delta_1 \cong \Delta_2 \cong K_3$.

Take $C_4 = (N, E)$ where $N = \{v_1, v_2, v_3, v_4\}$ and $E = \{e_{12}, e_{23}, e_{34}, e_{41}\}$. C_4 is an edge sum in two different ways: it is $\Delta_{123} \oplus_{e_{13}} \Delta_{134}$ and $\Delta_{124} \oplus_{e_{24}} \Delta_{234}$. Here Δ_{ijk} denotes the triangle with node set $\{v_i, v_j, v_k\}$. Let $\gamma \geq 4$ and let $\gamma \cdot \Delta_{123}$ and $\gamma \cdot \Delta_{134}$ be biased expansions that are not both group expansions by the same group. (That is, one or both is not a group expansion, or $\gamma \cdot \Delta_{123} = \langle \mathfrak{G} \Delta_{123} \rangle$ and

$\gamma \cdot \Delta_{134} = \langle \mathfrak{H}\Delta_{134} \rangle$ where $\mathfrak{G} \not\cong \mathfrak{H}$.) Then $\Omega = (\gamma \cdot \Delta_{123}) \oplus_{e_{13}} (\gamma \cdot \Delta_{134})$ is a biased expansion of C_4; also, $\Omega_{13} = (\gamma \cdot \Delta_{123}) \cup_{e_{13}} (\gamma \cdot \Delta_{134})$ is a biased expansion of $K_4 \setminus e_{24}$. Thus, Ω extends to e_{13}.

Although $C_4 = \Delta_{124} \oplus_{e_{24}} \Delta_{234}$, Ω cannot be an expanded edge sum of the form $(\gamma \cdot \Delta_{124}) \oplus_{e_{24}} (\gamma \cdot \Delta_{234})$. We prove this by contradiction. Suppose it were; then Ω would extend to e_{24}. By Proposition 3.3, it extends to $\gamma \cdot K_4$ having as minors both $\gamma \cdot \Delta_{123}$ and $\gamma \cdot \Delta_{134}$. These are not isomorphic, but by Lemma 4.2 $\gamma \cdot K_4$ is a group expansion and therefore all its triangular minors are isomorphic. We have a contradiction.

One wants to know that a multiple edge amalgamation is independent of the order of amalgamation. It suffices to treat two amalgamations.

Theorem 5.2. *Let $\Omega_i \downarrow \Delta_i$ for $i = 1, 2, 3$, where*

$$E(\Delta_1 \cap \Delta_2) = \{e\}, \quad E(\Delta_2 \cap \Delta_3) = \{f\}, \quad E(\Delta_1 \cap \Delta_3) = \{e\} \cap \{f\},$$

and Δ_1, Δ_2, Δ_3 are pairwise node-disjoint except as required by shared edges. Suppose given bijections $\alpha : p_1^{-1}(e) \to p_2^{-1}(e)$ and $\beta : p_3^{-1}(f) \to p_2^{-1}(f)$. Then

$$\Omega_1 \cup_{e,\alpha} (\Omega_2 \cup_{f,\beta} \Omega_3) = (\Omega_1 \cup_{e,\alpha} \Omega_2) \cup_{f,\alpha} \Omega_3. \tag{5.1}$$

Proof. The only question is the balance of circles in the amalgamation. Let Ω_L and Ω_R be the biased expansions of $\Delta_1 \cup_e \Delta_2 \cup_f \Delta_3$ on the left and right sides of (5.1). Consider a circle $\tilde{C}$ that meets both $\Omega_1 \setminus p_1^{-1}(e)$ and $\Omega_3 \setminus p_3^{-1}(f)$; thus $\tilde{C}$ consists of $\tilde{P}_1 = \tilde{C} \cap E(\Omega_1)$, $\tilde{P}_3 = \tilde{C} \cap E(\Omega_3)$ (both of which are paths) and $\tilde{Q} = \tilde{C} \cap E(\Omega_2)$. The latter may consist of two, one, or (if $N(e) = N(f)$) no paths.

We may use α and β to identify $p_1^{-1}(e)$ with $p_2^{-1}(e)$ and $p_2^{-1}(f)$ with $p_3^{-1}(f)$.

The case $e = f$ is easy, since we are really looking at $\Omega_1 \cup_{e,\alpha\circ\beta^{-1}} \Omega_3$ in both Ω_L and Ω_R.

When $e \neq f$, choose $\tilde{e}$ and $\tilde{f}$ so $\tilde{P}_1 \cup \tilde{e}$ and $\tilde{P}_3 \cup \tilde{f}$ are balanced in Ω_1 and Ω_3, respectively. Then $\tilde{C}$ is balanced in $\Omega_L \iff \tilde{P}_3 \cup \tilde{Q} \cup \tilde{e}$ is balanced in $\Omega_2 \cup_{f,\beta} \Omega_3$ (because $\tilde{P}_1 \cup \tilde{e}$ is balanced) $\iff \tilde{Q} \cup \{\tilde{e}, \tilde{f}\}$ is balanced in Ω_2 (because $P_3 \cup \tilde{f}$ is balanced). Similarly, $\tilde{C}$ is balanced in $\Omega_R \iff \tilde{Q} \cup \{\tilde{e}, \tilde{f}\}$ is balanced in Ω_2. It follows that balance of $\tilde{C}$ is the same in Ω_L and Ω_R. □

The theorem implies that one can define a multiple expanded edge amalgamation directly, even one with an infinite number of amalgamations, because defining balance of any particular circle $\tilde{C}$ in the result only involves a finite number of amalgamations, so is order independent by Theorem 5.2 and induction. For more on the definition of multiple amalgamation see after Theorem 6.2.

When we amalgamate two group expansions, whether we get a group expansion or not depends on the nature of the identification function β. As a mapping $p_1^{-1}(e) \to p_2^{-1}(e)$, β induces a mapping of groups by composition with the gain functions, namely

$$\bar{\beta} = \left[\varphi_1\big|_{p_1^{-1}(e)}\right]^{-1} \circ \beta \circ \left[\varphi_2\big|_{p_2^{-1}(e)}\right] : \mathfrak{G}_1 \to \mathfrak{G}_2,$$

or in a more compact expression,

$$\bar{\beta}(\varphi_1(\tilde{e})) = \varphi_2(\beta(\tilde{e})) \qquad \text{for} \quad \tilde{e} \in p_1^{-1}(e). \tag{5.2}$$

Note that $\bar{\beta}$ depends on the choice of gains; if we used different gain functions φ_i' we would get a different bijection $\bar{\beta}'$.

We shall need to know the effect on $\bar{\beta}$ of switchings η_i and group automorphisms α_i applied to Φ_1 and Φ_2. We write $\varphi_i' = \varphi_i^{\eta_i \alpha_i}$. The definition of $\bar{\beta}'$, in full, is

$$\begin{aligned}\varphi_2'(\beta(\tilde{e})) &= [\eta_2(v)^{-1}\varphi_2(\beta(\tilde{e}))\eta_2(w)]^{\alpha_2} \\ &= \eta_2(v)^{-\alpha_2}\bar{\beta}(\varphi_1(\tilde{e}))^{\alpha_2}\eta_2(w)^{\alpha_2}.\end{aligned}$$

Since $\varphi_1^{\eta_1\alpha_1}(\tilde{e}) = [\eta_1(v)^{-1}\varphi_1(\tilde{e})\eta_1(w)]^{\alpha_1}$, we can substitute

$$\varphi_1(\tilde{e}) = \eta_1(v)[\varphi_1'(\tilde{e})]^{\alpha_1^{-1}}\eta_1(w)^{-1}$$

in the previous equation, getting

$$\bar{\beta}'(\varphi_1'(\tilde{e})) = \varphi_2'(\beta(\tilde{e})) = \eta_2(v)^{-\alpha_2}\bar{\beta}[\eta_1(v)\varphi_1'(\tilde{e})^{\alpha_1^{-1}}\eta_1(w)^{-1}]^{\alpha_2}\eta_2(w)^{\alpha_2}.$$

Here $\varphi_1'(\tilde{e})$ can be any group element; therefore we can rewrite the equation as

$$\bar{\beta}'(g) = \eta_2(v)^{-\alpha_2}\bar{\beta}[\eta_1(v)g^{\alpha_1^{-1}}\eta_1(w)^{-1}]^{\alpha_2}\eta_2(w)^{\alpha_2}. \tag{5.3}$$

A *pseudoisomorphism* of groups (or quasigroups) is any mapping $\mathfrak{G}_1 \to \mathfrak{G}_2$ that has the form $g \mapsto g^\alpha c$ where $\alpha : \mathfrak{G}_1 \to \mathfrak{G}_2$ is an isomorphism and $c \in \mathfrak{G}_2$. The (quasi)groups must be isomorphic for such a mapping to exist. The pseudoautomorphisms of a group form a group, which we denotate $\operatorname{PsAut}\mathfrak{G}$.

Theorem 5.3. *Let $\Delta = \Delta_1 \cup_e \Delta_2$, where Δ_1 and Δ_2 are 2-connected simple graphs, and let Ω_1 and Ω_2 be biased expansions of Δ_1 and Δ_2 with the same multiplicity. The expanded edge amalgamation $\Omega = \Omega_1 \cup_{e,\beta} \Omega_2$ and the expanded edge sum $\Omega_0 = \Omega_1 \oplus_{e,\beta} \Omega_2$ are group expansions (of Δ and $\Delta \setminus e$, respectively) if and only if $\Omega_1 = \langle \mathfrak{G}_1\Delta_1 \rangle$ and $\Omega_2 = \langle \mathfrak{G}_2\Delta_2 \rangle$ and β, after suitable switching of $\mathfrak{G}_1\Delta_1$ and $\mathfrak{G}_2\Delta_2$, induces an isomorphism $\mathfrak{G}_1 \to \mathfrak{G}_2$.*

If $\mathfrak{G}_1 \cong \mathfrak{G}_2$, the condition on β is equivalent to $\bar{\beta}$'s being a pseudoisomorphism $\mathfrak{G}_1 \to \mathfrak{G}_2$.

