```
> with(linalg):
```

```
Warning, the protected names norm and trace have been redefined and
unprotected
```

Maximum value of k in all calculations.

```
> maxk:=60:
```

Latin homogeneous 2x3 rectangles

This is the raw data calculated by a simple method:

```
> for k from 1 to maxk do
    w[k]:=0:
    s[k]:=0:
     for x[2] from 1 to (k-1) do
      for x[3] from 1 to (k-1) do
       for x[4] from 1 to (k-1) do
>        x[1]:=2*x[2]+2*x[3]-3*x[4]:
         if ( (x[1]>0) and (x[1]<k) ) then
>         cs:=x[1]+x[4]:
>         x[5]:=cs-x[2]:
          x[6]:=cs-x[3]:
>           if ( (x[5]>0) and (x[5]<k) and (x[6]>0) and (x[6]<k)
  ) then
             w[k]:=w[k]+1:
             s[k]:=s[k]+1:
>            if ( (x[1]=x[2]) or (x[1]=x[3]) or (x[2]=x[3]) or
  (x[4]=x[5]) or (x[4]=x[6]) or (x[5]=x[6]) or (x[1]=x[4]) or (x
  [2]=x[5]) or (x[3]=x[6]) ) then s[k]:=s[k]-1: fi:
            fi:
          fi:
>      od:
      od:
     od:
    print(k,w[k],s[k]):
  od:
```

(1)

1, 0, 0
2, 1, 0
3, 2, 0
4, 9, 0
5, 16, 0
6, 35, 0
7, 54, 0
8, 91, 12
9, 128, 24
10, 189, 48
11, 250, 72
12, 341, 120
13, 432, 168
14, 559, 240
15, 686, 312
16, 855, 420
17, 1024, 528
18, 1241, 672
19, 1458, 816
20, 1729, 1008
21, 2000, 1200
22, 2331, 1440
23, 2662, 1680
24, 3059, 1980
25, 3456, 2280
26, 3925, 2640
27, 4394, 3000
28, 4941, 3432
29, 5488, 3864
30, 6119, 4368
31, 6750, 4872
32, 7471, 5460

33, 8192, 6048

34, 9009, 6720

35, 9826, 7392

36, 10745, 8160

37, 11664, 8928

38, 12691, 9792

39, 13718, 10656

40, 14859, 11628

41, 16000, 12600

42, 17261, 13680

43, 18522, 14760

44, 19909, 15960

45, 21296, 17160

46, 22815, 18480

47, 24334, 19800

48, 25991, 21252

49, 27648, 22704

50, 29449, 24288

51, 31250, 25872

52, 33201, 27600

53, 35152, 29328

54, 37259, 31200

55, 39366, 33072

56, 41635, 35100

57, 43904, 37128

58, 46341, 39312

59, 48778, 41496

60, 51389, 43848

We expect a quasipolynomial for w and a quasipolynomial for s, both of degree 3. We don't know the period; the following calculations are set up to do any desired period. The variables:
p = assumed period of quasipolynomial,
r (1<=r<=p) = constituent residue,
d = degree of polynomial, dp = d+1.

The first step sets up the period and degree.

```
> p:=4;
```

$$p := 4 \quad (2)$$

```
> dp := 3+1;
```

$$dp := 4 \quad (3)$$

Arrays to hold the coefficients of the weak and strong polynomials. "coef" is a temporary working array.

```
> coef:=array(1..dp);
  wcoeff:=array(1..p,1..dp);
  scoeff:=array(1..p,1..dp);
```

$$coef := array(1..4, [\,]) \quad (4)$$

$$wcoeff := array(1..4, 1..4, [\,])$$

$$scoeff := array(1..4, 1..4, [\,])$$

The following procedure will generate all the p different weak polynomials and p different strong polynomials, factor them, and test by substituting the next value of the argument, comparing to the raw data of the surplus period that was calculated in the first procedure). The polynomials will be saved in "wpoly[r]" and "spoly[r]".

```
> for r from 1 to p do
```

