Maximal and minimal polyhexes

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Abstract

The minimum perimeter of a polyhex with n hexagons is $2\lceil\sqrt{12n-3}\rceil$. To prove this result, we first obtain a lower bound on the perimeter by considering maximal polyhexes (i.e., polyhexes with a given perimeter and a maximum number of hexagons). We then show how to construct minimal polyhexes that attain the perimeter lower bounds.

A polyhex has even perimeter. If p is even, then the maximum number of hexagons in a polyhex with perimeter p is round $\left(p^2/48\right)$.

AMS Subject Classifications: 05A99, 68R05

1 Introduction

A *polyhex* is a connected planar set of congruent regular hexagons in which the edges of adjacent hexagons line up exactly (are not staggered) [2, 3, 6]. An n-*polyhex* is a polyhex with n hexagons. See Figure 1. We ignore rotations and reflections in considering polyhexes. We assume that each edge of each hexagon has length 1 and define the *perimeter* of a polyhex to be the total length of its exposed edges.

A polyhex is *minimal* (or *optimal*) iff it has min perimeter with respect to all polyhexes with the same number of hexagons. (The use of the term "optimal" comes from the domain decomposition problem with polyominos, discussed below.) A polyhex is *maximal* iff it has the maximum number of hexagons with respect to all polyhexes with the same perimeter. Minimal polyhexes are useful in the solution to domain decomposition problems in scientific computation (see [9] for an illustration of the use of domain decomposition in conjunction with triangulations).

Note that there are 2 other obvious optimization possibilities: we could maximize the perimeter subject to a fixed number of hexagons, or minimize the

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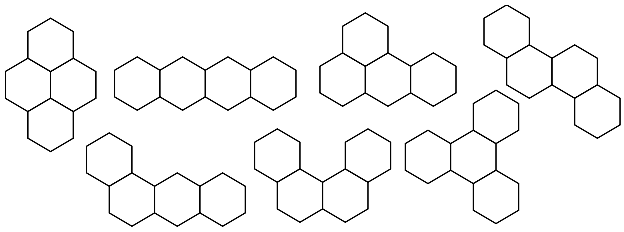

Figure 1: The 7 polyhexes with 4 hexagons.

number of hexagons subject to a fixed perimeter. But these problems are trivial and have the same solution: the polyhex shaped like a stick.

- **Maximal polyhex formula**: Let p be even. The max number of hexagons in a polyhex with perimeter p is round $\left(p^2/48\right)$.
- **Maximal polyhex algorithm**: Let p be even. This algorithm constructs a polyhex with perimeter p and the most hexagons.
- **Minimal polyhex formula**: The min perimeter of an n-polyhex is $2\left\lceil\sqrt{12n-3}\right\rceil$.
- **Minimal polyhex algorithm**: This algorithm constructs an n-polyhex with min perimeter.

These results are related as follows: a lower bound on perimeter is obtained using maximal polyhexes, and then attainment of the lower bound is demonstrated by using the minimal polyhex algorithm.

A recent result somewhat related to polyhexes is the honeycomb theorem [4, 7], which states that a hexagonal grid partitions a plane into regions of equal area with min total perimeter. This result is concerned mainly with infinite regions (such as a plane) instead of finite regions (such as polyhexes). It had been a conjecture for over 2000 years.

Motivation for this paper came from the domain decomposition problem with polyominos, which is a whole topic by itself [1, 5, 9, 10]. The many approaches to this problem include branch-and-bound, genetic algorithms, knapsack algorithms, and stripe algorithms.

This paper considers regular hexagons and polyhexes; its structure and results are similar to those of [11], which considers equilateral triangles and polyiamonds and has the following main results:

- **Maximal polyiamond formula**: Let $p \geq 3$. The max number of triangles in a polyiamond with perimeter p is

$$\text{round}\left(\frac{p^2}{6}\right) - \begin{cases} 0 & p \equiv 0 \pmod 6 \\ 1 & p \not\equiv 0 \pmod 6 \end{cases}$$

- **Maximal polyiamond algorithm**: Let $p \geq 3$. This algorithm constructs a polyiamond with perimeter p and the most triangles.
- **Minimal polyiamond formula**: The min perimeter of an n-polyiamond is whichever of $\lceil\sqrt{6n}\rceil$ or $\lceil\sqrt{6n}\rceil + 1$ has the same parity as n.
- **Minimal polyiamond algorithm**: This algorithm constructs an n-polyiamond with min perimeter.

Below, we briefly discuss well-known results in the domain decomposition problem with polyominos. These results for polyominos, along with the polyiamond results above from [11], will motivate the polyhex results in this paper.

2 Domain decomposition problem with polyominos

A *polyomino* is a connected planar set of congruent squares in which the edges of adjacent squares line up exactly (are not staggered) [2, 3, 6]. Equivalently, if we move a rook on a chessboard of any finite size, then the set of squares touched by the rook is a polyomino. "Polyomino" is a generalization of "domino". An *n-polyomino* is a polyomino with n squares. See Figure 2.

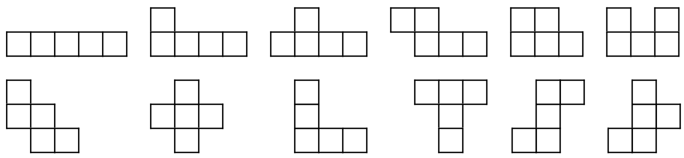

Figure 2: The 12 polyominos with 5 squares. All have perimeter 12 except for one shaped like a 1×1 square joined to a 2×2 square, which has perimeter 10.