What we mean by suitable switching is that there exist switching functions θ_1 and θ_2 such that when β is applied to $(\mathfrak{G}_1\Delta_1)^{\theta_1}$ and $(\mathfrak{G}_2\Delta_2)^{\theta_2}$, then β induces an isomorphism $\mathfrak{G}_1 \to \mathfrak{G}_2$. In terms of the original, unswitched gains, the induced mapping is $\left[\varphi_1^{\theta_1}\big|_{p_1^{-1}(e)}\right]^{-1} \circ \beta \circ \left[\varphi_2^{\theta_2}\big|_{p_2^{-1}(e)}\right]$. We call β *twisted* if no such switchings exist, or equivalently if $\bar{\beta}$ is not a pseudoisomorphism, and in particular if the groups are not isomorphic in the first place.

Proof. This is one of those theorems that seem obvious but have a complicated proof. The beginning is easy: according to Theorem 5.1 the expanded edge amalgamation or sum can be a group expansion only if $\Omega_1 = \langle \mathfrak{G}\Delta_1 \rangle$ and $\Omega_2 = \langle \mathfrak{G}\Delta_2 \rangle$ for some group. Let us therefore assume this is so and write $\Phi_i = \mathfrak{G}\Delta_i$. What we need to prove is the equivalence of the following properties:

(i) Ω_0 is a $\mathfrak{G}$-expansion of $\Delta_0 = \Delta \setminus e$: that is, $\Omega_0 \cong \langle \mathfrak{G}\Delta_0 \rangle$.
(ii) $\bar{\beta}(g) = g^{\alpha} c$ for some $c \in \mathfrak{G}$ and $\alpha \in \operatorname{Aut}\mathfrak{G}$.
(iii) There are switchings $(\mathfrak{G}\Delta_1)^{\theta_1}$ and $(\mathfrak{G}\Delta_2)^{\theta_2}$ such that

$$\alpha = \left[\varphi_1^{\theta_1}\big|_{p_1^{-1}(e)}\right]^{-1} \circ \beta \circ \left[\varphi_2^{\theta_2}\big|_{p_2^{-1}(e)}\right]$$

is an automorphism of $\mathfrak{G}$.
(iv) $\Omega \cong \langle \mathfrak{G}\Delta \rangle$.

We show that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv).

Assume, then, that $\Omega_0 \cong \langle \Phi_0 \rangle$ where $\Phi_0 = \mathfrak{G}\Delta_0$, and also, by prior switchings η_i' of Φ_i to Φ_i' (for $i = 1, 2$), that

$$\varphi_1'(1e) = 1 \quad \text{and} \quad \varphi_2'(\beta(1e)) = 1. \tag{5.4}$$

We may choose $\eta_1' \equiv 1$ and $\eta_2'(v) = 1$. Now, if we take paths $\tilde{P}_i$ in Φ_i such that $\tilde{P}_1 \cup \{1e\}$ and $\tilde{P}_2 \cup \{\beta(1e)\}$ are balanced circles, then $\varphi_1'(\tilde{P}_1) = \varphi_1'(1e) = 1$ and $\varphi_2'(\tilde{P}_2) = \varphi_2'(\beta(1e)) = 1$. (Here we orient $1e$, $\beta(1e)$, $\tilde{P}_1$, and $\tilde{P}_2$ similarly, from one endpoint v of e to the other endpoint w.) By construction, $\tilde{P}_1 \cup \tilde{P}_2$ is balanced in Ω_0; thus $\varphi_0'(\tilde{P}_1) = \varphi_0'(\tilde{P}_2)$; consequently, we may assume by prior switching of Φ_0 that $\varphi_0\big|_{\tilde{P}_1} \equiv 1$ and $\varphi_0\big|_{\tilde{P}_2} \equiv 1$. Note, though, that φ_0 need not agree with φ_1' even though $\Omega_1 \setminus p_1^{-1}(e) \subseteq \Omega_0$, and the same for φ_2'.

Nevertheless, Ω_1 is isomorphic to a minor Ω_{01} of Ω_0 that can be found by following the proof of Theorem 5.1. In that proof choose $\tilde{f}_2 \in \tilde{P}_2$ and $\tilde{Q}_2 = \tilde{P}_2 \setminus \tilde{f}_2$. The proof constructs Ω_{01} with underlying graph $\|\Omega_0\| = \|\Omega_1\| \setminus p_1^{-1}(e) \cup p_2^{-1}(f_2)$. Because $\tilde{P}_2$ has all identity gains, the correspondence ε_1 preserves gains. Therefore, $\Omega_{01} = \langle \Phi_{01} \rangle$ where Φ_{01} is a minor of Φ_0 with gains $\varphi_{01} = \varphi_0\big|_{E(\Phi_{01})}$.

Since $\langle \Phi_1 \rangle \cong \langle \Phi_{01} \rangle$, by uniqueness of gains [37, Theorem V.2.1(c)] $\varphi_{01} = (\varphi_1')^{\eta_1 \alpha_1} \circ \varepsilon_1$, where η_1 is a switching function and $\alpha_1 \in \operatorname{Aut}\mathfrak{G}$. Without loss of generality we may assume that $\eta_1(v) = 1$. Then

$$1 = \varphi_0'(\tilde{P}_1) = (\varphi_1')^{\eta_1\alpha_1}(\tilde{P}_1) = [\eta_1(v)^{-1}\varphi_1'(\tilde{P}_1)\eta_1(w)]^{\alpha_1} = [1 \cdot 1 \cdot \eta_1(w)]^{\alpha_1}$$

implies $\eta_1(w) = 1$.

Similarly we construct $\Omega_{02} = \langle \Phi_{02} \rangle$, a minor of Ω_0 that is isomorphic to Ω_2 with $\varphi_{02} = (\varphi_2')^{\eta_2\alpha_2} \circ \varepsilon_2$ where $\eta_2(v) = \eta_2(w) = 1$.

Applying Equation (5.3) to the special circumstances of $\Phi_i^{\eta_i\alpha_i}$ where $\eta_i(v) = \eta_i(w) = 1$, we see that

$$\bar{\beta}'' = \alpha_1^{-1} \circ \bar{\beta}' \circ \alpha_2.$$

Consequently, $\bar{\beta}' \in \operatorname{Aut}\mathfrak{G} \iff \bar{\beta}'' \in \operatorname{Aut}\mathfrak{G}$.

The next step is to prove the (surprising) fact that $\bar\beta''$ is the identity. If $\tilde P_1 \cap p_1^{-1}(f_1) = \{\tilde f_1\}$, and if we write $\tilde f_1^* = \varepsilon_2(\tilde e)$ for $\tilde e \in p_1^{-1}(e)$ and $\tilde P_1^* = \tilde P_1 \setminus \{\tilde f_1\} \cup \{\tilde f_1^*\}$, then $\tilde P_1^* \cup \{\tilde e\}$ is balanced. Similarly, $\tilde P_2^* \cup \{\beta(\tilde e)\}$ is balanced, so $\tilde P_1^* \cup \tilde P_2^*$ is also balanced. It follows, from balance of each of these circles in turn, that

$$\begin{aligned}
\varphi_1''(\tilde e) &= \varphi_{01}(\tilde f_1^*) = \varphi_0(\tilde f_1^*),\\
\varphi_2''(\beta(\tilde e)) &= \varphi_{02}(\tilde f_2^*) = \varphi_0(\tilde f_2^*),\\
\varphi_0(\tilde f_1^*) &= \varphi_0(\tilde P_1^*) = \varphi_0(\tilde P_2^*) = \varphi_0(\tilde f_2^*),
\end{aligned}$$

where $\varphi_i'' = (\varphi_i')^{\eta_i\alpha_i}$. Hence, $\varphi_1''(\tilde e) = \varphi_2''(\beta(\tilde e))$. This means that $\bar\beta''$ is the identity mapping.

Therefore $\bar\beta'$ is an automorphism of $\mathfrak{G}$; in fact, $\bar\beta' = \alpha_1 \circ \alpha_2^{-1}$.

The course of the proof so far may be summarized in a diagram. In it, $\tilde\eta_i$ is the permutation of $p_i^{-1}(e)$ induced by η_i; that is, $(\varphi_i')^{\eta_i} = \tilde\eta_i \circ \varphi_i'$. (In the description and diagram φ_i', $\tilde\eta_i$, etc. stand for $\varphi_i'\big|_{p_i^{-1}(e)}$, etc.; so that all maps are bijections.) The first square is commutative because $\tilde\eta_i$ is the identity on $p_i^{-1}(e)$, a consequence of having $\eta_i(v) = \eta_i(w) = 1$. The triangles commute by the definition of $\tilde\eta_i$. The square $\varphi_1' \circ \bar\beta'$ vs. $\beta \circ \varphi_2'$ commutes by the definition of $\bar\beta'$ and the rectangle $(\varphi_1')^{\eta_1} \circ \alpha_1 \circ \bar\beta''$ vs. $\beta \circ (\varphi_2')^{\eta_2} \circ \alpha_2$ commutes by the definition of $\bar\beta''$. From this it follows that the entire diagram commutes; since $\bar\beta'' \in \operatorname{Aut}\mathfrak{G}$, then $\bar\beta' \in \operatorname{Aut}\mathfrak{G}$.

$$\begin{tikzcd}
p_1^{-1}(e) \arrow[r, "\tilde\eta_1"'] \arrow[d, "\beta"'] \arrow[rr, bend left, "(\varphi_1')_1^{\eta_1}"] & p_1^{-1}(e) \arrow[r, "\varphi_1'"'] \arrow[d, dashed, "\beta"'] & \mathfrak{G} \arrow[r, "\alpha_1"'] \arrow[d, dashed, "\bar\beta'"'] & \mathfrak{G} \arrow[d, dashed, "\bar\beta''"'] \\
p_2^{-1}(e) \arrow[r, "\tilde\eta_2"] \arrow[rr, bend right, "(\varphi_2')^{\eta_2}"'] & p_2^{-1}(e) \arrow[r, "\varphi_2'"] & \mathfrak{G} \arrow[r, "\alpha_2"] & \mathfrak{G}
\end{tikzcd}$$

The reason $\bar\beta'$ is an automorphism is that we did the right kind of switching. First we switched Φ_i by η_i' so that $\varphi_1'(1e) = 1$ and $\varphi_2'(\beta(1e)) = 1$, then we switched Φ_i' by η_i. The overall effect is that of switching Φ_1 by $\theta_1 = \eta_1$ and Φ_2 by $\theta_2 = \eta_2'\eta_2$. We also switched Φ_0, but that is unimportant because the gains on Φ_0 were not given in advance like those on $\Phi_1 = \mathfrak{G}\Delta_1$ and $\Phi_2 = \mathfrak{G}\Delta_2$.

