# SIX LITTLE SQUARES AND HOW THEIR NUMBERS GROW (SUPPLEMENT: ALTERNATIVE AFFINE SEMIMAGIC COUNT)

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In this supplementary note to [1] we derive an explicit formula, that we call the *supernormalized formula*, for the number of $3 \times 3$ supernormalized semimagic squares with specified magic sum $s$ and specified value of a parameter $\alpha$. This implies, through two more steps, an explicit formula for the number of $3 \times 3$ semimagic squares with specified magic sum $t$, i.e., the affine count in [1, Proposition 3.1]. The supernormalized formula has the form of a single elaborate expression, with floor functions and with separate rules for various cases. It is complicated but the numbers it produces have been (partially) checked against other methods and appear to be correct. We think it is interesting that one can derive any such formula. The same thing should be feasible, though with more difficulty, for affine magilatin enumeration, thereby enabling one to derive the complete quasipolynomial from data as a check on the geometrical Ehrhart method.

## 3. Semimagic squares of order 3

We quote from [1, Proposition 3.1] the general form of a reduced, normalized $3 \times 3$ semimagic square, in which the magic sum is $s = 2\alpha + 2\beta + \gamma$:

(3.1)

| | | |
|---|---|---|
| $0$ | $\beta$ | $2\alpha+\beta+\gamma$ |
| $\alpha+\beta$ | $\alpha+\beta+\gamma-\delta$ | $\delta$ |
| $\alpha+\beta+\gamma$ | $\alpha+\delta$ | $\beta-\delta$ |

**Proposition 3.1.** *A reduced and normalized* $3 \times 3$ *semimagic square has the form* (3.1) *with the restrictions*

$$0 < \alpha, \beta, \gamma;$$
$$0 < \delta < \beta; \tag{3.2}$$

*and*

$$\delta \neq \begin{cases} \frac{\beta-\alpha}{2},\ \frac{\beta}{2},\ \frac{\beta+\gamma}{2},\ \frac{\beta+\alpha+\gamma}{2}; \\ \beta-\alpha,\ \alpha+\gamma; \\ \gamma. \end{cases} \tag{3.3}$$

### 3.2. **Semimagic squares: Affine count (by magic sum) (continued from [1]).**

3.2.7. *Counting and interpolation.* In this alternate method we develop a formula for the number of supernormalized, reduced squares with a fixed value of a certain parameter, that is fast enough to let us compute enough actual values of $n(s)$ to determine the inside-out quasipolynomial $n(s)$ by interpolation.

The simple, direct method of generating all integral lattice points inside $sQ$ and counting those that do not lie in any of the forbidden planes, would take thousands of years in Maple on our computers. The advantage of such a method is that it would be easy to program and verify. For a practical calculation, however, we have to take an indirect approach. We speed up the calculation by summming over two variables, $\delta$ and $\beta$. Thus we compute $n(s,\alpha)$—which counts integral points in a cross-section of $sQ$ taken perpendicular to the $x$-axis—by an explicit formula and then express $n(s)$ as a sum over $\alpha$, as described in the following basic result. All congruences are modulo 2.

**Proposition 3.2** (Supernormalize Formula)**.** *Let $n(s,\alpha)$ be the number of supernormalized squares with the specified value of the parameter $\alpha$ in the standard form of* (3.1)*. Define*

$$R := \left\lfloor \frac{s-1}{2} \right\rfloor - \alpha,$$
$$\varepsilon := \begin{cases} 0 & \text{if } s - \left\lfloor \frac{s-2\alpha}{3} \right\rfloor \text{ is even,} \\ 1 & \text{if } s - \left\lfloor \frac{s-2\alpha}{3} \right\rfloor \text{ is odd,} \end{cases}$$
$$\varepsilon' := \begin{cases} 0 & \text{if } s - \alpha - \left\lfloor \frac{s-\alpha}{3} \right\rfloor \text{ is even,} \\ 1 & \text{if } s - \alpha - \left\lfloor \frac{s-\alpha}{3} \right\rfloor \text{ is odd.} \end{cases}$$

*Then*

$$n(s) = \sum_{\alpha=1}^{\lfloor \frac{s-3}{2} \rfloor} n(s,\alpha), \tag{3.18}$$