The *domain decomposition problem* is a special case of graph partitioning problems which involve partitioning the vertices of a graph into equal-size sets as to minimize the number of edges connecting vertices in different sets. One version of the domain decomposition problem is as follows.

Problem 1. **(Domain decomposition problem with polyominos)** *Let n divide A. Tile a given set of A squares with n-polyominos. What is the min total perimeter of the polyominos in such a tiling?*

This paper arose when we asked what would happen if we worked with regular hexagons instead of squares. The domain decomposition problem with polyominos has motivation from parallel computation; think of the following analogy:

square	job that needs to communicate with adjacent jobs
n-polyomino	n jobs assigned to a processor
polyomino edge	expensive communication between jobs in different processors

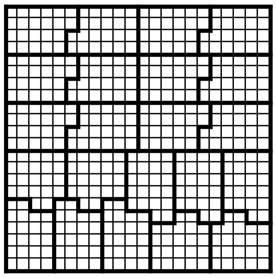

Figure 3: A solution of the domain decomposition problem with polyominos for a 22×22 board tiled by 22 polyominos, each of which has perimeter 20. By Yackel-Meyer-Christou's theorem, the min perimeter of a 22-polyomino is 20. So the total perimeter lower bound is $22 \times 20 = 440$ and is attained by this tiling.

In the domain decomposition problem, note that if each polyomino in the tiling has min perimeter, then the problem is solved. In a tiling of an arbitrary domain, in general, not all polyominos can have min perimeter, and solutions involve approximating the "all-min-perimeter" situation. Yackel-Meyer-Christou [10] found a simple formula for the min perimeter of a polyomino.

Theorem 1. **(Yackel-Meyer-Christou)** *The min perimeter of an n-polyomino is* $2\lceil 2\sqrt{n}\rceil$.

Idea. A polyomino has min perimeter if it is a square, or closely resembles a square. An n-polyomino has area n. If we shape this n-polyomino into a square having the same area, then the square has side $\sqrt{n}$ and perimeter $4\sqrt{n}$. So a lower bound for the min perimeter of an n-polyomino can be shown to be $4\sqrt{n}$. But this lower bound is not always integer. It turns out that $2\lceil 2\sqrt{n}\rceil$ is a lower bound and is always attainable. □

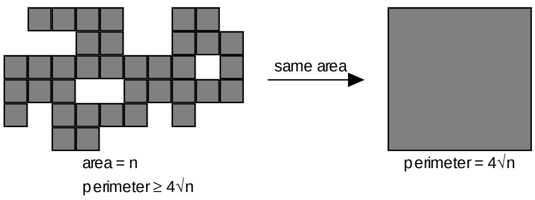

Figure 4: Relation between polyomino and square of same area.

3 Polyomino slices

The perimeter of a polyomino is related to its numbers of "subslices". A *slice* is a row or column containing squares (the squares need not be connected).

A *subslice* is a maximal connected set of squares in a slice. A *slice-gap* is an absence of squares between subslices in a row or column. A slice is *convex* iff the set of squares in the slice is convex; the slice has no gaps. A polyomino is *slice-convex* iff every slice is convex.

Theorem 2. **(Polyomino subslices theorem)** *The perimeter of a polyomino is 2 times the number of subslices.*

Proof. Every subslice contributes 2 boundary edges. See Figure 5. □

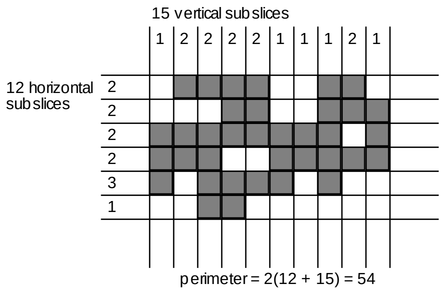

Figure 5: The perimeter of a polyomino is 2 times the number of subslices.

The following theorem follows immediately.

Theorem 3. **(Polyomino slices theorem)** *The perimeter of a slice-convex polyomino is 2 times the number of slices.*

4 Domain decomposition problem with polyhexes

We now begin consideration of the minimal polyhexes that provide a lower bound for the optimal value of the following domain decomposition problem:

Problem 2. **(Domain decomposition problem with polyhexes)** *Let n divide A. Tile a given set of A regular hexagons with n-polyhexes. What is the min total perimeter of the polyhexes in such a tiling?*

5 Polyhex slices

We generalize the slice approach used with polyominos. With polyominos, we have 2 kinds of slices: horizontal and vertical. But with polyhexes, we have 3 kinds of slices: horizontal, antidiagonal, and diagonal ("HAD"). For brevity, we say that a polyhex has *HAD slices (or dimensions)* (h, a, d) iff it has h horizontal slices, a antidiagonal slices, and d diagonal slices.