Expressed in terms of the original gains φ_i, the definition of $\bar\beta'$ is $\bar\beta'(\varphi_1(\tilde e)) = \varphi_2^{\eta_2'}(\beta(\tilde e))$. Substituting the values of $\eta_2'(v)$ and $\eta_2'(w)$, this becomes

$$\bar\beta'(\varphi_1(\tilde e)) = \varphi_2(\beta(\tilde e))\eta_2'(w) = \bar\beta(\varphi_1(\tilde e))\eta_2'(w).$$

Setting $\tilde e = ge$, we see that $\bar\beta(g) = g^\alpha c$ for $\alpha = \bar\beta' \in \operatorname{Aut}\mathfrak{G}$ and $c = \eta_2'(w)^{-1} \in \mathfrak{G}$, thereby proving (ii) from (i).

We know (ii) $\Longrightarrow$ (iii) because we can produce the necessary switching functions: $\theta_1 \equiv 1$ for Φ_1 and θ_2 with $\theta_2(v) = 1$ and $\theta_2(w) = c^{-1}$ for Φ_2.

Proving (iii) $\implies$ (iv) is easy. We may assume $\mathfrak{G}\Delta_1$ and $\mathfrak{G}\Delta_2$ switched and α previously applied to Φ_1 so that, in effect, α becomes the identity. Then $\beta : p_1^{-1}(e) = \mathfrak{G} \times \{e\} \to p_2^{-1}(e) = \mathfrak{G} \times \{e\}$ is the identity, so the amalgamation is $\langle \mathfrak{G}_1\Delta_1 \rangle \cup_{e,\mathrm{id}} \langle \mathfrak{G}_2\Delta_2 \rangle$, which is simply $\langle \mathfrak{G}\Delta \rangle$. □

Theorem 5.3 helps answer some questions about the existence of biased expansions that do not have gains. One question is whether an expanded edge amalgamation or sum of two $\mathfrak{G}$-expansions is itself a group expansion. That depends in part on whether or not $\operatorname{Aut}\mathfrak{G}$ is the full symmetric group of $\mathfrak{G} \setminus \{1\}$.

Lemma 5.4 ([37, Corollary V.3.4])**.** *Assuming Δ is a block of order at least 3, $\operatorname{Aut}_p\langle \mathfrak{G}\Delta \rangle$ acts as the symmetric group on a fiber $p^{-1}(e)$ if and only if $\mathfrak{G} = \mathbb{Z}_\gamma$ for $\gamma \leq 3$ or $\mathfrak{G} = \mathfrak{V}_4$.*

Corollary 5.5. *Suppose Δ_1 and Δ_2 are 2-connected simple graphs of order at least 3. An expanded edge amalgamation or expanded edge sum of group expansions $\mathfrak{G}\Delta_1$ and $\mathfrak{G}\Delta_2$ is necessarily a group expansion if and only if $\mathfrak{G} = \mathbb{Z}_\gamma$ for $\gamma \leq 3$ or $\mathfrak{G} = \mathfrak{V}_4$; and then it is a $\mathfrak{G}$-expansion.*

Proof. This is immediate from Lemma 5.4, which tells us that it is possible to find a bijection β for which $\bar{\beta}$, after suitable switching, is still not an automorphism if and only if $\mathfrak{G}$ is any group other than $\mathbb{Z}_\gamma$, $\gamma \leq 3$, and $\mathfrak{V}_4$. □

The application to multiary quasigroups is Corollary 9.6.

Another question resolved by Theorem 5.3 is whether it might be possible to ensure that an edge amalgamation or sum is a group expansion by putting a restriction on triangular expansion minors. For any group expansion, all expansion minors are expansions by the same group (Proposition 2.4). We might conjecture a kind of converse: that $\Omega = \langle \mathfrak{G}\Delta_1 \rangle \cup_e \langle \mathfrak{G}\Delta_2 \rangle$ or $\langle \mathfrak{G}\Delta_1 \rangle \oplus_e \langle \mathfrak{G}\Delta_2 \rangle$ is a $\mathfrak{G}$-expansion if every triangular expansion minor is isomorphic to $\langle \mathfrak{G}K_3 \rangle$. However, in general this is false.

Corollary 5.6. *It is possible to have a biased expansion $\gamma \cdot \Delta$, where Δ is a 2-connected but 2-separable simple graph, such that every triangular expansion minor is isomorphic to $\langle \mathfrak{G}K_3 \rangle$ for a fixed group $\mathfrak{G}$ but $\gamma \cdot \Delta$ is not a group expansion, except when $\mathfrak{G} = \mathbb{Z}_\gamma$ for $\gamma \leq 3$ or $\mathfrak{G} = \mathfrak{V}_4$. Furthermore, $\gamma \cdot \Delta$ can be taken to be an edge amalgamation of group expansions.*

Lemma 5.7. *Let $\mathfrak{G}$ be a fixed group. Suppose Ω, a biased expanison of a 2-connected graph Δ, is obtained by expanded edge summations and amalgamations from $\mathfrak{G}$-expansions of inseparable graphs. Then every triangular expansion minor of Ω is isomorphic to $\langle \mathfrak{G}K_3 \rangle$.*

Proof. We use induction on the order of Ω. Suppose in the construction of Ω that the last step is to assemble $\Omega_1 \downarrow \Delta_1$ and $\Omega_2 \downarrow \Delta_2$ into $\Omega = \Omega_1 \cup_{e,\beta} \Omega_2$ (or $\Omega = \Omega_1 \oplus_{e,\beta} \Omega_2$, but it suffices to consider the case of amalgamation). Consider a triangular expansion minor Ω_3 of Ω whose edge set is $E_3 = p^{-1}(\{e_1, e_2, e_3\})$.

Let Ω_3' be the corresponding subgraph of Ω; that is, the subgraph induced by the edge set E_3. If $e_1, e_2, e_3 \in E(\Delta_i)$, then Ω_3 is an expansion minor of $\langle \mathfrak{G}\Delta_i \rangle$ and the desired conclusion follows from Proposition 2.4. Otherwise, we may assume $e_1, e_2 \in E(\Delta_1 \setminus e)$ and $e_3 \in E(\Delta_2 \setminus e)$. By the definition of a minor, there is a circle C in Δ that contains all three edges such that $R = C \setminus \{e_1, e_2, e_3\}$ has a lift $\tilde{R}$ for which $\Omega_3 = (\Omega_3' \cup \tilde{R})/\tilde{R}$. Let $E_{30} = E_3 \cup p^{-1}(e)$, let Ω_{30}' be the corresponding subgraph of Ω, and let $\Omega_{30} = (\Omega_{30}' \cup \tilde{R})/\tilde{R}$. Then Ω_{30} is a biased expansion of the graph Δ_{30} consisting of the triangle $\{e_1, e_2, e_3\}$ and an edge e parallel to e_3. By Example 3.1, $\Omega_3 = \Omega_{30} \setminus p_{30}^{-1}(e)$ is isomorphic to $\Omega_{30} \setminus p_{30}^{-1}(e_3)$. The latter is an expansion minor of Ω_1, hence isomorphic to $\langle \mathfrak{G}K_3 \rangle$. $\square$

Proof of Corollary 5.6. The exceptional cases are covered by Corollary 5.5. For other groups, by Corollary 5.5 Ω need not be a group expansion. However, by the lemma, every triangular expansion minor is a $\mathfrak{G}$-expansion. $\square$

Corollary 5.6 might suggest that it is difficult to say from a criterion based on small minors whether Ω is or is not a group expansion. But that is not correct; minors of order four suffice; see Theorem 7.2.

A question that is not answered so far is that of reducibility of arbitrary biased expansions of 2-connected, 2-separable graphs. The methods of Theorems 5.1 and 5.3 produce only nongroup expansions that are 2-separable and have a 2-separation whose nodes are adjacent or can be made adjacent in an extended biased expansion. They will not give an example in which no 2-separating node pairs can be made adjacent: for instance, a biased expansion $4 \cdot C_4$ in which it is not possible to add a chord of the C_4. Any irreducible n-ary quasigroup $\mathfrak{Q}$ with $n \geq 3$ provides such an example in the form of the expansion $\mathfrak{Q}C_{n+1}$. By the results of Section 6, that is the only way.

We want criteria to decide when a biased expansion of a 2-separable graph Δ is an expanded edge amalgamation or sum along an edge (not necessarily in Δ) whose endpoints separate Δ.

Corollary 5.8 (Test for Decomposability across a 2-Separation)**.** *Suppose $\Omega \downarrow \Delta$, where Δ is 2-connected, and $\{v, w\}$ is a 2-separation of Δ into subgraphs Δ_1 and Δ_2. Let $\Omega_i = \Omega\big|_{\Delta_i}$. If v and w are adjacent by an edge e_{vw}, then Ω has the form of an expanded edge amalgamation $\Omega_1 \cup_{e_{vw}} \Omega_2$. If they are not adjacent, choose an arbitrary circle $C \subseteq E(\Delta)$ through v and w. Then Ω is an expanded edge sum $\Omega_1' \oplus_{e_{vw}} \Omega_2'$ if and only if $\Omega\big|_C$ extends to e_{vw}.*

Proof. The first part is obvious. In the second part, if Ω is an edge sum, then it extends to $\Omega' \downarrow \Delta \cup e_{vw}$, formed by amalgamating instead of summing. Conversely, if $\Omega\big|_C$ extends to e_{vw}, then Ω extends, by Proposition 3.9, and therefore is an expanded edge sum. $\square$

Belousov and Sandik have a criterion for extendibility of $\Omega\big|_C$ to a chord e_{vw}, expressed in terms of factorizability of a multiary quasigroup (which is equivalent by Proposition 1.1). Let P and Q be the paths into which v and w divide C. Translated to biased expansions, the criterion says:

Proposition 5.9 ([7, Lemma 6])**.** *If there exist lifts $\tilde{P}$, $\tilde{P}^*$, $\tilde{Q}$, and $\tilde{Q}^*$ such that $\tilde{P} \cup \tilde{Q}$, $\tilde{P}^* \cup \tilde{Q}$, $\tilde{P} \cup \tilde{Q}^*$ are balanced but $\tilde{P}^* \cup \tilde{Q}^*$ is not, then $\Omega\big|_C$ does not extend. Otherwise, it extends.*

6. The structure of biased expansions

We have two main structure theorems. One is about maximal biased expansions, and translates directly into a structural description of multiary quasigroups (Corollary 9.4). The other describes all biased expansion graphs. We want to make it very clear that these theorems are proved only for expansions of base graphs that are 2-*connected* and have *finite order*. The former is an insignificant restriction in general: when expanding an arbitrary graph, the expansion of each block is unrelated to that of any other block, so it is inevitable that a theorem can only refer to 2-connected graphs (but for regular expansions see Proposition 6.1). The restriction to finite order is due to the absence of a 3-decomposition theory of infinite graphs. (I see no reason why such a theory should not exist.) Another necessity for our structural theorems is a Menger theorem for 2-separation of nodes in infinite graphs; for this see, e.g., [11, Proposition 8.4.1]. Our results should follow for arbitrary infinite orders once a 3-decomposition theorem is proved.