*where*

$$\text{(3.19a)} \quad n(s,\alpha) = \binom{R}{2}$$

$$\text{(3.19b)} \qquad - \left\{ \left\lfloor \frac{R}{2} \right\rfloor + \left( R - \left\lfloor \frac{s-2\alpha}{3} \right\rfloor \right) + \left\lfloor \frac{R + \varepsilon - \lfloor \frac{s-2\alpha}{3} \rfloor}{2} \right\rfloor \right\}$$

$$\text{(3.19c)} \qquad - \left\{ (R-\alpha) + \left\lfloor \frac{R-\alpha}{2} \right\rfloor + \left( R - \left\lfloor \frac{s-\alpha}{3} \right\rfloor \right) + \left\lfloor \frac{R + \varepsilon' - \lfloor \frac{s-\alpha}{3} \rfloor}{2} \right\rfloor \right\} \text{ if } \alpha < \frac{s}{4}$$

$$\text{(3.19d)} \qquad + 1 \text{ if } \alpha < \frac{s}{6}$$

$$\text{(3.19e)} \qquad + 1 \text{ if } 3 \mid s - \alpha \text{ and } \alpha < \frac{s}{4}$$

$$\text{(3.19f)} \qquad + 2 \text{ if } 3 \mid s \text{ and } \alpha < \frac{s}{6}$$

$$\text{(3.19g)} \qquad + 1 \text{ if } 3 \mid s + \alpha \text{ and } \alpha < \frac{s}{8}$$

$$\text{(3.19h)} \qquad + 1 \text{ if } 5 \mid s - 2\alpha$$

$$\text{(3.19i)} \qquad + 1 \text{ if } 5 \mid s + \alpha \text{ and } \alpha < \frac{s}{4}$$

$$\text{(3.19j)} \qquad + 1 \text{ if } 5 \mid s - \alpha \text{ and } \alpha < \frac{s}{6}$$

$$\text{(3.19k)} \qquad + 1 \text{ if } 5 \mid s + 2\alpha \text{ and } \alpha < \frac{s}{8}$$

$$\text{(3.19l)} \qquad - 1 \text{ if } \alpha = \frac{s}{7}$$

$$\text{(3.19m)} \qquad - 1 \text{ if } \alpha = \frac{s}{14}.$$

Our computer program implements Formulas (3.18) and (3.19) to calculate $n(s)$ for $1 \leq s \leq 4d$ (where $d = 4620$ is the denominator), which is enough values to fit a cubic quasipolynomial. The result should agree with the quasipolynomial found by geometry and inside-out Ehrhart theory in [1]. Then we have three kinds of check: the leading coefficients should all equal $1/48$, the quasipolynomial should have zeroes at the smallest values of $s$ (where the number of squares is zero), and $n(0)$, using the calculated 0th constituent, should be a positive integer, the number of regions of the arrangement in $Q$ that corresponds to the restrictions (3.3).

In the computer program we use the alternative forms on the right sides of the following equations:

$$\left\lfloor \frac{R + \varepsilon - \lfloor \frac{s-2\alpha}{3} \rfloor}{2} \right\rfloor = \frac{R - 1 + \eta - \lfloor \frac{s-2\alpha}{3} \rfloor}{2},$$
$$\left\lfloor \frac{R + \varepsilon' - \lfloor \frac{s-\alpha}{3} \rfloor}{2} \right\rfloor = \frac{R - 1 + \eta' - \lfloor \frac{s-\alpha}{3} \rfloor}{2},$$

where

$$\eta := \text{ number among the variables } s, R \text{ that are } \not\equiv \left\lfloor \frac{s-2\alpha}{3} \right\rfloor,$$
$$\eta' := \text{ number among the variables } s, R \text{ that are } \not\equiv \left\lfloor \frac{s-\alpha}{3} \right\rfloor.$$

*Proof.* The basic counting formula for $n(s, \alpha)$ is

$$n(s,\alpha) = \sum_{\beta} \sum_{\delta} 1,$$

summed over all pairs that, with the given values of $s$ and $\alpha$, satisfy (3.2) and (3.3). We can reduce this to a sum over $\beta$ by finding the number of valid $\delta$ values for a given $\beta$. In general, we have $\beta - 1$ possible $\delta$'s, due to (3.2), with up to seven exclusions, depending on which of the quantities on the right hand of (3.3) are integers in the interval $(0, \beta)$. Let's write $e(\beta)$ for the number of excluded values of $\delta$, taking $s$ and $\alpha$ as fixed, so that

$$n(s,\alpha) = \sum_{\beta} \big[\beta - 1 - e(\beta)\big].$$

Our first task is to evaluate $e(\beta)$.