The following procedure will generate the matrix of values for numbers mod r of the period for degree d with any coefficients.

```
> V2:=array(1..dp,1..dp):
> for n from 1 to dp do
    for k from 1 to dp do
>     V2[k,n]:=(p*(k-1)+r)^(n-1):
    od:
> od:
> print(V2);
```

This part assumes degree 3.

```
> print([w[r],w[r+p],w[r+2*p],w[r+3*p]]);
> coef:=linsolve(V2,[w[r],w[r+p],w[r+2*p],w[r+3*p]]);
> for j from 1 to dp do
>   wcoeff[r,j]:=coef[j]:
  od;

>  coef:=linsolve(V2,[s[r],s[r+p],s[r+2*p],s[r+3*p]]);
> for j from 1 to dp do
>   scoeff[r,j]:=coef[j]:
  od;
```

```
> wpoly[r]:=wcoeff[r,4]*x^3+wcoeff[r,3]*x^2+wcoeff[r,2]*x+wcoeff
  [r,1]; subs(x=r+4*p,wpoly[r]); factor(wpoly[r]);

> spoly[r]:=scoeff[r,4]*x^3+scoeff[r,3]*x^2+scoeff[r,2]*x+scoeff
  [r,1]; subs(x=r+4*p,spoly[r]); factor(spoly[r]);

> od;
```

$$V2 := array(1..4, 1..4, [\,]) \tag{5}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 5 & 25 & 125 \\ 1 & 9 & 81 & 729 \\ 1 & 13 & 169 & 2197 \end{bmatrix}$$

$$[0, 16, 128, 432]$$

$$coef := \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & -\frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

$$coef := \begin{bmatrix} -\frac{15}{2} & \frac{41}{4} & -3 & \frac{1}{4} \end{bmatrix}$$

$$wpoly_1 := \frac{1}{4}\,x^3 - \frac{3}{4}\,x^2 + \frac{3}{4}\,x - \frac{1}{4}$$

$$1024$$

$$\frac{(x-1)^3}{4}$$

$$spoly_1 := \frac{1}{4}\,x^3 - 3\,x^2 + \frac{41}{4}\,x - \frac{15}{2}$$

$$528$$

$$\frac{(x-1)\,(x-5)\,(x-6)}{4}$$

$$V2 := array(1..4, 1..4, [\,])$$

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 6 & 36 & 216 \\ 1 & 10 & 100 & 1000 \\ 1 & 14 & 196 & 2744 \end{bmatrix}$$

$$[1, 35, 189, 559]$$

$$coef := \left[\begin{array}{cccc} -1 & \frac{3}{2} & -\frac{3}{4} & \frac{1}{4} \end{array}\right]$$

$$coef := \left[\begin{array}{cccc} -12 & 11 & -3 & \frac{1}{4} \end{array}\right]$$

$$wpoly_2 := \frac{1}{4}\, x^3 - \frac{3}{4}\, x^2 + \frac{3}{2}\, x - 1$$

$$1241$$

$$\frac{(x-1)\,(x^2-2\,x+4)}{4}$$

$$spoly_2 := \frac{1}{4}\, x^3 - 3\, x^2 + 11\, x - 12$$

$$672$$

$$\frac{(x-6)\,(x-2)\,(x-4)}{4}$$

$$V2 := array(1..4, 1..4, [\ ])$$

$$\left[\begin{array}{cccc} 1 & 3 & 9 & 27 \\ 1 & 7 & 49 & 343 \\ 1 & 11 & 121 & 1331 \\ 1 & 15 & 225 & 3375 \end{array}\right]$$

$$[2, 54, 250, 686]$$

$$coef := \left[\begin{array}{cccc} -\frac{1}{4} & \frac{3}{4} & -\frac{3}{4} & \frac{1}{4} \end{array}\right]$$

$$coef := \left[\begin{array}{cccc} -\frac{21}{2} & \frac{41}{4} & -3 & \frac{1}{4} \end{array}\right]$$