Theorem 4. **(Polyhex subslices theorem)** *The perimeter of a polyhex is 2 times the number of subslices.*

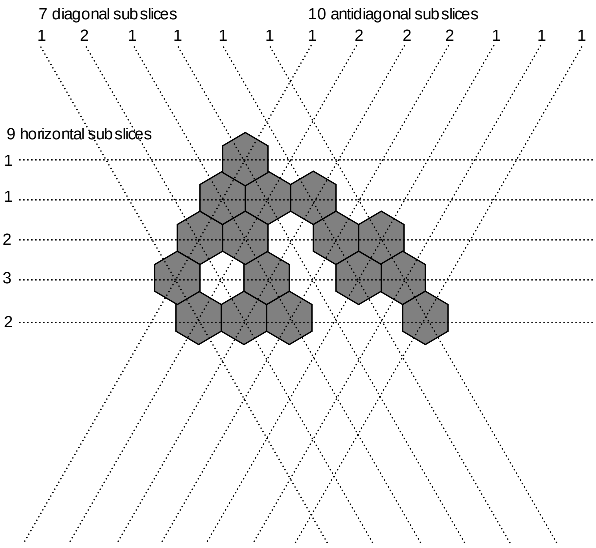

Figure 6: The perimeter of a polyhex is 2 times the number of subslices.

Proof. See Figure 6. Every subslice contributes 2 boundary edges. □

The following theorem follows immediately.

Theorem 5. **(Polyhex slices theorem)** *The perimeter of a slice-convex polyhex is* 2 *times the number of slices.*

In order to construct maximal polyhexes (polyhexes with given perimeter and the most hexagons), we start with polyhexes with given HAD dimensions and the most hexagons. Using the HAD capacity algorithm (Theorem 6) and the HAD capacity formula (Theorem 7), we show that size (number of hexagons) is maximized for a given perimeter $p = 2(h+a+d)$ by "balancing" the dimensions (choosing them as close together as possible).

Theorem 6. **(HAD capacity algorithm)** *Let $a \leq d$. To construct a polyhex with given HAD dimensions (h, a, d) and the most hexagons, do the following:*

- *Draw a parallelogram with HAD dimensions $(a + d - 1, a, d)$.*
- *Pick the h horizontal slices with the most hexagons.*

Such a polyhex is unique, ignoring rotation and reflection.

Proof. See Figure 7. Note that a polyhex with a antidiagonal slices and d diagonal slices fits inside a unique parallelogram with a antidiagonal slices and d diagonal slices (this parallelogram is the "AD parallelogram hull", analogous to the convex hull). It is easily seen that the parallelogram has $a+d-1$ horizontal slices, so we must have $h \leq a+d-1$. Note that for the number of hexagons to be maximized, the polyhex must have no gaps. Constructing the polyhex as described ensures no gaps and ensures uniqueness. □

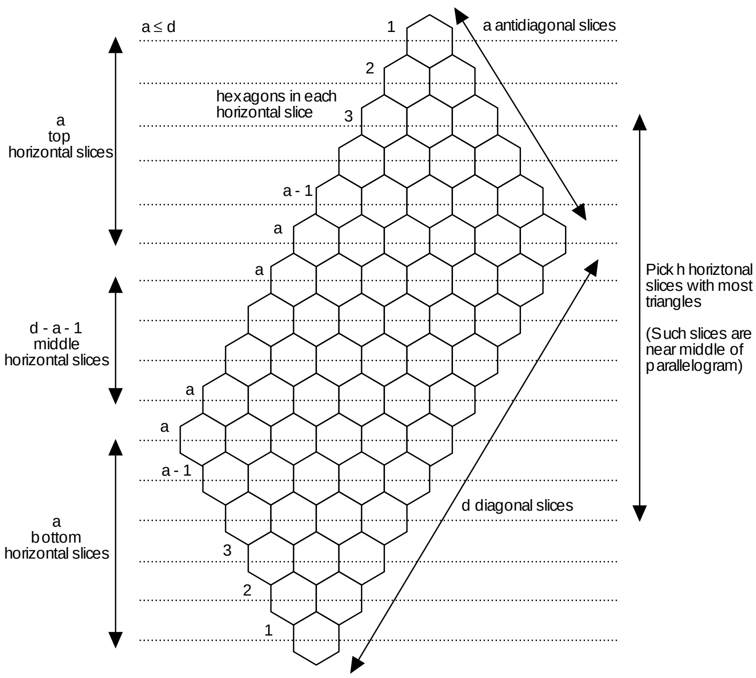

Figure 7: Constructing a polyhex with HAD dimensions (h, a, d) and the most hexagons.

To simplify the presentation, we assume for the remainder of the paper that the HAD dimensions satisfy $h \leq a \leq d$. For example, the polyhex of Figure 6 does not satisfy these inequalities because its HAD dimensions are $(h, a, d) = (5, 7, 6)$. However, if we do a vertical reflection to switch the antidiagonal and diagonal slices, then $h \leq a \leq d$. For an arbitrary polyhex, it is easy to see that the inequalities $h \leq a \leq d$ can be attained by rotation and reflection.

Theorem 7. **(HAD capacity formula)** *Let $h \leq a \leq d$. Let*

$$A = \frac{1}{2}(ad + ah + dh) - \frac{1}{4}(a^2 + d^2 + h^2).$$

The maximum number of hexagons in a polyhex with HAD dimensions (h, a, d) is

$$\text{capacity}\,(h,a,d) \quad = \quad \begin{cases} ah & h - (d - a - 1) < 0 \\ A + 1/4 & h - (d - a - 1) \geq 0 \textit{ and is even} \\ A & h - (d - a - 1) \geq 0 \textit{ and is odd} \end{cases}$$

Proof. There are 3 cases.

- Case: $h - (d - a - 1) < 0$. So $h < d - a - 1$. See Figure 7. The $d - a - 1$ horizontal slices in the middle of the parallelogram have the most hexagons; each slice has a hexagons. Pick h of these slices to construct a polyhex with ah hexagons.

