Semimagic generating functions and sequences

(general form, with cubic data)

Notation:

S: semimagic squares (all positive values).

s: normalized squares (symmetry types).

R: reduced squares (least element is 0).

r: reduced normalized squares (reduced symmetry types).

n: semimagic r.

gf: generating function in some form.

gfsum: generating function as a sum of simple terms.

c: Cubic (fixed strict upper bound; weak upper bound for reduced).

a: Affine (fixed magic sum).

p: Period of the quasipolynomial (known from geometry). (Period of the truncated quasipolynomial, in the affine count.)
d: Dimension of the geometry = degree of the quasipolynomials.
RtoSfactor: the rational function that multiplies Rgf to Sgf and rgf to sgf.

This is for **cubic**.

```
> d:=5; p:=60;
  RtoSfactor:=x^2/(1-x)^2;
```

$$d := 5$$

$$p := 60$$

$$RtoSfactor := \frac{x^2}{(1 - x)^2}$$

The number of terms desired of each sequence is "enddegree". Comment: The slow part of the program is the series expansion (the following four commands).

```
> enddegree:=500;
```

$$enddegree := 500$$

We start by computing r_s=rsgf from the semimagic count. From the Latte results we get the closed Ehrhart g.f. of each flat, which depends on whether we're doing cubic or affine.

Set up the simplex data for the faces and intersection polytopes in the semimagic series. These are **cubic** data.

```
> simplexname[1]:="OABC": ehrgf[1]:= 1/((1-x)^3*(1-x^2)) : dimen[1]:=3:
  simplexname[2]:="OEE2": ehrgf[2]:= 1/((1-x)*(1-x^2)*(1-x^3)) :
  dimen[2]:=2:
  simplexname[3]:="OAE2": ehrgf[3]:= 1/((1-x)*(1-x^2)^2) : dimen[3]:=2:
  simplexname[4]:="ADE2": ehrgf[4]:= 1/((1-x^2)^3) : dimen[4]:=2:
  simplexname[5]:="DE1E2": ehrgf[5]:= 1/((1-x^2)^2*(1-x^3)) : dimen[5]:=2:
  simplexname[6]:="OCE": ehrgf[6]:= 1/((1-x)^2*(1-x^3)) : dimen[6]:=2:
  simplexname[7]:="BDE1": ehrgf[7]:= 1/((1-x)*(1-x^2)*(1-x^3)) :
```

```
dimen[7]:=2:
simplexname[8]:="ABD": ehrgf[8]:= 1/((1-x)*(1-x^2)^2) : dimen[8]:=2:
simplexname[9]:="FG1": ehrgf[9]:= 1/((1-x^3)*(1-x^5)) : dimen[9]:=1:
simplexname[10]:="EF": ehrgf[10]:= 1/((1-x^3)^2) : dimen[10]:=1:
simplexname[11]:="OG": ehrgf[11]:= 1/((1-x)*(1-x^4)) : dimen[11]:=1:
simplexname[12]:="FG": ehrgf[12]:= 1/((1-x^3)*(1-x^4)) : dimen[12]:=1:
simplexname[13]:="AF": ehrgf[13]:= 1/((1-x^2)*(1-x^3)) : dimen[13]:=1:
simplexname[14]:="DG": ehrgf[14]:= 1/((1-x^2)*(1-x^4)) : dimen[14]:=1:
simplexname[15]:="DG2": ehrgf[15]:= 1/((1-x^2)*(1-x^5)) : dimen[15]:=1:
simplexname[16]:="DE": ehrgf[16]:= 1/((1-x^2)*(1-x^3)) : dimen[16]:=1:
simplexname[17]:="H": ehrgf[17] := 1/(1-x^5) : dimen[17]:=0:
```

The closed E.g.f. is converted to the open E.g.f. The first step is to compute the Mobius function of the intersection poset.

```
> for n from 1 to 17 do
    mu[n]:=(-1)^(dimen[1]-dimen[n]):
  od:
  mu[14]:=2*mu[14]:
  for n from 1 to 17 do
    openehrgf[n]:=simplify(-(-1)^dimen[n]*subs(x=1/x,ehrgf[n])):
  od:
```

Get the g.f. of reduced, normalized squares.

```
> for n from 1 to 17 do
    rgfterm[n]:=openehrgf[n]:
  od:
  rgfsum:=sum(mu[nn]*rgfterm[nn],nn=1..17):
  rgf:=simplify(rgfsum):
```

Get the g.f. of reduced squares.

```
> Rgfsum:=72*rgfsum:
  Rgf:=simplify(Rgfsum):
```

Hence S, the g.f. of the number of semimagic squares, equals

```
> Sgf:=simplify(RtoSfactor*Rgf):
```

The g.f. of the total number of symmetry types, l_ml ("lgf"):

```
> sgf:=simplify(RtoSfactor*rgf):
```

Generate the labelled sequence of magilatin square numbers of all four kinds. The first step is to compute the degree of the first non-zero term.

```
> Sgfdegree:=ldegree(numer(Sgf),x);
  Rgfdegree:=ldegree(numer(Rgf),x);
  sgfdegree:=ldegree(numer(sgf),x);
  rgfdegree:=ldegree(numer(rgf),x);
```

$$Sgfdegree := 10$$

$$Rgfdegree := 8$$

$$sgfdegree := 10$$

$$rgfdegree := 8$$

Generate the series expansions of the g.f.'s.