Now, here are the main results, beginning with a simple regularity property.

Proposition 6.1. *A regular biased expansion that is maximal is necessarily inseparable.*

Proof. Suppose a regular biased expansion $\Omega \downarrow \Delta$ has a cutpoint v, so that $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 = \{v\}$; let $\Delta_i = p(\Omega_i)$. Choose $e_i \in E(\Delta_i)$ incident with v and take any biased expansion $\Omega_3 \downarrow K_3$ whose multiplicity equals that of Ω. Identify e_1 and e_2 with different edges of the K_3 and amalgamate edges to form, first, $\Omega_1 \cup_{e_1} \Omega_3$ and then $(\Omega_1 \cup_{e_1} \Omega_3) \cup_{e_2} \Omega_2$. This is a proper extension of Ω. The disconnected case is similar. $\square$

Theorem 6.2 (Structure of Maximal Biased Expansions)**.** *Any* 2*-connected maximal biased expansion graph $\Omega \downarrow \Delta$ of finite order $n \geq 3$ is obtained by expanded edge amalgamation of group expansions of complete graphs of order at least* 3 *and irreducible, nongroup circle expansions of order at least* 3*, all of which are restriction subgraphs $\Omega\big|_{\Delta'}$ of Ω. The group expansions and circle expansions are uniquely determined as the maximal complete subgraphs and the maximal chordless circle expansions contained in Ω.*

Any such edge amalgamation is a biased expansion. It is maximal if and only if, for any two group expansions that are amalgamated along an expanded edge, the attachment map is twisted.

The last part calls for explanation. Twist was defined at Theorem 5.3. Let Δ be the base graph of Ω. The theorem is saying, in part, that Δ is obtained by amalgamating circles and complete graphs. In the second half, several complete graphs may be amalgamated along the same edge, either one at a time or all at once (to be explained momentarily). Call these $\Delta_1, \ldots, \Delta_r$ and the common edge e, and let $\Omega_i = \Omega\big|_{\Delta_i}$. There are many ways to amalgamate one step at a time, each described by a rooted binary tree with leaves $\Omega_1, \ldots, \Omega_r$. We might amalgamate first, for instance, Ω_1 and Ω_2 by way of a bijection $\beta_{12} : p_1^{-1}(e) \to p_2^{-1}(e)$, then Ω_3 and Ω_6 by β_{36}, then Ω_5 to $\Omega_1 \cup_e \Omega_2$ via β_{15}, etc. All these ways have the same outcome, by Theorem 5.2. Instead, we could amalgamate all at once by means of commuting bijections $\beta_{ij} : p_i^{-1}(e) \to p_j^{-1}(e)$, that is, $\beta_{ik} = \beta_{ij} \circ \beta_{jk}$ and $\beta_{ij}^{-1} = \beta_{ji}$. The theorem means that, if $\Omega_i = \mathfrak{G}_i \Delta_i$ and $\Omega_j = \mathfrak{G}_j \Delta_j$ for groups $\mathfrak{G}_i \cong \mathfrak{G}_j$, then β_{ij} should not have the form that, according to Theorem 5.3, makes $\Omega_i \cup_{e,\beta_{ij}} \Omega_j$ into a group expansion. (We discuss this further at Corollary 6.6.)

Theorem 6.3 (Structure of Biased Expansions)**.** *Any* 2*-connected biased expansion* Ω *of a simple graph* Δ *of finite order at least* 3 *is obtained by operations of expanded edge sum and amalgamation from* 3*-connected group expansions and nongroup irreducible quasigroup expansions of circles, each of which is uniquely determined and is an expansion minor of* Ω.

Note that C_3 and $\gamma \cdot C_3$ are considered to be 3-connected.

Call the group and circle expansions the *3-constituents* of Ω. (In Theorem 6.3 they may not be uniquely determined.) Note that K_3 is considered to be 3-connected. In the construction of Ω it may be that an edge e in Δ belongs to several 3-constituents. Then $p^{-1}(e)$ is the subject of several expanded amalgamations, and we could carry them all out at once as described previously. Similarly, if an edge e not in Δ belongs to several 3-constituents, then it is the subject of several edge sums; which means that all of the copies of $p^{-1}(e)$, except one, are amalgamated, and the last one is summed with the amalgamation of the others. Then we could carry out, instead, a multiple (expanded) edge sum, similar to the multiple edge amalgamation we described.

For the proofs we need Tutte's theory of decomposition of an inseparable graph into 3-blocks. We outline this theory (from [35, Chapter IV, Sections 3 and 4], originally in [34]). Let Δ be a 2-connected graph. If Δ is 3-connected, it is its own unique 3-block. If it is not 3-connected, we define a *cleavage* to be a 2-separation $\{x, y\}$ together with a bridge B of $\{x, y\}$, such that B is inseparable and not a single edge. (Then the complement of B has at least two edges, since Δ is 2-connected and 2-separable.) Choose a cleavage, and split Δ into two graphs: $B \cup e_{xy}$ and $B^c \cup e_{xy}$, where B^c is the union of the other bridges of $\{x, y\}$ and e_{xy} is a new edge, called a *virtual edge*. One continues this process on the resulting graphs until one obtains graphs $\Delta_1, \ldots, \Delta_k$ without cleavages. These are the *3-blocks* of Δ. Each virtual edge appears exactly twice and represents an edge sum; if all the indicated sums are carried out, the 3-blocks are reassembled into Δ. Each 3-block is either

3-connected, or a circle graph of order three or more, or a *multilink* of size three or more (that is, a graph consisting of at least three parallel links and their two nodes). There is a graph of 3-blocks, in which the nodes are the 3-blocks and two 3-blocks are adjacent when they share a virtual edge. Tutte's theorem is, first, that the 3-blocks are uniquely determined by Δ, and second, that the graph of 3-blocks is a tree, called the *3-block tree* of Δ (see Fig. 23).

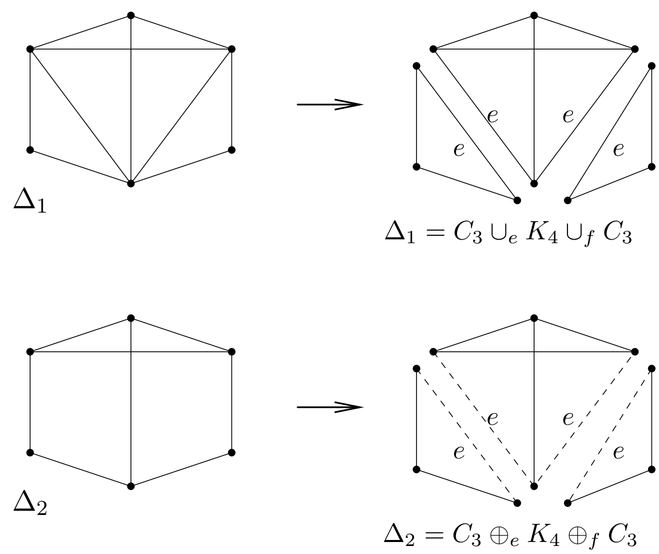


FIGURE 23. Two graphs with their Tutte 3-decompositions. The dashed lines in Δ_2 are virtual edges, eliminated by 2-summation in the assembly process. The corresponding line pairs in Δ_1 are combined into single edges by 2-amalgamation.

Suppose Δ is simple. Then a multilink Δ_0 contains at most one real edge (i.e., an edge of Δ). Suppose Δ_0 does contain a real edge, e, and virtual edges $e_1, \ldots, e_k$. If $\Delta_1, \ldots, \Delta_k$ are the 3-blocks that contain the other copies of $e_1, \ldots, e_k$, then $\Delta_0 \oplus_{e_1} \Delta_1 \oplus_{e_2} \cdots \oplus_{e_k} \Delta_k$ is the same as the amalgamation $\Delta_1 \cup_e \Delta_2 \cup_e \cdots \cup_e \Delta_k$ if we treat all the e_i as copies of e. Thus, by amalgamating rather than summing we can dispense with Δ_0. If Δ_0 contains only virtual edges, then $\Delta_0 \oplus_{e_1} \Delta_1 \oplus_{e_2} \cdots \oplus_{e_k} \Delta_k$ is the same as $(\Delta_1 \cup_{e_k} \cdots \cup_{e_k} \Delta_{k-1}) \oplus_{e_k} \Delta_k$ if we treat all the e_i as copies of e_k; so again we can dispense with Δ_0. (Or, again, we can treat this as a simultaneous edge sum.) The conclusion is that, for simple graphs Δ, the multilinks are not needed if we modify the 3-blocks and permit amalgamation. This is what we shall do.

For the proof of Theorem 6.2 we need the definition of a theta-complete graph from Section 3, and the following characterization of such graphs (when simple and 2-connected).

Proposition 6.4. *Any theta-complete simple,* 2*-connected graph* Δ *is obtained by edge amalgamation of complete and circle subgraphs of* Δ*, and conversely such an amalgamation is theta-complete.*

Proof. Consider the 3-blocks of Δ in Tutte's unmodified system. We show that every multilink 3-block Δ_0 contains a real edge. That is the same as saying that the two nodes of a cleavage are adjacent. This comes from theta-completeness and a lemma.

Lemma 6.5. *In Tutte's* 3*-decomposition of any* 2*-connected graph* Δ*, two nodes* x*,* y *of a cleavage are the trivalent nodes of a theta subgraph.*

Proof. If $\{x, y\}$ has more than two bridges, this is trivial. If it has only two bridges, B and B^c, then we know (by definition of a cleavage) that B is 2-connected. Hence, Δ contains two internally disjoint xy-paths in B and one more in B^c. □

Since every multilink 3-block does contain a real edge, it can be eliminated in favor of amalgamation. And, because every 2-separating pair of nodes is adjacent, every virtual edge lies in a 3-block that is a multilink. Consequently, when we modify Tutte's 3-decomposition all edge sums are replaced by amalgamations.