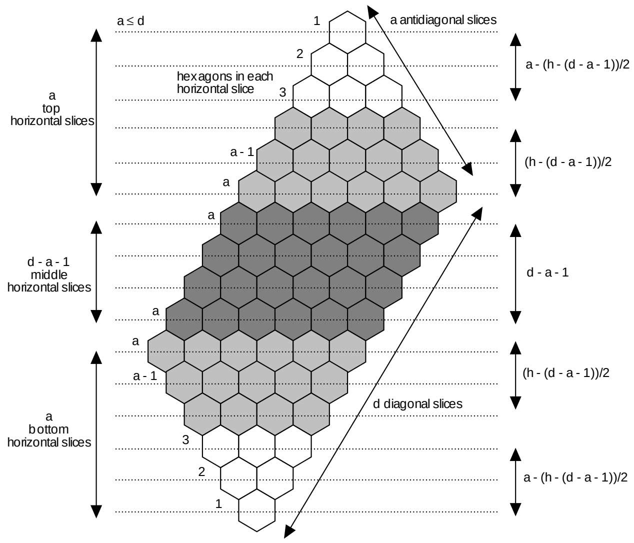

Figure 8: Counting hexagons in a polyhex with HAD dimensions (h, a, d) and the most hexagons, where $h - (d - a - 1) \geq 0$ and is even.

- Case: $h - (d - a - 1) \geq 0$ and is even. Note $h \geq d - a - 1$. See Figure 8.

The number of hexagons in the polyhex is

$$
\begin{aligned}
& a(d-a-1) \\
& +2\left(\frac{1}{2}a(a+1)-\frac{1}{2}\left(\frac{a+d-h-1}{2}\right)\left(\frac{a+d-h-1}{2}+1\right)\right) \\
= \; & \frac{1}{2}(ad+ah+dh)-\frac{1}{4}(a^2+d^2+h^2)+\frac{1}{4}.
\end{aligned}
$$

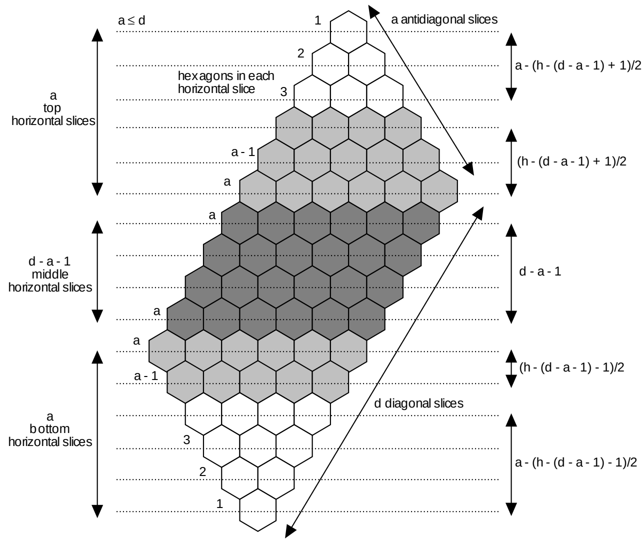

Figure 9: Counting hexagons in a polyhex with HAD dimensions (h, a, d) and the most hexagons, where $h-(d-a-1) \geq 0$ and is odd.

- Case: $h-(d-a-1) \geq 0$ and is odd. Note $h \geq d-a-1$. See Figure 9. The number of hexagons in the polyhex is

$$
\begin{aligned}
& a(d-a-1) \\
& +\left(\frac{1}{2}a(a+1)-\frac{1}{2}\left(\frac{a+d-h-2}{2}\right)\left(\frac{a+d-h-2}{2}+1\right)\right) \\
& +\left(\frac{1}{2}a(a+1)-\frac{1}{2}\left(\frac{a+d-h}{2}\right)\left(\frac{a+d-h}{2}+1\right)\right) \\
= \; & \frac{1}{2}(ad+ah+dh)-\frac{1}{4}(a^2+d^2+h^2).
\end{aligned}
$$

□

6 Maximal polyhex formula

Theorem 8. **(Maximal polyhex dimensions and capacity theorem)**

- *Let* capacity (p) *be the max number of hexagons in a polyhex with perimeter* p. *Let* p *not be odd, and let* $p \neq 2,\ 4,$ *or* 8 *because there are no polyhexes with these perimeters.*
- *Let* (h, a, d) *be the HAD dimensions of such a polyhex. (Without loss of generality, let* $h \leq a \leq d$. *We can rotate and reflect the polyhex if necessary to get these inequalities.)*

Then capacity (p) *and* (h, a, d) *are as follows.*

p	h	a	d	capacity (p)
$6q$	q	q	q	$\lceil \frac{1}{4}(3q^2) \rceil$
$6q+2$	q	q	$q+1$	$\lceil \frac{1}{4}(3q^2+2q-1) \rceil$
$6q+4$	q	$q+1$	$q+1$	$\lceil \frac{1}{4}(3q^2+4q) \rceil$

Also, the capacity formulas in the preceding table can be consolidated as follows:

$$\text{capacity}\,(p) = \left\lceil \frac{p^2}{48} - \delta \right\rceil, \quad \delta = \begin{cases} 0 & p \equiv 0 \pmod 6 \\ 1/3 & p \not\equiv 0 \pmod 6 \end{cases}$$

Proof. We will give an integer programming problem that maximizes the number of hexagons in a polyhex with perimeter exactly p. We note the following:

- To maximize the number of hexagons, the polyhex should have no gaps. By the Polyhex slices theorem (Theorem 5), $2(h + a + d) = p$.
- Without loss of generality, let $h \leq a \leq d$ (rotate and reflect the polyhex to get these inequalities). We use these inequalities in the proof.
- A maximal polyhex must have $h \geq d - a - 1$, by the following reasoning. If $h < d - a - 1$, then there is some diagonal slice that does not intersect the h horizontal slices of the polyhex (see Figure 7). We can remove this diagonal slice and add a horizontal slice to the polyhex. The polyhex now has HAD dimensions $(h + 1, a, d - 1)$, has more hexagons, and has the same perimeter, contradicting maximality.
- Because $h \geq d - a - 1$, by the HAD capacity formula (Theorem 7), the number of hexagons in the polyhex is as follows, where $c(h, a, d)$ is a correction term.

$$\begin{aligned}\text{number of hexagons} &= \frac{1}{2}(ad+ah+dh) - \frac{1}{4}(a^2+d^2+h^2) + c(h,a,d) \\ c(h,a,d) &= \begin{cases} 1/4 & h-(d-a-1) \geq 0 \text{ and is even} \\ 0 & h-(d-a-1) \geq 0 \text{ and is odd} \end{cases}\end{aligned}$$

The problem of maximizing the number of hexagons in a polyhex with perimeter p can therefore be expressed as follows:

$$\left[\begin{array}{llcll} \max & \frac{1}{2}(ad+ah+hd) - \frac{1}{4}(a^2+d^2+h^2) & + & c(h,a,d) & \\ \text{s.t.} & 2(h+a+d) & = & p & \\ & h & \leq & a & \leq \; d \\ & h,a,d & \in & \mathbf{Z} & \end{array}\right]$$

Simplify the objective function by using the constraint $2(h+a+d) = p$.

$$\left[\begin{array}{llcll} \max & \frac{1}{16}p^2 - \frac{1}{2}(a^2+d^2+h^2) & + & c(h,a,d) & \\ \text{s.t.} & 2(h+a+d) & = & p & \\ & h & \leq & a & \leq \; d \\ & h,a,d & \in & \mathbf{Z} & \end{array}\right]$$

Express the problem in terms of $x^T = (h,a,d)$ and $e^T = (1,1,1)$, and consider the relaxed problem obtained by dropping the constraint $h \leq a \leq d$.

$$\left[\begin{array}{llcl} \max & \frac{1}{16}p^2 - \frac{1}{2}x^Tx & + & c(x) \\ \text{s.t.} & 2e^Tx & = & p \\ & x & \in & \mathbf{Z^3} \end{array}\right]$$

Bring out the constant summand $p^2/16$ and the constant factor -1, and change the max to a min.

$$\frac{p^2}{16} - \left[\begin{array}{llcl} \min & \frac{1}{2}x^Tx & - & c(x) \\ \text{s.t.} & 2e^Tx & = & p \\ & x & \in & \mathbf{Z^3} \end{array}\right]$$

Drop the integer constraints (it turns out that we will be able to get integer solutions without them). Also, multiply the equality constraint by $1/2$. We have a relaxed problem.

$$\mathrm{RP} = \frac{p^2}{16} - \left[\begin{array}{llcl} \min & \frac{1}{2}x^Tx & - & c(x) \\ \text{s.t.} & e^Tx & = & p/2 \end{array}\right]$$

We will derive an alternative expression of the correction term $c(x) = c(h, a, d)$. At the beginning of this proof, we derived $h \geq d - a - 1$. So $h - (d - a - 1) \geq 0$. Note $h - (d - a - 1)$ is even iff $h + a + d = e^T x = p/2$ is even. We can express the correction term $c(h, a, d)$ as follows:

$$c(h, a, d) = \begin{cases} 1/4 & h - (d - a - 1) \geq 0 \text{ and is even} \\ 0 & h - (d - a - 1) \geq 0 \text{ and is odd} \end{cases} = \begin{cases} 1/4 & p/2 \text{ odd} \\ 0 & p/2 \text{ even} \end{cases}$$

The relaxed problem RP branches into 2 relaxed problems, one for the case $p/2$ odd and one for the case $p/2$ even.

$$\begin{aligned} \text{RP\_ODD} &= \frac{p^2}{16} - \left[\begin{array}{lllll} \min & \frac{1}{2}x^T x & - & 1/4 & \\ \text{s.t.} & e^T x & = & p/2 & \text{odd} \end{array} \right] \\ \text{RP\_EVEN} &= \frac{p^2}{16} - \left[\begin{array}{lllll} \min & \frac{1}{2}x^T x & & & \\ \text{s.t.} & e^T x & = & p/2 & \text{even} \end{array} \right] \end{aligned}$$

In each of these relaxed problems, the objective function is strictly convex. So any solution of these relaxed problems (and the related restricted problems considered below) is unique.

There are 3 cases: p can have the form $6q$, $6q + 2$, or $6q + 4$. In each case, it turns out that RP_ODD and RP_EVEN have solutions of the same form when expressed in terms of p.

For example, let $p = 6q + 4$. It turns out that if $p/2$ is odd, then $x_{\text{odd}} = (q, q+1, q+1)$ solves RP_ODD. If $p/2$ is even, then $x_{\text{even}} = (q, q+1, q+1)$ solves RP_EVEN. Note $x_{\text{odd}} = x_{\text{even}}$, in the sense that they have the same form.