$$wpoly_3 := \frac{1}{4}\, x^3 - \frac{3}{4}\, x^2 + \frac{3}{4}\, x - \frac{1}{4}$$

$$1458$$

$$\frac{(x-1)^3}{4}$$

$$spoly_3 := \frac{1}{4}\, x^3 - 3\, x^2 + \frac{41}{4}\, x - \frac{21}{2}$$

$$816$$

$$\frac{(x-2)\,(x-3)\,(x-7)}{4}$$

$$V2 := array(1..4, 1..4, [\ ])$$

$$\begin{bmatrix} 1 & 4 & 16 & 64 \\ 1 & 8 & 64 & 512 \\ 1 & 12 & 144 & 1728 \\ 1 & 16 & 256 & 4096 \end{bmatrix}$$

$$[9, 91, 341, 855]$$

$$coef := \begin{bmatrix} -1 & \frac{3}{2} & -\frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

$$coef := \begin{bmatrix} -12 & 11 & -3 & \frac{1}{4} \end{bmatrix}$$

$$wpoly_4 := \frac{1}{4}\,x^3 - \frac{3}{4}\,x^2 + \frac{3}{2}\,x - 1$$

$$1729$$

$$\frac{(x-1)\,(x^2-2\,x+4)}{4}$$

$$spoly_4 := \frac{1}{4}\,x^3 - 3\,x^2 + 11\,x - 12$$

$$1008$$

$$\frac{(x-6)\,(x-2)\,(x-4)}{4}$$

```
> for r from 1 to p do: r: spoly[r]: factor(spoly[r]): od;
```

$$1 \tag{6}$$

$$\frac{1}{4}\,x^3 - 3\,x^2 + \frac{41}{4}\,x - \frac{15}{2}$$

$$\frac{(x-1)\,(x-5)\,(x-6)}{4}$$

$$2$$

$$\frac{1}{4}\,x^3 - 3\,x^2 + 11\,x - 12$$

$$\frac{(x-6)\,(x-2)\,(x-4)}{4}$$

$$3$$

$$\frac{1}{4}x^3 - 3x^2 + \frac{41}{4}x - \frac{21}{2}$$

$$\frac{(x-2)(x-3)(x-7)}{4}$$

$$4$$

$$\frac{1}{4}x^3 - 3x^2 + 11x - 12$$

$$\frac{(x-6)(x-2)(x-4)}{4}$$

```
> r:=0;
> for k from 1 to maxk do
>   r:=r+1:
>   if (r>p) then r:=1: fi:
>   eval(spoly[r],x=k)-s[k]:
> od;
```

$$r := 0 \tag{7}$$

$$r := 1$$

$$0$$

$$r := 2$$

$$0$$

$$r := 3$$

$$0$$

$$r := 4$$

$$0$$

$$r := 5$$

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$$r := 2$$

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```
> for r from 1 to p do: r: wpoly[r]: factor(wpoly[r]): od;
```

$$1 \tag{8}$$
$$\frac{1}{4}x^3 - \frac{3}{4}x^2 + \frac{3}{4}x - \frac{1}{4}$$
$$\frac{(x-1)^3}{4}$$
$$2$$
$$\frac{1}{4}x^3 - \frac{3}{4}x^2 + \frac{3}{2}x - 1$$
$$\frac{(x-1)(x^2 - 2x + 4)}{4}$$
$$3$$
$$\frac{1}{4}x^3 - \frac{3}{4}x^2 + \frac{3}{4}x - \frac{1}{4}$$
$$\frac{(x-1)^3}{4}$$
$$4$$
$$\frac{1}{4}x^3 - \frac{3}{4}x^2 + \frac{3}{2}x - 1$$
$$\frac{(x-1)(x^2 - 2x + 4)}{4}$$

```
> r:=1;for k from 1 to maxk do
    eval(wpoly[r],x=k)-w[k]:
>   r:=3-r:
  od;
```

$$r := 1 \tag{9}$$
$$0$$
$$r := 2$$
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$$r := 1$$

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