Conversely, we have to prove the amalgamation is theta-complete. This is obvious. □

Proof of Theorem 6.2. Assume $\Omega \downarrow \Delta$ is maximal. Theorem 3.17 says that Δ is theta-complete. The rest is obvious.

Conversely, suppose $\Omega \downarrow \Delta$ is the result of expanded edge amalgamations applied to group expansions $\mathfrak{G}_1 K_{n_1}, \dots, \mathfrak{G}_r K_{n_r}$ and nongroup irreducible circle expansions $\Omega_1 \downarrow C_{l_1}, \dots, \Omega_s \downarrow C_{l_s}$. These are the 3-constituents of Ω and the K_{n_i}, C_{l_j} are the 3-constituents of Δ. By Tutte's 3-decomposition theorem they are unique. We have to prove Ω cannot be extended to any edge e not in Δ, the base graph constructed by the amalgamations.

Suppose it did extend to some $e \notin E(\Delta)$, and let $\Omega' \downarrow \Delta \cup e$ be the extension. The endpoints of e cannot be contained within one 3-constituent, because each K_{n_i} is complete, and if Ω extended to a chord of C_{l_j}, then Ω_j would be reducible (by Theorem 9.1). It follows that, if we take a path in the 3-block tree of Δ joining a 3-block containing x to a 3-block containing y, the path has positive length. Let $\Delta_1, \dots, \Delta_r$ be the shortest such path, with x in Δ_1 and y in Δ_r, and set

$$\Delta'' = \Delta_1 \cup \cdots \cup \Delta_r \cup e.$$

Then x and y are connected by two internally disjoint paths in $\Delta'' \setminus e$ and therefore by three in Δ''.

If $\Delta_1, \ldots, \Delta_r$ are all complete graphs, then Δ'' is 3-connected, because the only 2-separations of Δ'' are those at cleavages of Δ where a Δ_{h-1} and Δ_h share an edge. But if Δ'' is 3-connected, then $\Omega'\big|_{\Delta''}$ is a group expansion, and therefore $\Omega\big|_{\Delta_1}$ and $\Omega\big|_{\Delta_2}$ are group expansions, amalgamated by an attaching bijection that makes $\Omega\big|_{\Delta_1 \cup \Delta_2}$ a group expansion, contrary to hypothesis. So, some Δ_h is a circle of length $l \geq 4$.

We may assume by choice of indices that $h > 1$, so that Δ_h amalgamates with Δ_{h-1} along an edge uv. Also, either $y \in N(\Delta_h)$, or $h < r$ and Δ_h shares with Δ_{h+1} an edge $u'v'$. It is easy to verify that one can name the nodes so that u and y, in the former case, or u and u', in the latter, are not adjacent. In the former case let $u' = y$. Consider the two internally disjoint xy-paths in $\Delta'' \setminus e$. One must pass through u but not v; call P_1 its portion from x to u. One must pass through u' but not v'; call P_2 its portion from u' to y. (This is a trivial path if $u' = y$.) P_1 and P_2 are internally disjoint from Δ_h. Consequently, the uu'-path $P_1 \cup e \cup P_2$ is internally disjoint from Δ_h, and in combination with the two uu' paths in the circle Δ_h, it forms a theta graph with trivalent nodes $u, u' \in N(\Delta \cup e)$. By the previous section, then, Ω' extends to $\Delta \cup e \cup e_{uu'}$. Because u and u' are not adjacent in Δ_h, hence not in Δ either, we have contradicted the irreducibility of Δ_h.

Since in either case we deduce a contradiction, Ω is indeed maximal. $\square$

As an example, the expanded edge amalgamation of two maximal biased expansions is maximal if (but not only if) for any group $\mathfrak{G}$ the two expansions contain at most one 3-constituent that is a $\mathfrak{G}$-expansion.

Proof of Theorem 6.3. The trick is to extend Ω to edges e_{xy} for all cleavages $\{x, y\}$. We know from Lemma 6.5 that this is possible, but we also need to know that the cleavages are the same in the extended base graph Δ'. Clearly, Δ' has all the cleavages of Δ. On the other hand, in a cleavage $(\{x, y\}, B')$ of Δ', $\{x, y\}$ is a 2-separation of Δ and $B = B' \cap \Delta$ is connected, is a bridge of $\{x, y\}$, and has at least two edges; and the same holds for any other bridge B_1' and $B_1 = B_1' \cap \Delta$ unless B_1' is an edge. These facts are a consequence of Lemma 6.5. The conclusion is that $(\{x, y\}, B)$ is a cleavage of Δ. That is, Δ and Δ' have the same cleavages.

Consequently, they have the same 3-blocks (in Tutte's sense) except that the 3-blocks in Δ' may have additional edges. Ω' is obviously obtained from its 3-constituents by expanded edge amalgamation, and Ω is the same except for deletion of the amalgamated fibers $(p')^{-1}(e)$ for each additional edge e. This deletion simply converts an amalgamation to a sum; thus Ω is obtained by edge sum and amalgamation from 3-connected group expansions and circle expansions. Each circle expansion, if reducible, is an edge sum of smaller circle expansions; thus Ω does have the form stated in the theorem.

That all the 3-constituents are expansion minors follows from Theorem 5.1. $\square$

Two questions remain. First, what are the graphs that support maximal expansions? Second, which graphs have nongroup expansions (a question raised in [37, Example III.3.8]). Theorems 6.2 and 6.3 suggest the answers, but there are details to attend to. Let us call a complete graph *large* if it has at least four nodes.

Corollary 6.6. *A finite simple graph Δ has a biased expansion that is maximal if and only if it is inseparable and is obtained by edge amalgamation of complete graphs and circles.*

Let N_1, resp. N_0, be the maximum number of large, resp. all, complete 3-constituents of Δ that contain any one edge. The possible multiplicities of a maximal finite biased expansion $\gamma \cdot \Delta$ include every composite number $\gamma \geq 5$ such that $(\gamma-1)! \geq 2N_1$, as well as $\gamma = 4$ if $N_0 \leq 3$.

Proof. The form of Δ is entailed by Theorem 6.2, but it is necessary to produce examples. The general idea is to expand each 3-constituent Δ_i and amalgamate. We assume $N \geq 2$. Belousov and Sandik [7], Frenkin [18], and Borisenko [8] demonstrated the existence of an irreducible n-ary quasigroup with γ elements for every $n \geq 3$ and composite $\gamma \geq 4$ (see [2]). We also know there is a binary quasigroup of every order $\gamma \geq 5$ that is not isotopic to a group (by [10, Theorem 1.5.1] for $\gamma \neq 6$, [10, Figure 1.3.1] for $\gamma = 6$). As for a complete graph, it has group expansions of every multiplicity. The difficulty is to assemble the expansions into a maximal expansion.

Consider some complete 3-constituents $\Delta_1, \ldots, \Delta_r$ that share an amalgamating edge e. Expand them all by a group $\mathfrak{G}$ of order γ to construct $\Omega_i = \langle \mathfrak{G}\Delta_i \rangle$. Now we need attachment maps $\beta_{ij} : p_j^{-1}(e) \to p_j^{-1}(e)$. (We include $\beta_{ii} = \mathrm{id}$.) Since we want the amalgamated expansion to be maximal, none of the $\bar{\beta}_{ij}$ can be a pseudoautomorphism of $\mathfrak{G}$, except of course for the $\bar{\beta}_{ii}$. Factoring $\bar{\beta}_{ij} = \bar{\beta}_{1i}^{-1} \circ \bar{\beta}_{1j}$, we conclude that the mappings $\bar{\beta}_{1i}$ for $i = 1, 2, \ldots, r$ must belong to different cosets of $\operatorname{PsAut} \mathfrak{G}$, the group of pseudodautomorphisms, in the symmetric group of $\mathfrak{G}$. This condition is necessary and sufficient for maximality of the amalgamation.

In the simplest case we expand every large complete 3-constituent on e by the cyclic group $\mathbb{Z}_\gamma$. The number of cosets of $\operatorname{PsAut} \mathbb{Z}_\gamma$ is $(\gamma-1)!/2$, so we can accommodate $r \leq (\gamma-1)!/2$ different complete 3-constituents. If $\gamma \geq 5$ we expand the K_3 3-constituents by binary nongroup isotopes so we can take $r = N_1$. If $N_0 \leq 3$ we can expand every complete 3-constituent by $\mathbb{Z}_4$ and take $r = N_0$. The corollary follows easily. $\square$

The list of achievable multiplicities can be improved in special cases. If all 3-constituents are complete, they can all be expanded by a group so γ need not be composite; however, then we have to take $r = N_0$. If Δ is a circle, γ can be any composite number ≥ 4. If Δ is complete, γ can be any positive integer. In some situations we could handle larger N_0 or N_1 by using more than one gain group.

One would have liked to say that any two maximal biased expansions, $\gamma \cdot \Delta_1$ and $\gamma \cdot \Delta_2$, with a common base edge e and the same multiplicity, can be amalgamated into a maximal expansion by choosing β appropriately, but this is not true. For one reason, there could be a group $\mathfrak{G}$ that is the gain group of several 3-constituents, for which the combined number of 3-constituents in both graphs that are $\mathfrak{G}$-expansions and cover e exceeds the number of cosets of $\operatorname{PsAut}\mathfrak{G}$. It is possible to describe the exact conditions under which an expanded amalgamation is maximal, in terms of double cosets of pseudoautomorphism groups of groups of order γ, but the description is excessively complicated.

Corollary 6.7. *A finite simple graph Δ has a regular biased expansion that is not a group expansion if and only if it is not a forest and is not 3-connected.*

The possible finite multiplicities of a regular nongroup expansion $\gamma \cdot \Delta$ include every $\gamma \geq 4$, except that when every block is 3-connected with at least four nodes γ cannot be prime.

Proof. If Δ is separable we can expand two different blocks by two different groups of order γ with the exception noted. In a 2-separable block we can expand every 3-constituent by $\mathbb{Z}_\gamma$ and make sure to attach one of them, whether by edge summation or edge amalgamation, so as to produce a nongroup expansion. □

7. Four-node minors

A biased expansion graph may have gains for fairly special reasons. As we mentioned in connection with Corollary 5.6, gainability of minors of order four suffices to imply that Ω is a group expansion. Partially for that reason, a biased expansion may be forced to have gains in a group simply because its multiplicity is very small.