Expressing the rational function with standard denominator gives an orders-of-magnitude speedup in the series expansion.

Standard denominator (1-x^p)^{d+1}.

```
> pdenom:=(1-x^p):
  standenom:=pdenom^(d+1);
```

$$standenom := (1 - x^{60})^6$$

G.f. as rational function with standard denominator.

```
> Sgfstandnum:=simplify(numer(Sgf)*standenom/denom(Sgf)):
  Sgf:=Sgfstandnum/standenom;
```

$$Sgf := \frac{1}{(1 - x^{60})^6} (72 (1 - x + x^3 - x^5 + x^6 + x^{15} + x^{12} - x^{13} - x^{17} + x^{18} + x^{54} - x^{53} + x^{51} - x^{49} + x^{48} + x^{42} - x^{41} + x^{39} - x^{37} + x^{36} + x^{30} - x^{29} + x^{27} - x^{25} + x^{24}) (x^{57} + x^{54} + x^{51} + x^{48} + x^{45} + x^{42} + x^{39} + x^{36} + x^{33} + x^{30} + x^{27} + x^{24} + x^{21} + x^{18} + x^{15} + x^{12} + x^9 + x^6 + x^3 + 1) (1 + x^2 + x^4 + x^6 + x^8 + x^{10} + x^{12} + x^{14} + x^{16} + x^{18} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20}) (1 + x + x^2 + x^3 + x^5 + x^4 + x^6 + x^7 + x^8 + x^{10} + x^{15} + x^{12} + x^9 + x^{13} + x^{11} + x^{19} + x^{14} + x^{16} + x^{17} + x^{18} + x^{59} + x^{57} + x^{58} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20})^2 (1 - x + x^2 - x^3 + x^4 + x^{10} + x^{12} - x^{13} - x^{11} + x^{14} + x^{54} - x^{53} + x^{52} - x^{51} + x^{50} + x^{44} - x^{43} + x^{42} - x^{41} + x^{40} + x^{34} - x^{33} + x^{32} - x^{31} + x^{30} + x^{24} - x^{23} + x^{22} - x^{21} + x^{20}) x^{10} (18 x^9 + 46 x^8 + 69 x^7 + 74 x^6 + 65 x^5 + 46 x^4 + 26 x^3 + 11 x^2 + 4 x + 1))$$

G.f. as rational function with standard denominator.

```
> Rgfstandnum:=simplify(numer(Rgf)*standenom/denom(Rgf)):
  Rgf:=Rgfstandnum/standenom;
```

$$Rgf := \frac{1}{(1 - x^{60})^6} (72 (18 x^9 + 46 x^8 + 69 x^7 + 74 x^6 + 65 x^5 + 46 x^4 + 26 x^3 + 11 x^2 + 4 x + 1) x^8 (x^{57} + x^{54} + x^{51} + x^{48} + x^{45} + x^{42} + x^{39} + x^{36} + x^{33} + x^{30} + x^{27} + x^{24} + x^{21} + x^{18} + x^{15} + x^{12} + x^9 + x^6 + x^3 + 1)^2 (-1 + x - x^2 + x^3 + x^5 - x^4 - x^6 + x^7 - x^8 - x^{10} + x^{15} - x^{12} + x^9 + x^{13} + x^{11} + x^{19} - x^{14} - x^{16} + x^{17} - x^{18} + x^{59} + x^{57} - x^{58} - x^{56} + x^{55} - x^{54} + x^{53} - x^{52} + x^{51} - x^{50} + x^{49} - x^{48} + x^{47} - x^{46} + x^{45} - x^{44} + x^{43} - x^{42} + x^{41} - x^{40} + x^{39} - x^{38} + x^{37} - x^{36} + x^{35} - x^{34} + x^{33} - x^{32} + x^{31} - x^{30} + x^{29} - x^{28} + x^{27} - x^{26} + x^{25} - x^{24} + x^{23} - x^{22} + x^{21} - x^{20}) (-1 + x^{60}) (1 + x^2 + x^4 + x^6 + x^8 + x^{10} + x^{12} + x^{14} + x^{16} + x^{18} + x^{58} + x^{56} + x^{54} + x^{52} + x^{50} + x^{48} + x^{46} + x^{44} + x^{42} + x^{40} + x^{38} + x^{36} + x^{34} + x^{32} + x^{30} + x^{28} + x^{26} + x^{24} + x^{22} + x^{20}) (x^{52} - x^{51} + x^{48} - x^{46} + x^{44} - x^{41} + x^{40} + x^{32} - x^{31} + x^{28} - x^{26} + x^{24} - x^{21} + x^{20} + x^{12} - x^{11} + x^8 - x^6 + x^4 - x + 1))$$