- Case: $p = 6q$. Both RP_ODD and RP_EVEN have the solution $x = (q, q, q)$, which also solves the initial integer problem. The optimal value is

$$\text{capacity } (p) = \text{optimal value} = \frac{p^2}{16} - \frac{1}{2}x^T x + c(x) = \frac{1}{4}(3q^2) + c(x).$$

  To express the optimal value in a simpler form, note that if $p/2$ is even, then $c(x) = 0$ and $p^2/16 - x^T x/2$ is integer. If $p/2$ is odd, then $c(x) = 1/4$ and $p^2/16 - x^T x/2$ is an integer minus $1/4$. So the optimal value is

$$\left\lceil \frac{1}{4}(3q^2) \right\rceil.$$

- Case: $p = 6q + 2$. Because $h \leq a \leq d$, we cannot have $h = a = d$; else, $p = 2(h + a + d)$ is a multiple of 6. So we must have $h \leq d - 1$. There are 2 subcases.

– Subcase: $h = d-1$. Because p has the form $p = 6q+2$, we must also have $a = d - 1$. Add these constraints to RP_ODD and RP_EVEN. Both problems have the solution $x = (q, q, q+1)$, which also solves the integer problem. The optimal value is

$$\left\lceil \frac{1}{4}(3q^2 + 2q - 1) \right\rceil.$$

– Subcase: $h \leq d-2$. Add this constraint to RP_ODD and RP_EVEN. Both problems have the solution $x = (q-1, q+1, q+1)$, which also solves the integer problem. The optimal value is

$$\left\lceil \frac{1}{4}(3q^2 + 2q - 5) \right\rceil.$$

The subcase $h = d-1$ yields the solution because it gives the larger number of hexagons.

- Case: $p = 6q + 4$. Again, $h \leq a \leq d$ implies $h \leq d - 1$. Add this constraint to RP_ODD and RP_EVEN. Both problems have the solution $x = (q, q+1, q+1)$, which also solves the integer problem. The optimal value is

$$\left\lceil \frac{1}{4}(3q^2 + 4q) \right\rceil.$$

Summarizing all the cases, we get the table stated in the theorem. The statement about the consolidated capacity formula follows from the following calculations, in which

$$\delta = \begin{cases} 0 & p \equiv 0 \pmod 6 \\ 1/3 & p \not\equiv 0 \pmod 6 \end{cases}$$

p	p^2	$p^2/48$	δ	$p^2/48 - \delta$
$6q$	$36q^2$	$\frac{3}{4}q^2$	0	$\frac{3}{4}q^2$
$6q+2$	$36q^2 + 24q + 4$	$\frac{3}{4}q^2 + \frac{1}{2}q + \frac{1}{12}$	$\frac{1}{3}$	$\frac{3}{4}q^2 + \frac{1}{2}q - \frac{1}{4}$
$6q+4$	$36q^2 + 48q + 16$	$\frac{3}{4}q^2 + q + \frac{1}{3}$	$\frac{1}{3}$	$\frac{3}{4}q^2 + q$

□

A polyhex with HAD dimensions (h, a, d) is *balanced* iff it is slice-convex and h, a, d differ from one another by at most 1.

Theorem 9. **(Maximal-balanced equivalence theorem)** *A polyhex is maximal iff it is balanced.*

Proof. Use the Maximal polyhex dimensions and capacity theorem (Theorem 8). □

Theorem 10. **(Maximal polyhex formula)** *Let p be even and $p \neq 2$, 4, or 8. The max number of hexagons in a polyhex with perimeter p is*

$$\text{capacity}\,(p) = \text{round}\left(\frac{p^2}{48}\right).$$

Proof. Use the Maximal polyhex dimensions and capacity theorem (Theorem 8):

$$\text{capacity}\,(p) = \left\lceil \frac{p^2}{48} - \delta \right\rceil, \quad \delta = \begin{cases} 0 & p \equiv 0 \pmod 6 \\ 1/3 & p \not\equiv 0 \pmod 6 \end{cases}$$

Let $p = 12q + 2r$, where $r = 0, \ldots, 5$. Note the following equivalences.

$$\begin{array}{rrcl} & \text{capacity}\,(p) & = & \text{round}\left(\frac{p^2}{48}\right) \\ \iff & \left\lceil \frac{p^2}{48} - \delta \right\rceil & = & \text{round}\left(\frac{p^2}{48}\right) \\ \iff & \left\lceil 3q^2 + qr + \frac{r^2}{12} - \delta \right\rceil & = & \text{round}\left(3q^2 + qr + \frac{r^2}{12}\right) \\ \iff & 3q^2 + qr + \left\lceil \frac{r^2}{12} - \delta \right\rceil & = & 3q^2 + qr + \text{round}\left(\frac{r^2}{12}\right) \\ \iff & \left\lceil \frac{r^2}{12} - \delta \right\rceil & = & \text{round}\left(\frac{r^2}{12}\right). \end{array}$$

The last equality is easy verified by considering the cases $r = 0, \ldots, 5$. □

7 Maximal polyhex algorithm

We give 2 versions of the maximal polyhex algorithm, a slice version and a spiral version. The slice version contains the proof. The spiral version is an alternate approach.