**Lemma 3.3.** *The number of excluded values of $\delta$ is given by*

$$\begin{aligned}
e(\beta) = \quad & 1 \text{ if } \beta \equiv 0 \\
& + 1 \text{ if } \beta > \alpha \\
& + 1 \text{ if } \beta > \alpha \text{ and } \beta \equiv \alpha \\
& + 1 \text{ if } \beta > \gamma \\
& + 1 \text{ if } \beta > \gamma \text{ and } \beta \equiv \gamma \\
& + 1 \text{ if } \beta > \alpha + \gamma \\
& + 1 \text{ if } \beta > \alpha + \gamma \text{ and } \beta \equiv \alpha + \gamma \\
& - 1 \text{ if } \beta = 2\alpha \\
& - 1 \text{ if } \beta = \frac{s-\alpha}{3} \\
& - 2 \text{ if } \beta = \frac{s}{3} \\
& - 1 \text{ if } \beta = \frac{s+\alpha}{3} \\
& - 1 \text{ if } \beta = \frac{2s-4\alpha}{5} \\
& - 1 \text{ if } \beta = \frac{2s-3\alpha}{5} \\
& - 1 \text{ if } \beta = \frac{2s-2\alpha}{5} \\
& - 1 \text{ if } \beta = \frac{2s-\alpha}{5} \\
& + 1 \text{ if } \alpha = s/7 \text{ and } \beta = 2\alpha \\
& + 1 \text{ if } \alpha = s/14 \text{ and } \beta = 5\alpha.
\end{aligned} \tag{3.20}$$

*Proof.* Formula (3.3) shows seven forbidden values, but which of them are effective depends on $s$, $\alpha$, and $\beta$. An effective exclusion has to be an integer; that implies congruence criteria for the first four excluded values, e.g., for the first one to be effective it is necessary that $\beta \equiv \alpha \pmod 2$. (All congruences will be modulo 2.) The other criteria are in terms of the range of $\beta$. Since $1 \le \gamma = s - 2\alpha - 2\beta$, we see that

$$0 < \alpha \le \frac{s-3}{2} \qquad \text{and} \qquad 0 < \beta \le \frac{s-1}{2} - \alpha.$$

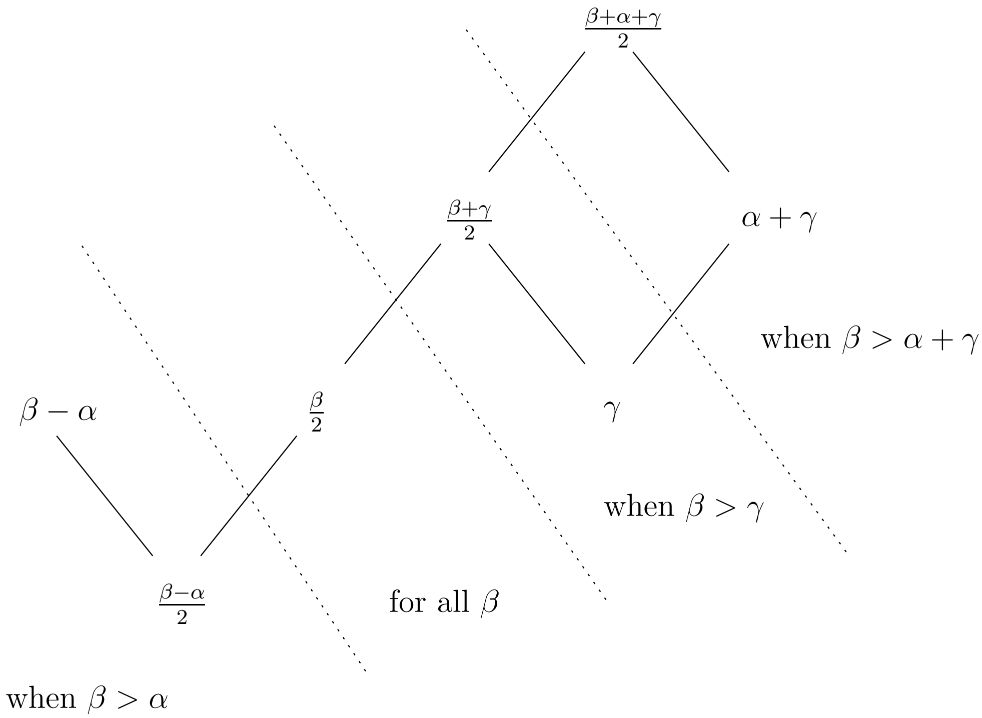
Now, ignoring the parity conditions for the moment,

($\delta$1) $\delta \neq \frac{\beta}{2}$ is effective for all $\beta$;
($\delta$2) $\delta \neq \beta - \alpha$, $\frac{\beta-\alpha}{2}$ are effective for $\beta > \alpha$;
($\delta$3) $\delta \neq \gamma$, $\frac{\beta+\gamma}{2}$ are effective for $\beta > \gamma$;
($\delta$4) $\delta \neq \alpha + \gamma$, $\frac{\beta+\alpha+\gamma}{2}$ are effective for $\beta > \alpha + \gamma$.