G.f. as rational function with standard denominator.

```
> sgfstandnum:=simplify(numer(sgf)*standenom/denom(sgf)):
  sgf:=sgfstandnum/standenom;
```

$$sgf := \frac{1}{(1 - x^{60})^6} ((1 - x + x^3 - x^5 + x^6 + x^{15} + x^{12} - x^{13} - x^{17} + x^{18} + x^{54} - x^{53} + x^{51} - x^{49} + x^{48} + x^{42} - x^{41} + x^{39} - x^{37} + x^{36} + x^{30} - x^{29} + x^{27} - x^{25} + x^{24}) (x^{57} + x^{54} + x^{51} + x^{48} + x^{45} + x^{42} + x^{39} + x^{36} + x^{33}$$

$$+x^{30}+x^{27}+x^{24}+x^{21}+x^{18}+x^{15}+x^{12}+x^{9}+x^{6}+x^{3}+1)\,(1+x^{2}+x^{4}+x^{6}+x^{8}+x^{10}+x^{12}+x^{14}+x^{16}+x^{18}+x^{58}+x^{56}+x^{54}+x^{52}+x^{50}+x^{48}+x^{46}+x^{44}+x^{42}+x^{40}+x^{38}+x^{36}+x^{34}+x^{32}+x^{30}+x^{28}+x^{26}+x^{24}+x^{22}+x^{20})\,(1+x+x^{2}+x^{3}+x^{5}+x^{4}+x^{6}+x^{7}+x^{8}+x^{10}+x^{15}+x^{12}+x^{9}+x^{13}+x^{11}+x^{19}+x^{14}+x^{16}+x^{17}+x^{18}+x^{59}+x^{57}+x^{58}+x^{56}+x^{55}+x^{54}+x^{53}+x^{52}+x^{51}+x^{50}+x^{49}+x^{48}+x^{47}+x^{46}+x^{45}+x^{44}+x^{43}+x^{42}+x^{41}+x^{40}+x^{39}+x^{38}+x^{37}+x^{36}+x^{35}+x^{34}+x^{33}+x^{32}+x^{31}+x^{30}+x^{29}+x^{28}+x^{27}+x^{26}+x^{25}+x^{24}+x^{23}+x^{22}+x^{21}+x^{20})^{2}\,(1-x+x^{2}-x^{3}+x^{4}+x^{10}+x^{12}-x^{13}-x^{11}+x^{14}+x^{54}-x^{53}+x^{52}-x^{51}+x^{50}+x^{44}-x^{43}+x^{42}-x^{41}+x^{40}+x^{34}-x^{33}+x^{32}-x^{31}+x^{30}+x^{24}-x^{23}+x^{22}-x^{21}+x^{20})\,x^{10}\,(18x^{9}+46x^{8}+69x^{7}+74x^{6}+65x^{5}+46x^{4}+26x^{3}+11x^{2}+4x+1))$$

G.f. as rational function with standard denominator.

```
> rgfstandnum:=simplify(numer(rgf)*standenom/denom(rgf)):
  rgf:=rgfstandnum/standenom;
```

$$rgf := \frac{1}{(1-x^{60})^{6}}((18x^{9}+46x^{8}+69x^{7}+74x^{6}+65x^{5}+46x^{4}+26x^{3}+11x^{2}+4x+1)\,x^{8}\,(x^{57}+x^{54}+x^{51}+x^{48}+x^{45}+x^{42}+x^{39}+x^{36}+x^{33}+x^{30}+x^{27}+x^{24}+x^{21}+x^{18}+x^{15}+x^{12}+x^{9}+x^{6}+x^{3}+1)^{2}\,(-1+x-x^{2}+x^{3}+x^{5}-x^{4}-x^{6}+x^{7}-x^{8}-x^{10}+x^{15}-x^{12}+x^{9}+x^{13}+x^{11}+x^{19}-x^{14}-x^{16}+x^{17}-x^{18}+x^{59}+x^{57}-x^{58}-x^{56}+x^{55}-x^{54}+x^{53}-x^{52}+x^{51}-x^{50}+x^{49}-x^{48}+x^{47}-x^{46}+x^{45}-x^{44}+x^{43}-x^{42}+x^{41}-x^{40}+x^{39}-x^{38}+x^{37}-x^{36}+x^{35}-x^{34}+x^{33}-x^{32}+x^{31}-x^{30}+x^{29}-x^{28}+x^{27}-x^{26}+x^{25}-x^{24}+x^{23}-x^{22}+x^{21}-x^{20})\,(-1+x^{60})\,(1+x^{2}+x^{4}+x^{6}+x^{8}+x^{10}+x^{12}+x^{14}+x^{16}+x^{18}+x^{58}+x^{56}+x^{54}+x^{52}+x^{50}+x^{48}+x^{46}+x^{44}+x^{42}+x^{40}+x^{38}+x^{36}+x^{34}+x^{32}+x^{30}+x^{28}+x^{26}+x^{24}+x^{22}+x^{20})\,(x^{52}-x^{51}+x^{48}-x^{46}+x^{44}-x^{41}+x^{40}+x^{32}-x^{31}+x^{28}-x^{26}+x^{24}-x^{21}+x^{20}+x^{12}-x^{11}+x^{8}-x^{6}+x^{4}-x+1))$$