Theorem 11. (**Maximal polyhex slice algorithm**) *See Figure 10. Let p be even. To construct a polyhex with perimeter p and the most hexagons, do the following:*

- *If p is odd or is 2, 4, or 8, stop. There is no polyhex with this perimeter.*
- *Find the HAD dimensions (h, a, d) in the following table.*

p	h	a	d
$6q$	q	q	q
$6q+2$	q	q	$q+1$
$6q+4$	q	$q+1$	$q+1$

- *Draw a parallelogram with HAD dimensions $(a + d - 1, a, d)$.*
- *Pick the h horizontal slices with the most hexagons.*

Such a polyhex is unique, ignoring rotation and reflection.

Proof. Use the Maximal-balanced equivalence theorem (Theorem 9) and the HAD capacity algorithm (Theorem 6). □

An alternative algorithm that produces polyhexes of the most hexagons is as follows; from now on, "maximal polyhex algorithm" will refer this spiral version:

Theorem 12. (**Maximal polyhex spiral algorithm**) *See Figure 11. Let p be even. To construct a polyhex with perimeter p and with the most hexagons, follow the spiral until the last appearance of a perimeter at most p.*

Note that by construction, the capacity is increasing.

8 Minimal polyhex algorithm

Theorem 13. (**Minimal polyhex algorithm**) *See Figure 11. To construct a polyhex with n hexagons and with min perimeter, follow the spiral for n hexagons.*

Proof. Note that if p is even, $p \geq 6$, and $p \neq 8$, there is some polyhex with perimeter p. See Figure 11.

Note that if p is even and $p \geq 12$, then p is the min perimeter of an n-polyhex iff capacity $(p-2) < n \leq$ capacity (p). This follows because polyhexes with perimeter $\leq p-2$ cannot contain an n-polyhex (because they do not have sufficient capacity), whereas polyhexes with perimeter p do have sufficient capacity. These latter polyhexes can be constructed using the Maximal polyhex spiral algorithm (Theorem 12). See Figure 11. □

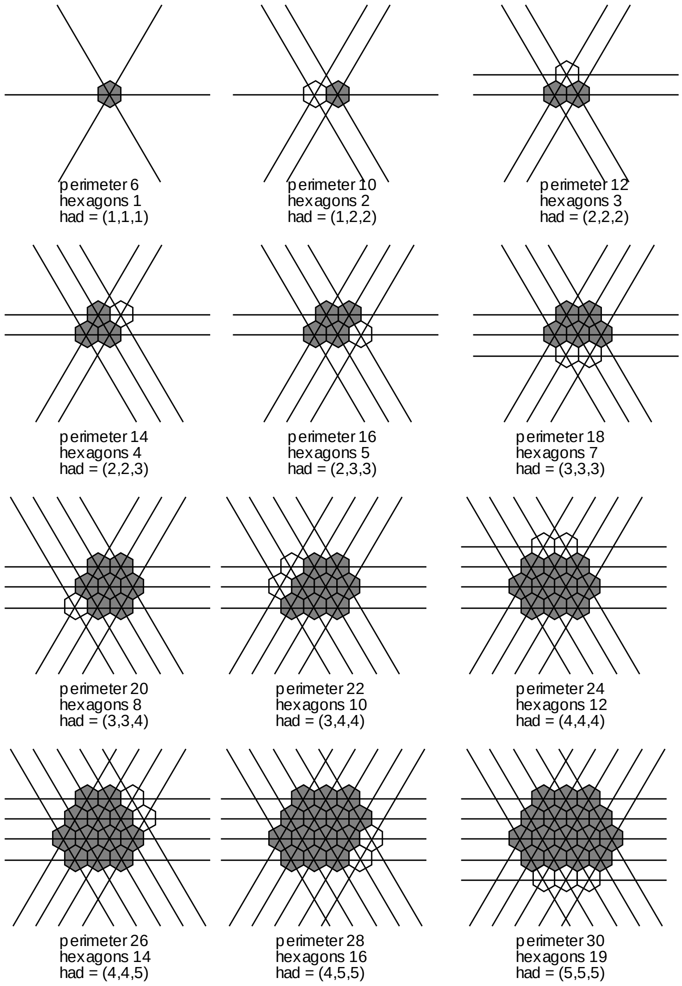

Figure 10: Maximal polyhex slice algorithm. Newly added hexagons are white, and old hexagons are shaded.

9 Minimal polyhex formula

To prove the Minimal polyhex formula (Theorem 18), we need the following definition. A honeycomb hexagon is a shape constructed by the following algorithm.

> **Honeycomb hexagon algorithm**: See Figure 12. At step 0, start with any hexagon H_0. At step k, let H_k be the union of H_{k-1} and all hexagons that share an edge with H_{k-1}. Each set H_k is a *honeycomb hexagon*.

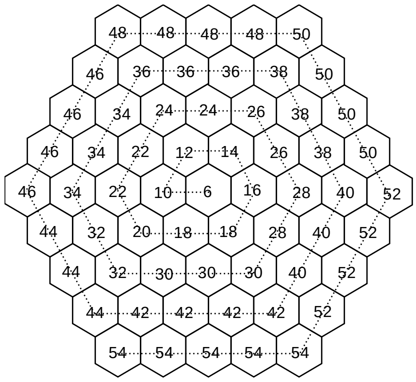

Figure 11: Add hexagons in a clockwise spiral. The number in a hexagon is the perimeter of the polyhex constructed so far. It is easy to see that the perimeter stays the same along each edge of the spiral, and increases by 2 each time the spiral turns a corner. This spiral is used in the Maximal polyhex spiral algorithm (Theorem 12) and the Minimal polyhex algorithm (Theorem 13).