This can be summarize in the formula

$$\begin{aligned} e(\beta) = \quad & 1 \text{ if } \beta \equiv 0 \\ & + 1 \text{ if } \beta > \alpha \\ & + 1 \text{ if } \beta > \alpha \text{ and } \beta \equiv 0 \\ & + 1 \text{ if } \beta > \gamma \\ & + 1 \text{ if } \beta > \gamma \text{ and } \beta \equiv \gamma \\ & + 1 \text{ if } \beta > \alpha + \gamma \\ & + 1 \text{ if } \beta > \alpha + \gamma \text{ and } \beta \equiv \alpha + \gamma, \end{aligned} \tag{3.21}$$

which is valid if none of the seven exclusions coincide. Keeping in mind the ranges of validity, we can display the excluded values in a Hasse diagram:




Some of the exclusions might happen to be equal, but only those that are not comparable in this partial ordering. Table 3.1 is a complete list of possible equalities and the conditions under which they hold true.

We see in Table 3.1 that there are eight special values of $\beta$, each of which has to be examined for its effect on $e(\beta)$. We find that, in every case, the restrictions in Table 3.1 are superfluous.

($\beta$1) $\beta = 2\alpha$: Here $\beta - \alpha = \frac{\beta}{2}$. The restrictions are $\beta > \alpha$ and $\beta \equiv 0$. Obviously, whenever $\beta$ actually takes on the value $2\alpha$ the restrictions are satisfied; so they can be ignored. In this case we should deduct 1 from the value of $e(\beta)$ given by Formula 3.21 because the formula treats the one special value as two different ones.

($\beta$2) $\beta = \frac{s-\alpha}{3}$: Here $\beta - \alpha = \gamma$. The restrictions are $\beta > \alpha, \gamma$. If $\beta > \frac{s-2\alpha}{3}$, as it is in this case, then $\beta > s - 2\alpha - 2\beta = \gamma$. The condition under which $\beta > \alpha$ is $\alpha < \frac{s}{4}$, but this too is implied by $\beta = \frac{s-\alpha}{3}$. In this case we should deduct 1 from the value of $e(\beta)$ given by Formula 3.21.

($\beta$3) $\beta = \frac{s}{3}$: Here there are three exclusions that may coincide; see the second, fifth, and last lines of Table 3.1. Since $\gamma \equiv s = 3\beta$ in this case, the congruence restriction is automatically satisfied. Also, $\beta > \alpha + \gamma \iff \beta > s - \alpha - 2\beta \iff 0 > -\alpha$, so $\beta = \frac{s}{3} \implies \beta > \alpha + \gamma$. Therefore, when $\beta = \frac{s}{3}$ we should always deduct 2 from the value of $e(\beta)$ given by Formula 3.21.

($\beta$4) $\beta = \frac{s+\alpha}{3}$: Here $\beta - \alpha = \frac{\beta+\alpha+\gamma}{2}$. The restrictions, which may be written $\beta > \alpha + \gamma$ and $\beta \equiv s + \alpha$, follow much as in the previous case. In this case we should deduct 1 from the value of $e(\beta)$ given by Formula 3.21.

Table 3.1. Coinciding exclusions, the values of $\beta$ at which they occur, and the restrictions on $\beta$. Congruence is modulo 2. Note that $\gamma \equiv s$ always.

| Equal exclusions | $\beta$ value | $\beta$ restrictions |
|---|---|---|
| $\beta - \alpha = \frac{\beta}{2}$ | $\beta = 2\alpha$ | $\beta > \alpha$ and $\beta \equiv 0$ |
| $\beta - \alpha = \frac{\beta+\gamma}{2}$ | $\beta = \frac{s}{3}$ | $\beta > \alpha, \gamma$ and $\beta \equiv \gamma$ |
| $\beta - \alpha = \gamma$ | $\beta = \frac{s-\alpha}{3}$ | $\beta > \alpha, \gamma$ |
| $\beta - \alpha = \frac{\beta+\alpha+\gamma}{2}$ | $\beta = \frac{s+\alpha}{3}$ | $\beta > \alpha + \gamma$ and $\beta \equiv \alpha + \gamma$ |
| $\beta - \alpha = \alpha + \gamma$ | $\beta = \frac{s}{3}$ | $\beta > \alpha + \gamma$ |
| $\frac{\beta-\alpha}{2} = \gamma$ | $\beta = \frac{2s-3\alpha}{5}$ | $\beta > \alpha, \gamma$ and $\beta \equiv \alpha$ |
| $\frac{\beta-\alpha}{2} = \alpha + \gamma$ | $\beta = \frac{2s-\alpha}{5}$ | $\beta > \alpha + \gamma$ and $\beta \equiv \alpha$ |
| $\frac{\beta}{2} = \gamma$ | $\beta = \frac{2s-4\alpha}{5}$ | $\beta > \gamma$ and $\beta \equiv 0$ |
| $\frac{\beta}{2} = \alpha + \gamma$ | $\beta = \frac{2s-2\alpha}{5}$ | $\beta > \alpha + \gamma$ and $\beta \equiv 0$ |
| $\frac{\beta+\gamma}{2} = \alpha + \gamma$ | $\beta = \frac{s}{3}$ | $\beta > \alpha + \gamma$ and $\beta \equiv \gamma$ |

($\beta$5) $\beta = \frac{2s-4\alpha}{5}$: Here $\frac{\beta}{2} = \gamma$. The restrictions are $\beta > \frac{s-2\alpha}{3}$ and $\beta \equiv 0$. They follow easily. In this case we should deduct 1 from the value of $e(\beta)$ given by Formula 3.21.