Expand as a series to find the first few values of the number of squares and symmetry types.

```
> Sseries:=series(Sgf,x=0,enddegree+1):
  print("Series computed.");
```

"Series computed."

```
> sseries:=series(sgf,x=0,enddegree+1):
  print("Series computed.");
```

"Series computed."

Expand as a series to find the first few values of the number of reduced squares and reduced symmetry types.

```
> Rseries:=series(Rgf,x=0,enddegree+1):
  print("Series computed.");
```

"Series computed."

```
> rseries:=series(rgf,x=0,enddegree+1):
  print("Series computed.");
```

"Series computed."

Find the counting sequences

List the coefficients of each series, i.e., the terms of the counting sequences.
The comment symbol # is for controlling the output. With large "enddegree" the output is huge so it's

more convenient to run each sequence's output separately and copy it from the worksheet.

```
> for n from Sgfdegree to enddegree do
   co:=coeff(Sseries,x,n):
   printf("%d  %d \n",n,co);
  od:
  print("Coefficients complete.",n,co);
```

```
10  72
11  288
12  936
13  2592
14  5760
15  11520
16  20952
17  35712
18  57168
19  88272
20  131112
21  189504
22  265752
23  365760
24  492480
25  653040

26  851472
27  1096416
28  1392768
29  1751904
30  2178864
31  2687184
32  3283632
33  3983760
34  4794984
35  5736528
36  6816456
37  8056224
38  9466128
39  11070432
40  12880152
41  14924016
42  17213328
43  19780128
44  22638744
45  25823952
46  29350728
47  33259392
48  37565856
49  42313968
50  47522592
51  53239248
52  59483664
53  66309840
54  73738224
55  81826848
```

56 90600048
57 100119600
58 110410776
59 121543056
60 133542504
61 146482992
62 160394544
63 175355856
64 191398248
65 208608480
66 227018592
67 246720384
68 267750936
69 290207088
70 314126712
71 339616080
72 366714000
73 395532288
74 426115008
75 458579520
76 492971184
77 529417296
78 567964224
79 608745600

80 651813552
81 697307616
82 745280856
83 795884256
84 849172248
85 905302080
86 964334304
87 1026432720
88 1091659464
89 1160190432
90 1232088624
91 1307537136
92 1386605736
93 1469484576
94 1556244792
95 1647089568
96 1742091408
97 1841460768
98 1945277424
99 2053759680
100 2166988752
101 2285196768
102 2408465952
103 2537036784
104 2670999552
105 2810602512
106 2955937320
107 3107267136
108 3264685272

109	3428463456
110	3598702992
111	3775684320
112	3959510328
113	4150477296
114	4348689552
115	4554452448
116	4767879240
117	4989284064
118	5218781832
119	5456703744
120	5703166224
121	5958509904
122	6222860208
123	6496567632
124	6779759616
125	7072804080
126	7375829904
127	7689215088
128	8013098592
129	8347868496
130	8693665272
131	9050895792
132	9419702184
133	9800501904
134	10193447376
135	10598966640
136	11017214136
137	11448637200
138	11893392000
139	12351937248
140	12824439912
141	13311369648
142	13812895080
143	14329506672
144	14861375136
145	15409002240
146	15972569856
147	16552591344
148	17149250880
149	17763083280
150	18394274304
151	19043371008
152	19710570960
153	20396433312
154	21101157720
155	21825325728
156	22569139080
157	23333191632
158	24117697440
159	24923263248
160	25750105272
161	26598853440
162	27469725696

163 28363365360
164 29280003480
165 30220296192

166 31184476632
167 32173225200
168 33186777408
169 34225827264
170 35290623312
171 36381873312
172 37499828112
173 38645220672
174 39818304000
175 41019825168
176 42250051152
177 43509742848
178 44799169608
179 46119118752
180 47469861864
181 48852200736
182 50266420992
183 51713339328
184 53193244104
185 54706978800
186 56254833936
187 57837668112
188 59455786968
189 61110064224
190 62800807752
191 64528919424
192 66294709488
193 68099095440
194 69942402864
195 71825564880
196 73748909808
197 75713399424
198 77719364496
199 79767783216
200 81859002192
201 83994015600
202 86173172424
203 88397497008
204 90667341144
205 92983745520
206 95347078128
207 97758396288
208 100218071016
209 102727190448
210 105286127904
211 107895988800
212 110557163304
213 113270773968
214 116037213768
215 118857637008