Lemma 7.1. *If $\Omega \downarrow C_{n+1}$, where $n \geq 3$, and all expansion minors of order four that contain a specific edge fiber $p^{-1}(e_i)$ are group expansions (not necessarily of the same group), then Ω is a group expansion of C_{n+1}.*

Proof. We assume the reader is acquainted with contraction of gain and biased graphs (see [37, Sections I.2 and I.5]). We write $C = C_{n+1} = e_0 e_1 \cdots e_n$, with $N(e_i) = \{v_i, v_{i+1}\}$ where $v_0 = v_{n+1}$. The special edge in the statement of the lemma will be e_0. The case $n = 3$ being trivial, we assume $n \geq 4$.

Fix a balanced lift $\tilde{C}^0$. Some notation that will be convenient: $\tilde{C}^0(\tilde{e}_i, \tilde{e}_j)$ is $\tilde{C}^0$ with $\tilde{e}_i$ and $\tilde{e}_j$ replacing $\tilde{e}_i^0$ and $\tilde{e}_j^0$. Ω_{ijk} is the expansion minor of Ω whose edge set is $p^{-1}(\{e_i, e_j, e_k\})$ that is obtained by contracting $\tilde{C}^0 \setminus p^{-1}(\{e_i, e_j, e_k\})$; similarly, we write Ω_{0ijk}, Ω_{ij}.

The hypothesis is that each $\Omega_{0ijk} \cong \langle \mathfrak{G}_{ijk} C_4 \rangle$ for a group $\mathfrak{G}_{ijk}$. Ω_{0ij} is an expansion minor of both Ω_{0ijk} and Ω_{0ijl}. In the former capacity it is isomorphic to $\langle \mathfrak{G}_{ijk} C_3 \rangle$ and in the latter to $\langle \mathfrak{G}_{ijl} C_3 \rangle$ (by contracting $\tilde{e}_k^0$ and $\tilde{e}_l^0$, respectively). Since the gain group of a group expansion is unique, $\mathfrak{G}_{ijk} \cong \mathfrak{G}_{ijl}$. It follows that all groups $\mathfrak{G}_{ijk}$ are isomorphic to a single group $\mathfrak{G}$.

In the rest of the proof we construct a gain graph $\Phi = \mathfrak{G}C$ and prove that $\langle\Phi\rangle = \Omega$. For this purpose we consider e_i to be oriented from v_i to v_{i+1}.

Step 1. We define the gain mapping φ. Its identity-gain edge set will be $\tilde{C}^0$. Any isomorphism $\Omega_{0123} \cong \langle\mathfrak{G}C_4\rangle$ defines gains φ on $p^{-1}(\{e_0, e_1, e_2, e_3\})$; we choose φ so it is 1 on $\{\tilde{e}_0^0, \tilde{e}_1^0, \tilde{e}_2^0, \tilde{e}_3^0\}$. We extend φ to $p^{-1}(e_i)$ for $i > 3$ by $\varphi(\tilde{e}_i) = \varphi(\tilde{e}_0)^{-1}$, where $\tilde{e}_0$ is the lift of e_0 that makes $\tilde{C}^0(\tilde{e}_0, \tilde{e}_i)$ balanced. This rule can be expressed as choosing $\varphi\big|_{p^{-1}(e_i)}$ so that $\langle\Phi_{0i}\rangle = \Omega_{0i}$.

Note that, if we want to change $\varphi^{-1}(1)$ to be a different balanced lift, $\tilde{C}^1$, we can do it by switching φ.

Step 2. We next show that Φ is valid on expansion minors of order four that include $p^{-1}(e_0)$; that is, $\langle\Phi_{0ijk}\rangle = \Omega_{0ijk}$.

For Ω_{0123} that is a matter of definition.

For Ω_{012k} (where $k > 3$), because $\Omega_{012k} \cong \langle\mathfrak{G}C_4\rangle$ we can choose gains in $\mathfrak{G}$ for Ω_{012k}, and we may choose them so that, contracted by $\tilde{e}_3^0$ to Ω_{012}, they agree with Φ_{012}. Then the gains on Ω_{012k} are forced by the Ω_{0k} minor to be as in Φ. Thus, $\langle\Phi_{012k}\rangle = \Omega_{012k}$. We infer that $\langle\Phi_{01k}\rangle = \Omega_{01k}$.

Considering $\Omega_{01jk} \cong \langle\mathfrak{G}C_4\rangle$, the gains can be chosen to agree on Ω_{01j} with those of Φ_{01j}. The minor Ω_{0k} forces Ω_{01jk} to have gains as in Φ, so $\langle\Phi_{01jk}\rangle = \Omega_{01jk}$. We further conclude from this and the previous cases that $\langle\Phi_{0jk}\rangle = \Omega_{0jk}$.

Finally, $\Omega_{0ijk} \cong \langle\mathfrak{G}C_4\rangle$ and the gains on Ω_{0ij} can be chosen to agree with those of Φ_{0ij}. Again Ω_{0k} forces the gains of Ω_{0ijk} to be as in Φ_{0ijk}, so $\langle\Phi_{0ijk}\rangle = \Omega_{0ijk}$.

Step 3. We prove by induction on n that $\langle\Phi\rangle = \Omega$. The task is to prove that every lift $\tilde{C}^*$ is well behaved: it is balanced in Ω if and only if $\varphi(\tilde{C}^*) = \varphi(\tilde{e}_0^*)\varphi(\tilde{e}_1^*)\cdots\varphi(\tilde{e}_n^*) = 1$.

If $\tilde{C}^*$ has an edge $\tilde{e}_i^* = \tilde{e}_i^0$ with $i \neq 0$, then we contract Ω and Φ by $\tilde{e}_i^0$ and discard loops. This gives expansion minors $\Omega' \downarrow C_n$ and $\Phi' = \mathfrak{G}C_n$, in which $\tilde{C}^*$ becomes $\tilde{C}^*/\tilde{e}_j^0$ and φ' is the restriction of φ. The process of constructing gains in Ω' in Step 1 produces the gain function φ' if the various choices are made in agreement with those defining Φ. By induction, therefore, $\tilde{C}^*/\tilde{e}_i^0$ is balanced in Ω' if and only if $\varphi'(\tilde{C}^*/\tilde{e}_i^0) = 1$. However, $\varphi'(\tilde{C}^*/\tilde{e}_i^0) = \varphi(\tilde{C}^*)$ because $\varphi(\tilde{e}_i^0) = 1$, and by definition of contraction $\tilde{C}^*$ is balanced in Ω if and only if $\tilde{C}^*/\tilde{e}_i^0$ is balanced in Ω'. Therefore, $\tilde{C}^*$ is well behaved.

If $\tilde{C}^*$ fails to contain an edge $\tilde{e}_i^0$ with $i \neq 0$, we replace $\tilde{C}^0$ by a different balanced circle $\tilde{C}^1$ that does have an edge $\tilde{e}_i^1$ in common with $\tilde{C}^*$. We choose $\tilde{C}^1 = \tilde{C}^0(\tilde{e}_0^1, \tilde{e}_1^1)$ where $\tilde{e}_1^1 = \tilde{e}_1^*$ and $\tilde{e}_0^1$ is the edge that makes $\tilde{C}^1$ balanced; that is, $\varphi(\tilde{e}_0^1) = \varphi(\tilde{e}_1^1)^{-1}$, since $\tilde{C}^1$ is well behaved. Changing $\tilde{C}^0$ to $\tilde{C}^1$ alters the gain mapping φ, but under control: we simply switch it by a suitable switching function η. A valid choice for η is $\eta(v_i) = 1$ except $\eta(v_1) = \varphi(\tilde{e}_1^*)^{-1} = \varphi(\tilde{e}_0^1)$. Then $(\varphi^\eta)^{-1}(1) = \tilde{C}^1$, and because $\langle\Phi\rangle$ is invariant under switching φ^η is a suitable gain function with respect to $\tilde{C}^1$ in Step 1. By the previous case with $\tilde{C}^1$ in place of $\tilde{C}^0$, $\tilde{C}^*$ is well behaved. $\square$

Theorem 7.2. *Suppose $\Omega \downarrow \Delta$ is a 2-connected biased expansion graph of finite order at least 4, and $f \in E(\Delta)$. If every expansion minor of order 4 that contains the edge fiber $p^{-1}(f)$ is a group expansion, then so is Ω.*

Especially, if every expansion minor of order 4 is a group expansion, then Ω is a group expansion; but one may also deduce the conclusion from less information.

Proof. We first prove that every 3-constituent of Ω is a group expansion. From Theorems 5.1 and 6.3 every 3-constituent Ω_0 is an expansion minor of Ω; more precisely, there is an expansion minor Ω_0'' which is Ω_0 with the virtual edges replaced by real edges of Ω. In the amalgamations that prove Theorem 6.3, suppose Ω_0 is a minor of Ω_1 and $f \in E(\Omega_2)$. In the last part of the proof of Theorem 5.1 take $f_2 = f$; then $p^{-1}(f)$ will be an edge of the expansion minor corresponding to Ω_1. Tracking this process inductively shows that $p^{-1}(f) \subseteq \Omega_0'$. If Ω_0 is 3-connected and not a triangle, it is a group expansion by Theorem 4.1. If Ω_0 is an expansion of a circle that is not a triangle, then it is a group expansion by Lemma 7.1 because every minor of Ω_0' that contains $p^{-1}(f)$ is a minor of Ω that contains $p^{-1}(f)$. Now suppose $\Omega_0 \downarrow \Delta_0 \cong C_3$. We refer again to the end of the proof of Theorem 5.1. $Q_2' f_2 Q_2''$ is a path of length at least 2; thus, instead of contracting $\tilde{Q}_2$ we may contract all but one edge in it, say $\tilde{e}_2 \in p^{-1}(e_2)$. That gives a four-node circular expansion minor $\Omega_0'' \supseteq p^{-1}(\{e_2, f\})$ which contracts by $\tilde{e}_2$ to Ω_0'. As Ω_0'' is a group expansion by assumption, Ω_0 is also a group expansion. Therefore, every 3-constituent of Ω is a group expansion.