($\beta$6) $\beta = \frac{2s-3\alpha}{5}$: Here $\frac{\beta-\alpha}{2} = \gamma$. The restrictions are $\beta > \frac{s-\alpha}{3}$, $\beta > \alpha$, and $\beta \equiv \alpha$. The last holds because $5\beta \equiv 2s - 3\alpha$. The first two are both equivalent to $\alpha < \frac{s}{4}$, which follows from $0 < \gamma = s - 2\frac{2s-3\alpha}{5} - 2\alpha = \frac{s-4\alpha}{5}$. In this case we should deduct 1 from the value of $e(\beta)$ given by Formula 3.21.

($\beta$7) $\beta = \frac{2s-2\alpha}{5}$: Here $\frac{\beta}{2} = \alpha + \gamma$. The restrictions are $\beta > \alpha + \gamma$ and $\beta \equiv 0$. The latter is automatic from $5\beta = 2s - 2\alpha$. The former is equivalent to $s > \alpha$, which is trivially satisfied. In this case we should deduct 1 from the value of $e(\beta)$ given by Formula 3.21.

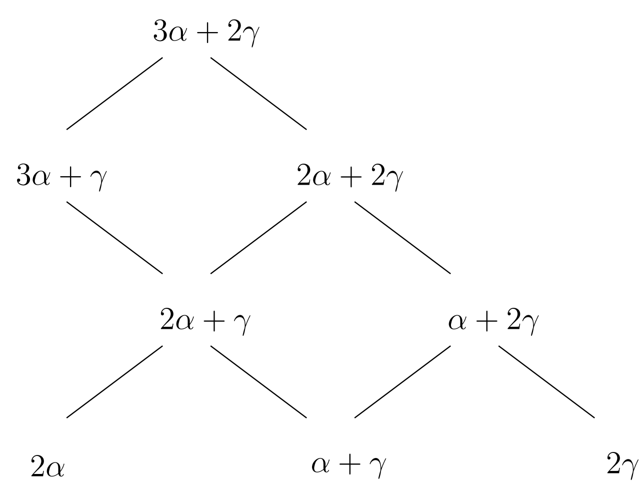
($\beta$8) $\beta = \frac{2s-\alpha}{5}$: Here $\frac{\beta-\alpha}{2} = \alpha + \gamma$. The restrictions are $\beta > \alpha + \gamma$ and $\beta \equiv \alpha$. These follow much as in the preceding case, but even more trivially. In this case we should deduct 1 from the value of $e(\beta)$ given by Formula 3.21.

These results are summarized in Table 3.2 along with the ranges of $\alpha$ in which the various special values are possible values of $\beta$.

There is another complication. Some of the special values of $\beta$ might coincide. The ways it might happen are shown by a Hasse diagram of special values:

Table 3.2. The special values of $\beta$, with the corresponding deduction from $e(\beta)$ and the range of $\alpha$ in which each special value can occur. The values of $\beta$ are stated in terms of $s$ and $\alpha$ and, for ease of comparison in the Hasse diagram, also in terms of $\alpha$ and $\gamma$.

| Special value of $\beta$ | Deduction | Range of $\alpha$ |
|---|---|---|
| $2\alpha$ | 1 | $\alpha < s/6$ |
| $\frac{s-\alpha}{3} = \alpha + \gamma$ | 1 | $\alpha < s/4$ |
| $\frac{s}{3} = 2\alpha + \gamma$ | 2 | $\alpha < s/6$ |
| $\frac{s+\alpha}{3} = 3\alpha + \gamma$ | 1 | $\alpha < s/8$ |
| $\frac{2s-4\alpha}{5} = 2\gamma$ | 1 | $\alpha < s/2$ |
| $\frac{2s-3\alpha}{5} = \alpha + 2\gamma$ | 1 | $\alpha < s/4$ |
| $\frac{2s-2\alpha}{5} = 2\alpha + 2\gamma$ | 1 | $\alpha < s/6$ |
| $\frac{2s-\alpha}{5} = 3\alpha + 2\gamma$ | 1 | $\alpha < s/8$ |




One can read off the possible equations amongst the values; they occur as shown in Table 3.3. In each case there is a certain number of distinct excluded values of $\delta$, which is given in the table; we found the numbers by writing down the excluded values and counting them. For instance, in the first line the excluded values of $\delta/s$ are $\frac{2}{13}$, $\frac{1}{13}$, $\frac{2}{13}$, $\frac{5}{26}$, $\frac{1}{13}$, $\frac{7}{26}$, $\frac{3}{13}$, of which there are three distinct values with denominator 13, applying to all $s$ that are divisible by 13, and two more values that apply only when $s$ is also even.