216	121732439472
217	124662792816
218	127649110176
219	130692581136
220	133793621712
221	136953454032
222	140172496560
223	143451989856
224	146792370528
225	150194896992
226	153660008664
227	157188997584
228	160782306264
229	164441245392
230	168166275552
231	171958726224
232	175819061016
233	179748643968
234	183747941568
235	187818337008
236	191960315784
237	196175279952
238	200463718104
239	204827068080
240	209265821424
241	213781435488
242	218374420896
243	223046254944
244	227797451712
245	232629524640
246	237542990688
247	242539383456
248	247619240064
249	252784114272
250	258034546152
251	263372126976
252	268797399912
253	274311976896
254	279916421472
255	285612366240
256	291400378200
257	297282127968
258	303258185712
259	309330243504
260	315498892392
261	321765845472
262	328131696888
263	334598199264
264	341165950272
265	347836723920
266	354611139120
267	361490991552
268	368476903872
269	375570711936

270	382773041424
271	390085750512
272	397509486768
273	405046130544
274	412696332936
275	420462015408
276	428343832584
277	436343728320
278	444462379632
279	452701753344
280	461062530072
281	469546718544
282	478155002544
283	486889414272
284	495750660696
285	504740796912
286	513860533416
287	523111968288
288	532495815840
289	542014197840
290	551667851712
291	561458923056
292	571388153040
293	581457731184
294	591668402256
295	602022379968
296	612520433136
297	623164799376
298	633956251320
299	644897071728
300	655988036904
301	667231454160
302	678628123920
303	690180378480
304	701889022440
305	713756433600
306	725783420160
307	737972385120
308	750324161880
309	762841178640
310	775524272472
311	788375918448
312	801396957456
313	814589890272
314	827955583200
315	841496562720
316	855213699312
317	869109566832
318	883185039648
319	897442718112
320	911883502512
321	926510019264
322	941323172472
323	956325637440

324	971518322520
325	986903929440
326	1002483392832
327	1018259441136
328	1034233013448
329	1050406887744
330	1066782006864
331	1083361176144
332	1100145365352
333	1117137407040
334	1134338275224
335	1151750852928
336	1169376118416
337	1187216982144
338	1205274449808
339	1223551459872
340	1242049022352
341	1260770126976
342	1279715787648
343	1298889022608
344	1318290873984
345	1337924387952
346	1357790610888
347	1377892641312
348	1398231530136
349	1418810404608
350	1439630343792
351	1460694503808
352	1482003968184
353	1503561946320
354	1525369526064
355	1547429946048
356	1569744323208
357	1592315925120
358	1615145873256
359	1638237489696
360	1661591900304
361	1685212456752
362	1709100314064
363	1733258853936
364	1757689236288
365	1782394897680
366	1807377002352
367	1832639017104
368	1858182136416
369	1884009857328
370	1910123378712
371	1936526253840
372	1963219686120
373	1990207259568
374	2017490208048
375	2045072146320
376	2072954313144

377	2101140380016
378	2129631590304
379	2158431647040
380	2187541824552
381	2216965856976
382	2246705023176
383	2276763115536
384	2307141417888
385	2337843754080
386	2368871439264
387	2400228329040
388	2431915743744
389	2463937597872
390	2496295216224
391	2528992545696
392	2562030943056
393	2595414387456
394	2629144240632
395	2663224541568
396	2697656656968
397	2732444658288
398	2767589944704
399	2803096620720
400	2838966090552
401	2875202519328
402	2911807315872
403	2948784678864
404	2986136050392
405	3023865662112
406	3061974961080
407	3100468240656
408	3139346953152
409	3178615425696
410	3218275143792
411	3258330468480
412	3298782890448
413	3339636833376
414	3380893792992
415	3422558227248
416	3464631666000
417	3507118601184
418	3550020567912
419	3593342121984
420	3637084803624
421	3681253203264
422	3725848895328
423	3770876505312
424	3816337613256
425	3862236908880
426	3908575977264
427	3955359543408
428	4002589227672
429	4050269790336
430	4098402856872