If Ω is not a group expansion, then at some point in the process of amalgamation and summation two group expansions, $\mathfrak{G}_1\Delta_1$ and $\mathfrak{G}_2\Delta_2$, are summed (or amalgamated, which is treated similarly) along an edge e by a twisted attachment map β to form a nongroup biased expansion. We may assume $f \in E(\Delta_1)$. There are expansion minors $\langle \mathfrak{G}_1 C_3 \rangle$ of $\langle \mathfrak{G}_1\Delta_1 \rangle$, containing $p^{-1}(\{e, f\})$, and $\langle \mathfrak{G}_2 C_3 \rangle$ of $\langle \mathfrak{G}_2\Delta_2 \rangle$ that contains $p^{-1}(e)$, and then $\Omega_4 := \langle \mathfrak{G}_1 C_3 \rangle \oplus_{e,\beta} \langle \mathfrak{G}_2 C_3 \rangle$ is a minor of $\langle \mathfrak{G}_1\Delta_1 \rangle \oplus_{e,\beta} \langle \mathfrak{G}_2\Delta_2 \rangle$ which contains $p^{-1}(f)$. Ω_4 is not a group expansion of $C_3 \oplus_e C_3 = C_4$ because β is twisted and twistedness is unaltered by taking minors. But Ω_4 is one of the four-node expansion minors of Ω that, by hypothesis, are group expansions. This contradiction demonstrates that β cannot be twisted. □

Problem 7.3. Can the list of order-four expansion minors in the hypotheses of Theorem 7.2 or Lemma 7.1 be further reduced?

8. Thin expansions

We now examine the case of small multiplicity.

Theorem 8.1. *Let Δ be a finite graph and $\Omega = \gamma \cdot \Delta$ a γ-fold biased expansion. Then $\Omega = \langle \pm\Delta \rangle$ if $\gamma = 2$ and $\Omega = \langle \mathbb{Z}_3\Delta \rangle$ if $\gamma = 3$.*

Proof. First we observe that $\gamma \cdot C_{n+1}$ is isomorphic to $\mathbb{Z}_\gamma C_{n+1}$. The proof is simplified if we think of $\gamma \cdot C_{n+1}$ as a Latin hypercube, i.e., as the operation table of $q : \{1, \ldots, \gamma\}^n \to \gamma$. The lemma amounts to saying that the hypercube is isotopic to the iterated addition table of $\mathbb{Z}_\gamma$. That can be proved readily by induction on n. (The case $\gamma = 3$ is mentioned in [17, proof of Corollary, p. 142]; also, it is [25, Exercise 13.15].[4])

It remains to solve the case in which Δ is 2-connected but not a circle. Let $\Omega = \gamma \cdot \Delta$. By the preceding case and Theorem 6.3, Ω is the expanded edge amalgamation and sum of various $\mathbb{Z}_\gamma$-expansions. By Corollary 5.5, Ω is a $\mathbb{Z}_\gamma$-expansion. $\square$

There are two reasons why the theorem is limited to multiplicities below four. The simpler is that in each order $\gamma > 3$ there exists a (binary) quasigroup that is not isotopic to a group. The other is that, for many graphs, one can combine expansions by the same group of order at least four so as to make a nongroup expansion (Corollary 6.7). Still, all such counterexamples are 2-separable since it is impossible to have a nongroup biased expansion, exept of K_3, that is 3-connected (Theorem 4.1).

9. Factorization and construction of multiary quasigroups

Let us discuss the consequences of our results for multiary quasigroups. From an n-ary quasigroup $\mathfrak{Q}$ with operation f (which we shall sometimes denote by $\mathfrak{Q}_f$) construct the factorization graph $\Delta(\mathfrak{Q})$; recall that this is the circle graph C_{n+1} on node set $\{v_0, v_1, \ldots, v_n\}$, whose edges $e_i = e_{i-1,i} = v_{i-1}v_i$ we call the *sides* of $\Delta(\mathfrak{Q})$, together with a chord $e_{ij} = v_iv_j$ whenever f has a factorization

$$f(x_1, \ldots, x_n) = g(x_1, \ldots, h(x_{i+1}, \ldots, x_j), \ldots, x_n). \tag{9.1}$$

Clearly, $\Delta(\mathfrak{Q}) = K_{n+1}$ if $\mathfrak{Q}$ is isotopic to an iterated group, and the converse has long been known (Lemma 4.2). A stronger converse follows from Theorem 4.1; that is Theorem 9.2. From Theorem 6.2 we further deduce a structural description of multiary quasigroups (Corollary 9.4) due to Belousov.

To obtain our results we need the connection between the factorization graph and the maximal extension of $\langle \mathfrak{Q}C_{n+1} \rangle$.

Theorem 9.1. *The unique maximal extension $\Omega(\mathfrak{Q})$ of the biased graph $\langle \mathfrak{Q}C_{n+1} \rangle$ corresponding to an n-ary quasigroup $\mathfrak{Q}$ is a biased expansion of the factorization graph $\Delta(\mathfrak{Q})$.*

Proof. By the theorems of Section 3 it suffices to prove the last part of Proposition 1.1: $\langle \mathfrak{Q}C_{n+1} \rangle$ extends to every chord in $\Delta(\mathfrak{Q})$ but to no other chord of C_{n+1}. (The uniqueness of the extension to each chord is obvious.)

Suppose $\langle \mathfrak{Q}C_{n+1} \rangle$ extends to a chord e_{ij}. Call the extension Ω. Let C' and C'' be the circles formed by the chord, with $e_0 \in C'$. Then $\Omega' = \Omega\big|_{C'}$ and $\Omega'' = \Omega\big|_{C''}$ define operations g and h satisfying (9.1) by the construction described in Section 1.2. Thus, e_{ij} belongs to $\Delta(\mathfrak{Q})$.

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[4] I thank a referee for the references.

Suppose on the other hand that f factors as in (9.1). Then $\langle\mathfrak{Q}_g C'\rangle \cup_{e_{ij},\beta} \langle\mathfrak{Q}_h C''\rangle$, which we call Ω, is a biased expansion of $C_{n+1} \cup e_{ij}$, where we take the amalgamating mapping $\beta : {p'}^{-1}(e_{ij}) \to {p''}^{-1}(e_{ij})$ to be the identity function $\beta(xe_{ij}) = xe_{ij}$. A circle $\{x_0e_0, x_1e_1, \ldots, x_ne_n\}$ is balanced in Ω if there is an edge xe_{ij} that makes $\{xe_{ij}, x_{i+1}e_{i+1}, \ldots, x_je_j\}$ and
$\{xe_{ij}, x_0e_0, x_1e_1, \ldots, x_ie_i, x_{j+1}e_{j+1}, \ldots, x_ne_n\}$ both balanced. In terms of g and h, this means that

$$x = h(x_{i+1}, \ldots, x_j)$$

and

$$x_0 = g(x_1, \ldots, x_i, x, x_{j+1}, \ldots, x_n).$$

It follows that $x_0 = f(x_1, \ldots, x_n)$, so $\Omega\big|_{C_{n+1}} = \langle\mathfrak{Q}C_{n+1}\rangle$. Thus, $\langle\mathfrak{Q}C_{n+1}\rangle$ extends to every chord e_{ij} in $\Delta(\mathfrak{Q})$. $\square$

We see in the proof of Theorem 9.1 that expanded edge amalgamation is the analog of functional composition.

We immediately obtain from Theorem 4.1 the promised strong characterization of iterated group isotopes.

Theorem 9.2. *Let $\mathfrak{Q}$ be an n-ary quasigroup with $n \geq 3$. $\mathfrak{Q}$ is isotopic to an iterated group if and only if $\Delta(\mathfrak{Q})$ is 3-connected.* $\square$

Therefore, if $\Delta(\mathfrak{Q})$ is 3-connected it is complete. We mentioned at Lemma 4.2 the long-known fact that completeness of $\Delta(\mathfrak{Q})$ implies that $\mathfrak{Q}$ is an iterated group isotope. The new result amounts to saying that one need not know $\Delta(\mathfrak{Q})$ completely to arrive at the same conclusion.

Example 9.1 (Left vs. Right). Suppose $2n-2$ binary quasigroups satisfy the identity

$$f_{n-1}(f_{n-2}(\cdots(f_2(f_1(x_1,x_2),x_3),\ldots,x_{n-1}),x_n) = g_1(x_1, g_2(x_2,\ldots,g_{n-1}(x_{n-1},x_n)\cdots)).$$

We see immediately that the n-ary operation defined by either side of this equation has 3-connected factorization graph. Therefore, all f_i and g_i are isotopic to one group.

Example 9.2 (Multiary Groups). A good illustration of Theorem 9.2 is its application to n-ary groups (with $n \geq 3$), which are quasigroups where Equation (1.6) holds. Let $\hat{\mathfrak{Q}}$ be the $(2n-1)$-ary quasigroup with operation $\hat{f}$ defined by (1.6). Multiary associativity means that $\Delta(\hat{\mathfrak{Q}})$ contains diameters $e_{i,i+n}$ for $i = 0, 1, \ldots, n-1$, so it is 3-connected. By Theorem 9.2, $\hat{\mathfrak{Q}}$ is an iterated group isotope. It follows that $\mathfrak{Q}$ is an iterated group isotope, either by an easy algebraic argument or by combinatorial reasoning: $\langle\mathfrak{Q}C_{n+1}\rangle$ is a subgraph of $\langle\hat{\mathfrak{Q}}C_{2n}\rangle$ extended to the chords (Fig. 9.2); the latter is a group expansion; therefore the former is a group expansion; therefore $\mathfrak{Q}$ is isotopic to an iterated group. This is the easy part of the Post–Hosszú–Gluskin theorem, as promised in the introduction.

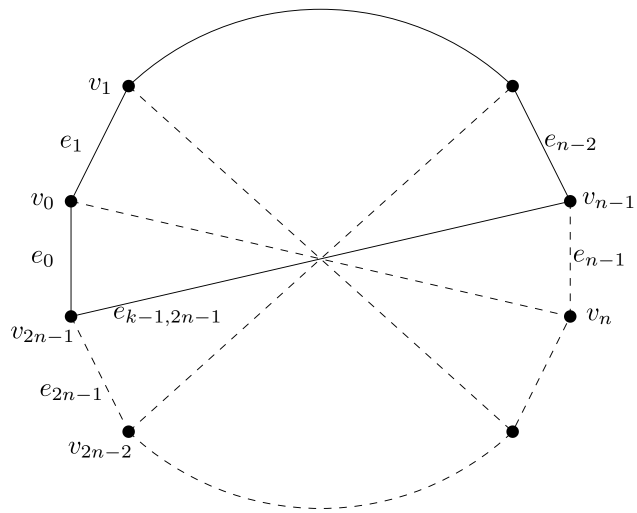

FIGURE 24. The sides and diametric chords of $\Delta(\hat{\mathfrak{Q}})$, showing the embedded C_{n+1} corresponding to $\mathfrak{Q}$ (solid lines).