TABLE 3.3. Values of $\alpha$ at which special $\beta$ values coincide, the corresponding values of $\beta$ and $\gamma$, the coinciding values of $\beta$, and the correct value of the deduction $e(\beta)$ with the applicable values of $s$. Congruences are modulo 2.

| $\frac{1}{s}(\alpha,\beta,\gamma)$ | Coincident $\beta$ values | Correct Deduction | When |
|---|---|---|---|
| $\frac{1}{13}(2,4,1)$ | $2\alpha=\alpha+2\gamma$ | $3+(\text{if } s\equiv 0)\ 2$ | $13 \mid s$ |
| $\frac{1}{11}(1,4,1)$ | $3\alpha+\gamma=2\alpha+2\gamma$ | $3+(\text{if } s\equiv 0)\ 2$ | $11 \mid s$ |
| $\frac{1}{9}(1,3,1)$ | $2\alpha+\gamma=\alpha+2\gamma$ | $2+(\text{if } s\equiv 0)\ 2$ | $9 \mid s$ |
| $\frac{1}{7}(1,2,1)$ | $2\alpha=\alpha+\gamma=2\gamma$ | $1+(\text{if } s\equiv 0)\ 2$ | $7 \mid s$ |
| $\frac{1}{14}(1,5,2)$ | $3\alpha+\gamma=\alpha+2\gamma$ | $4+(\text{if } 28 \mid s)\ 2$ | $14 \mid s$ |
| $\frac{1}{12}(1,4,2)$ | $2\alpha+\gamma=2\gamma$ | $3+(\text{if } 24 \mid s)\ 2$ | $12 \mid s$ |
| $\frac{1}{17}(1,6,3)$ | $3\alpha+\gamma=2\gamma$ | $3+(\text{if } s\equiv 0)\ 2$ | $17 \mid s$ |

The formula for $e(\beta)$ as developed so far, however, does not always give the correct number of exclusions. This is the formula obtained by taking, in (3.20), the first group of seven positive terms and the next group of eight negative terms. If we apply this formula to the special values of $\beta$ in Table 3.3 we get the numbers in Table 3.4. The table also shows how these numbers compare with the correct numbers. The difference is a correction which appears as the last group of three positive entries in (3.20). ■

It follows from the lemma that

$$
\begin{aligned}
n(s,\alpha) = {} & \sum_{\beta=1}^{R}(\beta-1) \\
& - \sum_{\substack{\beta=1\\ \beta\equiv 0}}^{R} 1 - \sum_{\beta=\alpha+1}^{R} 1 - \sum_{\substack{\beta=\alpha+1\\ \beta\equiv\alpha}}^{R} 1 - \sum_{\gamma<\beta\le R} 1 - \sum_{\substack{\gamma<\beta\le R\\ \beta\equiv\gamma}} 1 - \sum_{\alpha+\gamma<\beta\le R} 1 - \sum_{\substack{\alpha+\gamma<\beta\le R\\ \beta\equiv\alpha+\gamma}} 1 \\
& + \sum_{1\le\beta=2\alpha\le R} 1 + \sum_{1\le\beta=\frac{s-\alpha}{3}\le R} 1 + \sum_{1\le\beta=\frac{s}{3}\le R} 2 + \sum_{1\le\beta=\frac{s+\alpha}{3}\le R} 1 \\
& + \sum_{1\le\beta=\frac{2s-4\alpha}{5}\le R} 1 + \sum_{1\le\beta=\frac{2s-3\alpha}{5}\le R} 1 + \sum_{1\le\beta=\frac{2s-2\alpha}{5}\le R} 1 + \sum_{1\le\beta=\frac{2s-\alpha}{5}\le R} 1 \\
& - \sum_{\substack{1\le\beta=2s/7\le R\\ \alpha=s/7}} 1 - \sum_{\substack{1\le\beta=5s/14\le R\\ \alpha=s/14}} 1.
\end{aligned}
\tag{3.22}
$$

Table 3.4. Values of $\alpha, \beta, \gamma$ at which special $\beta$ values coincide, the deduction computed by the first two groups in Formula 3.20 (assuming applicable values of $s$), and the additional term needed in Formula 3.20 to get the correct deduction as in Table 3.3.