431	4146993253152
432	4196042609904
433	4245555788784
434	4295534456016
435	4345983509040
436	4396904619696
437	4448302751520
438	4500179581680
439	4552540110288
440	4605386050512
441	4658722438608
442	4712550993000
443	4766876817552
444	4821701636376
445	4877030589840
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447	4989211359264
448	5046070118760
449	5103446962320
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451	5219765499168
452	5278714283880
453	5338195335792
454	5398210469160
455	5458765073328
456	5519860968240
457	5581503580752
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460	5769741126672
461	5833603695600
462	5898029571216
463	5963024327040
464	6028589869920
465	6094731811392
466	6161452063992
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469	6365128261104
470	6434203643712
471	6503878377072
472	6574154423832
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475	6788637215568
476	6861361501512
477	6934708691568
478	7008680799288
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480	7158519738864
481	7234394613696
482	7310910491712
483	7388073436608
484	7465885513344

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486  7623477544896
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488  7783719579648
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490  7946644898952
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492  8112287072808
493  8196138061920
494  8280679927104
495  8365919042880
496  8451857578968
497  8538501985344
498  8625854437776
499  8713921427856
500  8802705172392
```

"Coefficients complete.", 501, 8802705172392

```
> for n from Rgfdegree to enddegree do
   co:=coeff(Rseries,x,n):
   printf("%d  %d \n",n,co);
  od:
  print("Coefficients complete.",n,co);
```

```
8  72
9  144
10  432
11  1008
12  1512
13  2592
14  3672
15  5328
16  6696
17  9648
18  11736
19  15552
20  17856
21  23760
22  26712
23  33840
24  37872
25  46512
26  51408
27  62784
28  67824
29  81360
30  88128
31  103680
32  111096
33  130320
34  138384
35  159840
36  170136
37  194400
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47 441648
48 460512
49 508032
50 527760
51 581760
52 602208
53 660240
54 684576
55 746352
56 771624
57 841104
58 867168
59 941040
60 971064
61 1049760
62 1081080
63 1167840
64 1199880
65 1291680
66 1328760
67 1425600
68 1463472
69 1569744
70 1608552
71 1720368
72 1764432
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74 1927152
75 2054448
76 2100816
77 2234448
78 2286576
79 2426112
80 2479176
81 2630160
82 2684592
83 2841840
84 2902392
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86 3128328
87 3304224
88 3367224

89 3550320
90 3620088

91 3810240
92 3881376
93 4084560
94 4157064
95 4367520
96 4447296
97 4665600
98 4746816
99 4978944
100 5061168
101 5301648
102 5391936
103 5640192
104 5731848
105 5995008
106 6088320
107 6360048
108 6461352
109 6741792
110 6844680
111 7140960
112 7245288
113 7550640
114 7663896
115 7978032
116 8092944
117 8424144
118 8540568
119 8881200
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124 9981360
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133 12573792
134 12728232
135 13175568
136 13331736
137 13790448
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143 15758640
144 15940512

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153 19443600
154 19645344
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156 20453256
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158 21276216
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160 22124088
161 22767408
162 22998456
163 23654592
164 23887728
165 24568128
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171 27437760
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173 28437840
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185 34979040
186 35284680
187 36158400
188 36466272
189 37368144
190 37678392
191 38595888
192 38921472
193 39854592
194 40182912
195 41144688
196 41475456
197 42453648
198 42800256

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205	47985552
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210	51313608
211	52436160
212	52829136
213	53983440
214	54379224
215	55550880
216	55964016
217	57153600
218	57569616
219	58791744
220	59210208
221	60450768
222	60887376
223	62145792
224	62585208
225	63877248
226	64319760
227	65630448
228	66091032
229	67420512
230	67884120
231	69248160
232	69714648
233	71097840
234	71583336
235	72985392
236	73473984
237	74911824
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239	76860720
240	77371344
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244	81393120
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248	85557672
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250	87692112
251	89304048

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254	92067192
255	93737808
256	94307976
257	96000048
258	96591096
259	98304192
260	98898336
261	100650960
262	101248632
263	103022640
264	103641552
265	105437232
266	106059888
267	107895744
268	108521424
269	110379600
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271	112907520
272	113558616
273	115480080
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275	118078560
276	118755576
277	120722400
278	121403016
279	123411744
280	124095528
281	126127728
282	126834696
283	128889792
284	129600288
285	131698368
286	132412680
287	134534448
288	135271872
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291	140348160
292	141092928
293	143306640
294	144075456
295	146313072
296	147085704
297	149368464
298	150144768
299	152452080
300	153252504
301	155584800
302	156389400
303	158767200
304	159575400
305	161978400

306 162811800
307 165240000
308 166077072
309 168552144
310 169393032
311 171893808
312 172760112
313 175286592
314 176157072
315 178730928
316 179605296
317 182205648
318 183105936
319 185732352
320 186636456
321 189311760
322 190220112
323 192921840
324 193856472
325 196584912
326 197524008
327 200301984
328 201244824
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330 205019928
331 207852480
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333 211709520
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337 219542400
338 220552416
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367	284601600
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389	339711120
390	341067888
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413	407474640