To get the same conclusion the various appearances of f need not represent the same operation; it is only necessary that for n-ary quasigroups $f_1, g_1, \dots, f_n, g_n$ the n compositions

$$f_i(x_1, \dots, g_i(x_i, \dots, x_{i+n-1}), \dots, x_{2n-1})$$

should be independent of i. Then each of the $2n$ operations is isotopic to the $n-1$-fold iteration of a single group operation; this is part of a theorem of Ušan [36].

The basis for Theorem 9.2 is that the factorization graph of any multiary quasigroup is theta-complete. (This is the quasigroup version of Theorem 3.17.) We are able to characterize factorization graphs completely.

Theorem 9.3. *For a simple graph Δ to be the factorization graph of a multiary quasigroup, a necessary and sufficient condition is that Δ be theta-complete and have a Hamiltonian circle. A second necessary and sufficient condition is that Δ be obtained by edge amalgamations of circles and complete graphs and have a Hamiltonian circle.*

Proof. Apply Theorem 6.2 in view of Theorem 9.1. □

The amalgamation in Theorem 6.2 corresponds to a decomposition of f into iterated group isotopes, irreducible multiary quasigroups of arity greater than 2, and nongroup binary quasigroups. Furthermore, the decomposition of f is unique because the 3-constituents of $\Omega(\mathfrak{Q})$ are unique. Thus we have:

Corollary 9.4 (Belousov; see [6, Section V.4]). *Every multiary quasigroup is in a unique way (up to isotopy) the composition of iterated group isotopes and irreducible, nongroup multiary quasigroups.*

Belousov deduces this through the algebra of multiary quasigroup composition. His key result about such composition, the solution of Equation (1.5), is our next corollary, which we prove by another application of theta completeness. The corollary treats the three possible relationships between the two chords v_iv_k and v_jv_l of C_{n+1} that represent h and h' in the factorization graph. In Case (a) the second chord lies between the endpoints of the first chord. In Case (b) the first chord precedes the second. Case (c), where the chords cross, gives the strongest conclusion.

Bear in mind that a unary quasigroup is merely a permutation of the set $\mathfrak{Q}$.

Corollary 9.5 ([5, Theorem 2.1], [6, Chapter IV]). *Suppose an n-ary quasigroup $\mathfrak{Q}$ has an $(i+1,k)$-factorization,*

$$f(x_1,\dots,x_n) = g(x_1,\dots,x_i,h(x_{i+1},\dots,x_k),\dots,x_n),$$

and a $(j+1,l)$-factorization,

$$f(x_1,\dots,x_n) = g'(x_1,\dots,x_j,h'(x_{j+1},\dots,x_l),\dots,x_n),$$

where $i<k$, $j<l$, and $i\le j$ (and $k<l$ if $i=j$), and g,h,g',h' are multiary quasigroups.

(a) If $k\ge l$, then

$$f(x_1,\dots,x_n) = g(x_1,\dots,a(x_{i+1},\dots,h'(x_{j+1},\dots,x_l),\dots,x_k),\dots,x_n),$$

where a is a multiary quasigroup such that

$$h(x_{i+1},\dots,x_k) = a(x_{i+1},\dots,h'(x_{j+1},\dots,x_l),\dots,x_k).$$

(b) If $k\le j$, then

$$f(x_1,\dots,x_n) = b(x_1,\dots,h(x_{i+1},\dots,x_k),x_{k+1},\dots,h'(x_{j+1},\dots,x_l),\dots,x_n),$$

where b is a multiary quasigroup.

(c) If $i<j<k<l$, then

$$f(x_1,\dots,x_n) = c(x_1,\dots,d(x_{i+1},\dots,x_j)\circ d'(x_{j+1},\dots,x_k)\circ d''(x_{k+1},\dots,x_l),\dots,x_n),$$

where c,d,d',d'' are multiary quasigroups, $\circ$ is a group multiplication, and

$$h(x_{i+1},\dots,x_k) = a'(d(x_{i+1},\dots,x_j)\circ d'(x_{j+1},\dots,x_k)),$$
$$h'(x_{j+1},\dots,x_l) = a''(d'(x_{j+1},\dots,x_k)\circ d''(x_{k+1},\dots,x_l)),$$

in which a',a'' denote unary quasigroups.

Proof. Parts (a) and (b) are immediate from Theorem 9.1. Part (c) is from that theorem, Theorem 9.3, and the case of order four in Lemma 4.2. $\square$

Corollary 9.5 shows that reductive associativity is related to non-crossing partitions [27]. This connection is not surprising; it is known that non-crossing partitions correspond to parenthesizations of a string of $n+1$ letters (i.e., $v_0, v_1, \dots, v_n$). Define an equivalence relation on $N = \{v_0, v_1, \dots, v_n\}$ by extension of $v_i \sim v_j$ if there is a chord $v_i v_j$ in $\Delta(\mathfrak{Q})$, i.e., if there is an $(i+1, j)$-factorization of f. The corollary states that the equivalence classes are the blocks of a non-crossing partition of N, because crossing chords put their nodes into the same equivalence class. Theorem 9.3 tells us that each block of the partition supports a complete subgraph of $\Delta(\mathfrak{Q})$.

Other of our results on biased expansions have quasigroup interpretations. Corollary 5.5 applied to multiary quasigroups is the following statement:

Corollary 9.6. *A composition of multiary quasigroups, all isotopic to iterates of a group $\mathfrak{G}$, is necessarily isotopic to an iteration of $\mathfrak{G}$ if $\mathfrak{G} = \mathbb{Z}_\gamma$ for $\gamma \leq 3$ or $\mathfrak{G} = \mathfrak{V}_4$, but otherwise need not be isotopic to any iterated group.*

By "need not be isotopic", I mean that by appropriate choices in the process of composition, one can make the composition not isotopic to any iterated group.

The quasigroup version of Corollary 5.6 requires a definition. Take an n-ary quasigroup $\mathfrak{Q}$. Lemma 2.5(a) implies that expansion minors of $\langle \mathfrak{Q} C_{n+1} \rangle$ of order $r+1$ correspond to r-ary retracts. Apply Construction XM, taking $S \cup T = C_{n+1}$. The choice of lift $\tilde{T}$ signifies fixing the values of the variables corresponding to edges of T. The variables of the retract are the variables that correspond to edges of S. Thus, expansion minors that contain the edge e_0 correspond to principal retracts of $\mathfrak{Q}$.

Corollary 9.7. *It is possible to have an n-ary quasigroup of any order $\gamma \geq 4$ and any arity $n \geq 3$ that is not isotopic to an iterated group but whose binary principal retracts are all isotopic to the same arbitrary group of order γ, except when the group is $\mathfrak{V}_4$.*

But raising the arity of the retract yields quite a different result. The quasigroup interpretation of Lemma 7.1 is:

Theorem 9.8. *If every ternary principal retract of an n-ary quasigroup $\mathfrak{Q}$ with arity $n \geq 3$ is isotopic to an iterated group (not necessarily the same group), then $\mathfrak{Q}$ is an iterated group isotope.*

10. Postscript

10.1. **Nontopological homotopy?** There is a perceptible flavor of homotopy about our combinatorial arguments. We treat balanced circles in a manner reminiscent of contractible circles in a topological space. A way of making this similarity exact is to embed the underlying graph in a topological space so that the balanced circles are precisely the graph circles that are contractible. That is possible if and only if the graph has gains, so it cannot be used to justify our reasoning. Nevertheless the analogy is suggestive. One has to wonder what lies behind it.

10.2. **Formulas and bijections.** In characterizing multiary groups and in Ušan's generalization (Example 9.2) our method yields a description up to circular paratopy, and this is typical of our results. Post's (Hosszú–Gluskin) and Ušan's theorems, however, give exact formulas for the multiary operations. I believe their formulae and some of the many other generalizations of the Post–Hosszú–Gluskin theorem can be reproduced and extended by the expansion-graph method when it is supplemented by attention to the exact bijections between the set $\mathfrak{Q}$ and the edge fibers.

10.3. **Infinitary quasigroups.** For biased expansion graphs we obtained an infinitary result, Theorem 4.1. It is not immediately obvious how to apply this to infinitary quasigroups. Difficulties arise in defining a factorization graph, an infinitely iterated group, and even an infinitary operation.

Making sense of reductive associativity in the infinitary case seems to require that an infinitary operation be a function $f : \mathfrak{Q}^I \to \mathfrak{Q}$ whose index set I is totally ordered. In order to define a factorization graph as in the introduction, it is necessary to assume that I has a minimum element $\hat{0}$ and a maximum element $\hat{1}$ and that every element other than $\hat{0}$ has a predecessor and every element other than $\hat{1}$ has a successor. The reason is that, in our definition for a finite set $I = \{1, 2, \ldots, k\}$, the nodes of the factorization graph (except for v_0 and v_k) correspond to the covering pairs $i < j$, i.e., pairs where there is no index l such that $i < l < j$. Then one can define reductive associativity by Equation (1.4). To treat all possible ordered index sets, however, the factorization graph may need additional nodes corresponding to elements of I without predecessors or successors and to infinite covering sequences $i_1 < i_2 < \cdots$ and $\cdots < i_2 < i_1$. The exact definition awaits further study.

10.4. **Multivalued quasigroups.** One is inspired by the design interpretation (Section 1.4) to wonder about generalizing to larger values of λ. In terms of biased expansions, suppose for each lift $\tilde{P}$ of $C \setminus e$ there are exactly λ edges $\tilde{e}$ that make a balanced circle, where $\lambda > 1$. Do our theorems generalize to this situation?

The operational view of this generalization is that we have a λ-valued n-ary operation, each of whose n inverse operations is also λ-valued. One has to modify the definition of biased expansion: $\mathcal{B}$ need no longer be a linear class; instead, only each separate lift of a base theta subgraph would be subject to the linearity condition that its number of balanced circles be different from two. The value of λ cannot be a constant, because, in operational language, the composition of λ_1-valued and λ_2-valued operations is $\lambda_1\lambda_2$-valued.

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