| $\frac{1}{s}(\alpha,\beta,\gamma)$ | Computed deductions *Positive* | *Negative* | *Net* | Correction |
|---|---|---|---|---|
| $\frac{1}{13}(2,4,1)$ | $5 + (\text{if } s \equiv 0)\ 2$ | 2 | $3 + (\text{if } s \equiv 0)\ 2$ | 0 |
| $\frac{1}{11}(1,4,1)$ | $5 + (\text{if } s \equiv 0)\ 2$ | 2 | $3 + (\text{if } s \equiv 0)\ 2$ | 0 |
| $\frac{1}{9}(1,3,1)$ | $5 + (\text{if } s \equiv 0)\ 2$ | 3 | $2 + (\text{if } s \equiv 0)\ 2$ | 0 |
| $\frac{1}{7}(1,2,1)$ | $5 + (\text{if } s \equiv 0)\ 2$ | 3 | $0 + (\text{if } s \equiv 0)\ 2$ | $+1$ |
| $\frac{1}{14}(1,5,2)$ | $5 + (\text{if } 28 \mid s)\ 2$ | 2 | $3 + (\text{if } 28 \mid s)\ 2$ | $+1$ |
| $\frac{1}{12}(1,3,4)$ | $5 + (\text{if } 24 \mid s)\ 2$ | 3 | $3 + (\text{if } 24 \mid s)\ 2$ | 0 |
| $\frac{1}{17}(1,6,3)$ | $5 + (\text{if } s \equiv 0)\ 2$ | 2 | $3 + (\text{if } s \equiv 0)\ 2$ | 0 |

The upper bound follows from the fact that $\beta = \frac{s-\gamma}{2} - \alpha \leq R$. The sums with upper limit, $U$, less than lower limit, $L$ (or, if one prefers, with $U < L-1$) have the value zero; they should be interpreted as missing terms. (It is optional whether or not to include a term where $U = L-1$. We simplify some formulas by making the right choice in each case.) In simplifying (3.22) we will need to determine which sums those are.

The first term sums to $\binom{\beta}{2}$. The second equals the number of even positive integers up to $R$, which is $\lfloor \frac{R}{2} \rfloor$. This accounts for the first two terms in Formula (3.19).

For the third and fourth terms we must decide when $R \geq \alpha$. (When $R = \alpha$, the term can be included or not since it has value 0.) This is easy:

$$R = \left\lfloor \frac{s-1}{2} \right\rfloor - \alpha \geq \alpha \iff \left\lfloor \frac{s-1}{2} \right\rfloor \geq 2\alpha \iff \frac{s-1}{2} \geq 2\alpha \iff \alpha < \frac{s}{4}.$$

Therefore, these terms appear if and only if $\alpha < s/4$. Then the first term is $R - \alpha$ and the second is $\lfloor \frac{R-\alpha}{2} \rfloor$. Thus we account for the first two terms in (3.19c).

For the next two terms of (3.22), note that $\beta > \gamma \iff \beta > \frac{s-2\alpha}{3}$ so the lower bound can be taken to be $\lfloor \frac{s-2\alpha}{3} \rfloor + 1$. Thus, the terms appear

$$\begin{aligned} &\iff \left\lfloor \frac{s-2\alpha}{3} \right\rfloor \leq R \iff \frac{s-2\alpha-2}{3} \leq \left\lfloor \frac{s-1}{2} \right\rfloor - \alpha \\ &\iff \begin{Bmatrix} \frac{s-2\alpha-2}{3} \leq \frac{s-1}{2} - \alpha & \text{if } s \text{ is odd} \\ \frac{s-2\alpha-2}{3} \leq \frac{s-2}{2} - \alpha & \text{if } s \text{ is even} \end{Bmatrix} \end{aligned}$$

$$\iff \begin{Bmatrix} \alpha \le \frac{s+1}{2} & \text{if } s \text{ is odd} \\ \alpha \le \frac{s-2}{2} & \text{if } s \text{ is odd} \end{Bmatrix},$$

which is always satisfied since $\alpha \le (s-3)/2$. The first of our terms now equals $R - \lfloor \frac{s-2\alpha}{3} \rfloor$. The second sum, which requires $\beta \equiv \gamma$, equivalently $\beta \equiv s$, has two cases depending on whether $s - \lfloor \frac{s-2\alpha}{3} \rfloor$ is even or odd: the sum equals

$$\begin{cases} \left\lfloor \dfrac{(R+1-\lfloor \frac{s-2\alpha}{3} \rfloor}{2} \right\rfloor & \text{if } s \not\equiv \lfloor \frac{s-2\alpha}{3} \rfloor, \\ \left\lfloor \dfrac{R-\lfloor \frac{s-2\alpha}{3} \rfloor}{2} \right\rfloor & \text{if } s \equiv \lfloor \frac{s-2\alpha}{3} \rfloor, \end{cases}$$

which is the same as

$$\begin{cases} \left\lfloor \dfrac{(R+1-\lfloor \frac{s-2\alpha}{3} \rfloor}{2} \right\rfloor & \text{if } R \not\equiv \lfloor \frac{s-2\alpha}{3} \rfloor \not\equiv s, \\ \left\lfloor \dfrac{R-\lfloor \frac{s-2\alpha}{3} \rfloor}{2} \right\rfloor & \text{if } R \equiv \lfloor \frac{s-2\alpha}{3} \rfloor \not\equiv s \text{ or } R \not\equiv \lfloor \frac{s-2\alpha}{3} \rfloor \equiv s, \\ \left\lfloor \dfrac{R-1-\lfloor \frac{s-2\alpha}{3} \rfloor}{2} \right\rfloor & \text{if } R \equiv \lfloor \frac{s-2\alpha}{3} \rfloor \equiv s. \end{cases}$$