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417 419587344
418 421127568
419 425718000
420 427292424
421 431917920
422 433497960
423 438187680
424 439772760
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426 446118120
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430 458960472
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457 554040000
458 555903216
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464 578311128
465 583993728
466 585912240
467 591633648

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470  601313400
471  607145760
472  609114168
473  614985840
474  616993416
475  622904112
476  624917664
477  630901584
478  632920968
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483  655269120
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488  673963272
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492  690876072
493  697250592
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496  708046056
497  714537648
498  716754456
499  723287232
500  725510016
```

"Coefficients complete.", 501, 725510016

```
> for n from sgfdegree to enddegree do
   co:=coeff(sseries,x,n):
   printf("%d  %d \n",n,co);
  od:
  print("Coefficients complete.",n,co);
```

```
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14  80
15  160
16  291
17  496
18  794
19  1226
20  1821
21  2632
22  3691
```

23 5080
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27 15228
28 19344
29 24332
30 30262
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32 45606
33 55330
34 66597
35 79674
36 94673
37 111892
38 131474
39 153756
40 178891
41 207278
42 239074
43 274724
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46 407649
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70 4362871
71 4716890
72 5093250
73 5493504
74 5918264
75 6369160
76 6846822

77 7353018
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79 8454800
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83 11053948
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85 12573640
86 13393532
87 14256010
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89 16113756
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91 18160238
92 19258413
93 20409508
94 21614511
95 22876244
96 24195714
97 25575844
98 27017742
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102 33450916
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106 41054685
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108 45342851
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111 52440060
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113 57645518
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150 255476032
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158 334968020
159 346156434
160 357640351
161 369428520
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206 1324264974
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213 1573205194
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221 1902131306
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266 4925154710
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268 5117734776
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454  74975145405
455  75816181574
456  76664735670
457  77520883066
458  78384649560
459  79256111054
460  80135293426
461  81022273550
462  81917077378
463  82819782320
464  83730414860
465  84649052936
466  85575723111
467  86510504310
468  87453423179
469  88404559182
470  89363939496
471  90331644126
472  91307700331
473  92292189116
474  93285137820
475  94286627994
476  95296687521
477  96315398494
478  97342788879
479  98378941786
480  99423885262
481  100477702968
482  101540423496
483  102612131064
484  103692854352
485  104782678600
486  105881632568
487  106989802056
488  108107216384
489  109233961912
490  110370068041
491  111515622172
492  112670653789
493  113835250860
494  115009443432
495  116193320040
496  117386910819
497  118590305352
498  119803533858
499  121026686498
500  122259794061
```

"Coefficients complete.", 501, 122259794061

```
> for n from rgfdegree to enddegree do
   co:=coeff(rseries,x,n):
   printf("%d  %d \n",n,co);
```

```
  od:
  print("Coefficients complete.",n,co);
```

```
8   1
9   2
10   6
11   14
12   21
13   36
14   51
15   74
16   93
17   134
18   163
19   216
20   248
21   330
22   371
23   470
24   526
25   646
26   714
27   872
28   942
29   1130
30   1224
31   1440
32   1543
33   1810
34   1922
35   2220
36   2363
37   2700
38   2853
39   3252
40   3409
41   3854
42   4053
43   4536
44   4744
45   5304
46   5525
47   6134
48   6396
49   7056
50   7330
51   8080
52   8364
53   9170
54   9508
55   10366
56   10717
57   11682
58   12044
59   13070
```

60 13487
61 14580
62 15015
63 16220
64 16665
65 17940
66 18455
67 19800
68 20326
69 21802
70 22341
71 23894
72 24506
73 26136
74 26766
75 28534
76 29178
77 31034
78 31758
79 33696
80 34433
81 36530
82 37286
83 39470
84 40311
85 42586
86 43449
87 45892
88 46767
89 49310
90 50279
91 52920
92 53908
93 56730
94 57737
95 60660
96 61768
97 64800
98 65928
99 69152
100 70294
101 73634
102 74888
103 78336
104 79609
105 83264
106 84560

107 88334
108 89741
109 93636
110 95065
111 99180
112 100629

113	104870
114	106443
115	110806
116	112402
117	117002
118	118619
119	123350
120	125092
121	129960
122	131730
123	136840
124	138630
125	143880
126	145810
127	151200
128	153151
129	158802
130	160776
131	166574
132	168691
133	174636
134	176781
135	182994
136	185163
137	191534
138	193853
139	200376
140	202718
141	209530
142	211901
143	218870
144	221396
145	228526
146	231084
147	238512
148	241092
149	248690
150	251434
151	259200
152	261973
153	270050
154	272852
155	281100
156	284073
157	292500
158	295503
159	304252
160	307279
161	316214
162	319423
163	328536
164	331774
165	341224
166	344495

167	354134
168	357586
169	367416
170	370900
171	381080
172	384594
173	394970
174	398678
175	409246
176	412987
177	423922
178	427694
179	438830
180	442797
181	454140
182	458145
183	469860
184	473895
185	485820
186	490065
187	502200
188	506476
189	519002
190	523311
191	536054
192	540576
193	553536
194	558096
195	571454
196	576048
197	589634
198	594448
199	608256
200	613103
201	627330
202	632216
203	646670
204	651781
205	666466
206	671619
207	686732
208	691917
209	707270
210	712689
211	728280
212	733738
213	749770
214	755267
215	771540
216	777278
217	793800
218	799578
219	816552
220	822364

221	839594
222	845658
223	863136
224	869239
225	887184
226	893330
227	911534
228	917931
229	936396
230	942835
231	961780
232	968259
233	987470
234	994213
235	1013686
236	1020472
237	1040442
238	1047269
239	1067510
240	1074602
241	1095120
242	1102260
243	1123280
244	1130460
245	1151760
246	1159220
247	1180800
248	1188301
249	1210402
250	1217946
251	1240334
252	1248161
253	1270836
254	1278711
255	1301914
256	1309833
257	1333334
258	1341543
259	1365336
260	1373588
261	1397930
262	1406231
263	1430870
264	1439466
265	1464406
266	1473054
267	1498552
268	1507242
269	1533050
270	1542044
271	1568160
272	1577203
273	1603890
274	1612982

275 1639980
276 1649383
277 1676700
278 1686153
279 1714052
280 1723549
281 1751774
282 1761593
283 1790136
284 1800004
285 1829144
286 1839065
287 1868534
288 1878776
289 1908576
290 1918870
291 1949280
292 1959624
293 1990370
294 2001048
295 2032126
296 2042857
297 2074562
298 2085344
299 2117390
300 2128507
301 2160900
302 2172075
303 2205100
304 2216325
305 2249700
306 2261275
307 2295000
308 2306626
309 2341002
310 2352681
311 2387414
312 2399446
313 2434536
314 2446626
315 2482374
316 2494518
317 2530634
318 2543138
319 2579616
320 2592173
321 2629330
322 2641946
323 2679470
324 2692451
325 2730346
326 2743389
327 2781972
328 2795067

329	2834030
330	2847499
331	2886840
332	2900368
333	2940410
334	2953997
335	2994420
336	3008388
337	3049200
338	3063228
339	3104752
340	3118834
341	3160754
342	3175228
343	3217536
344	3232069
345	3275104
346	3289700
347	3333134
348	3348121
349	3391956
350	3407005
351	3451580
352	3466689
353	3511670
354	3527183
355	3572566
356	3588142
357	3634282
358	3649919
359	3696470
360	3712512
361	3759480
362	3775590
363	3823320
364	3839490
365	3887640
366	3904230
367	3952800
368	3969451
369	4018802
370	4035516
371	4085294
372	4102431
373	4152636
374	4169841
375	4220834
376	4238103
377	4289534
378	4307233
379	4359096
380	4376858
381	4429530
382	4447361

383 4500470
384 4518736
385 4572286
386 4590624
387 4644992
388 4663392
389 4718210
390 4737054
391 4792320
392 4811233
393 4867330
394 4886312
395 4942860
396 4962293
397 5019300
398 5038803
399 5096652
400 5116219
401 5174534
402 5194563
403 5253336
404 5273434
405 5333064
406 5353235
407 5413334
408 5433966
409 5494536
410 5515240
411 5576680
412 5597454
413 5659370
414 5680618
415 5743006
416 5764327
417 5827602
418 5848994
419 5912750
420 5934617
421 5998860
422 6020805
423 6085940
424 6107955
425 6173580
426 6196085
427 6262200
428 6284776
429 6351802
430 6374451
431 6441974

432 6465116
433 6533136
434 6556356
435 6625294

436	6648588
437	6718034
438	6741828
439	6811776
440	6835643
441	6906530
442	6930476
443	7001870
444	7026321
445	7098226
446	7122759
447	7195612
448	7220217
449	7293590
450	7318709
451	7392600
452	7417798
453	7492650
454	7517927
455	7593300
456	7619098
457	7695000
458	7720878
459	7797752
460	7823704
461	7901114
462	7927598
463	8005536
464	8032099
465	8111024
466	8137670
467	8217134
468	8244311
469	8324316
470	8351575
471	8432580
472	8459919
473	8541470
474	8569353
475	8651446
476	8679412
477	8762522
478	8790569
479	8874230
480	8902822
481	8987040
482	9015720
483	9100960
484	9129720
485	9215520
486	9244840
487	9331200
488	9360601
489	9448002

```
490   9477486
491   9565454
492   9595501
493   9684036
494   9714171
495   9803754
496   9833973
497   9924134
498   9954923
499   10045656
500   10076528

                                    "Coefficients complete.", 501, 10076528
>
```