For the seventh and eight terms of (3.22) note that $\beta > \alpha + \gamma \iff \beta > \frac{s-\alpha}{3}$. Thus, we are summing over the range $\lfloor \frac{s-\alpha}{3} \rfloor + 1 \le \beta \le R$. The term appears, possibly as a zero sum,

$$\iff \left\lfloor \frac{s-\alpha}{3} \right\rfloor \le R \iff \left\lfloor \frac{s-\alpha}{3} \right\rfloor \le \frac{s-1}{2} - \alpha$$
$$\iff \begin{cases} \frac{s-\alpha}{3} < \frac{s}{2} - \alpha & \text{if } s \text{ is even,} \\ \frac{s-\alpha}{3} < \frac{s+1}{2} - \alpha & \text{if } s \text{ is odd} \end{cases}$$
$$\iff \begin{cases} \alpha < \frac{s}{4} & \text{if } s \text{ is even,} \\ \alpha < \frac{s+3}{4} & \text{if } s \text{ is odd.} \end{cases}$$

However, the value $\alpha = \frac{s+1}{4}$, the highest possible when $s$ is odd, gives $\left\lfloor \frac{s-\alpha}{3} \right\rfloor = \alpha$ and $R = \alpha - 1$; in this case the sum equals zero, so we may say that the seventh and eighth terms appear $\iff \alpha < \frac{s}{4}$. The value of the seventh term is $R - \left\lfloor \frac{s-\alpha}{3} \right\rfloor$. The eighth is

$$\sum_{\substack{\beta = \lfloor \frac{s-\alpha}{3} \rfloor + 1 \\ \beta \equiv s-\alpha}}^{R} 1 = \begin{cases} \left\lfloor \dfrac{R-\lfloor \frac{s-\alpha}{3} \rfloor}{2} \right\rfloor & \text{if } s-\alpha \equiv \lfloor \frac{s-\alpha}{3} \rfloor, \\ \left\lfloor \dfrac{R+1-\lfloor \frac{s-\alpha}{3} \rfloor}{2} \right\rfloor & \text{otherwise.} \end{cases}$$

Now we treat the group of sums in which $\alpha$ is constrained. (These terms are subtracted.) In the first of them, $\beta = 2\alpha \le R$. As

$$2\alpha \le R \iff 2\alpha \le \left\lfloor \frac{s-1}{2} \right\rfloor - \alpha \iff 2\alpha < \frac{s}{2} - \alpha \iff \alpha < \frac{s}{6},$$

this term sums to 1 if $\alpha < \frac{s}{6}$ and 0 otherwise. The next three terms have $\beta = \frac{s-k\alpha}{3}$, which must be an integer; each sum is 0 unless $3 \mid s - k\alpha$ and $\frac{s-k\alpha}{3} \le R$. The latter holds

$$\iff \frac{s-k\alpha}{3} < \frac{s}{2} - \alpha \iff \alpha < \frac{s}{6-2k}.$$

The cases are $k = 1, 0, -1$; the constraints are $\alpha < \frac{s}{4}, \frac{s}{6}, \frac{s}{8}$.

In the last four of these terms, $\beta = \frac{2s-k\alpha}{5}$ where $k = 4, 3, 2, 1$. Note that

$$\begin{aligned} 5 \mid 2s - 4\alpha &\iff 5 \mid s - 2\alpha, \\ 5 \mid 2s - 3\alpha &\iff 5 \mid s + \alpha, \\ 5 \mid 2s - 2\alpha &\iff 5 \mid s - \alpha, \\ 5 \mid 2s - \alpha &\iff 5 \mid s + 2\alpha. \end{aligned}$$

As for the requirement that $\frac{2s-k\alpha}{5} \le R$, it is equivalent to $\frac{2s-k\alpha}{5} < \frac{s}{2} - \alpha$, whose solution is $\alpha < \frac{s}{2(5-k)}$. When $k = 4$ this is $\alpha < s/2$, which is always valid, so no upper bound on $\alpha$ is needed in (3.19).

The very last sums are obvious. ■

## References

[1] Matthias Beck and Thomas Zaslavsky, Six little squares and how their numbers grow. In preparation.

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