Note that the Minimal polyhex algorithm (Theorem 13) gives another way to construct honeycomb hexagons: add hexagons in a clockwise spiral.

Note that in a honeycomb hexagon, the centers of the boundary hexagons lie on a hexagon (see Figure 12). The *index* of a honeycomb hexagon is the length of the side of this hexagon, where the measurement unit is the distance between the centers of 2 boundary hexagons. With this definition, the honeycomb hexagon H_k constructed at step k of the honeycomb hexagon algorithm has index k.

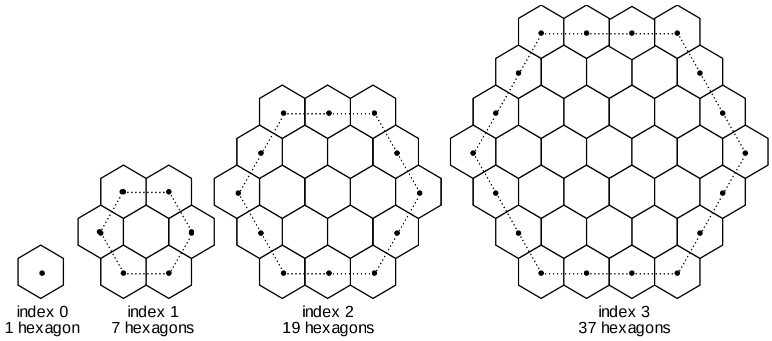

Figure 12: Honeycomb hexagons of indices 0, 1, 2, 3.

Theorem 14. (**Minimal polyhex intermediate perimeter theorem**) *In the Minimal polyhex algorithm (Theorem 13), if we add Δ hexagons to a honeycomb hexagon of index k, then the polyhexes constructed have the following perimeters:*

Δ	perimeter
$0 \leq \Delta \leq 0$	$12k+6$
$1 \leq \Delta \leq k$	$12k+8$
$k+1 \leq \Delta \leq 2k+1$	$12k+10$
$2k+2 \leq \Delta \leq 3k+2$	$12k+12$
$3k+3 \leq \Delta \leq 4k+3$	$12k+14$
$4k+4 \leq \Delta \leq 5k+4$	$12k+16$
$5k+5 \leq \Delta \leq 6k+6$	$12k+18$

Proof. See Figure 11 and Figure 13. Use induction. When we add hexagons, there are 2 cases. If the hexagon added is at a vertex of the honeycomb hexagon, the perimeter increases by 2. If the hexagon added is not at a vertex of the honeycomb hexagon, the perimeter stays the same. □

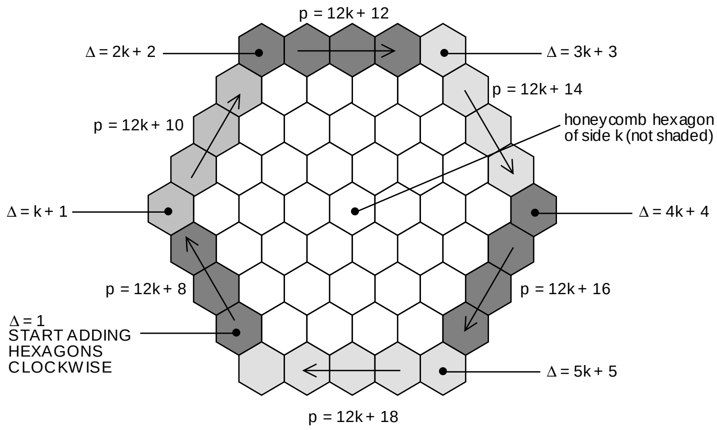

Figure 13: Perimeters of polyhexes constructed by the Minimal polyhex algorithm (Theorem 13); add hexagons clockwise along the boundary of a honeycomb hexagon of index k to construct a honeycomb hexagon of index $k+1$.

Theorem 15. (**Honeycomb hexagon size theorem**) *A honeycomb hexagon of index k has $3k^2+3k+1$ hexagons.*

Proof. See Figure 12. Let H_k be a honeycomb hexagon of index k. Let $|H_k|$ be the number of hexagons in H_k. Note $|H_{k+1}| = |H_k| + 6(k+1)$. Use induction. □

Theorem 16. (**Honeycomb hexagon integrality theorem**) *Let $n, k \in Z$. Then*

$$n = 3k^2 + 3k + 1 \iff \sqrt{12n-3} = 6k+3 \iff \sqrt{12n-3} \in Z.$$

Proof. The only implication that needs explanation is $\sqrt{12n-3} \in Z \implies \sqrt{12n-3} = 6k+3$. Let $\sqrt{12n-3} = m \in Z$. Then $n = (m^2+3)/12$. Note m must be odd for n to be integer. There are 3 cases: m has the form $6k+1$, $6k+3$, or $6k+5$. Only the case $m = 6k+3$ allows n to be integer. □

Theorem 17. (**Minimal polyhex ceiling theorem**) *In the Minimal polyhex algorithm (Theorem 13), if we add Δ hexagons to a honeycomb hexagon of index k, and if $n = (3k^2+3k+1)+\Delta$, then $\lceil\sqrt{12n-3}\rceil$ has the following values:*

Δ					$\lceil\sqrt{12n-3}\rceil$
0	$\leq$	Δ	$\leq$	0	$6k+3$
1	$\leq$	Δ	$\leq$	k	$6k+4$
$k+1$	$\leq$	Δ	$\leq$	$2k+1$	$6k+5$
$2k+2$	$\leq$	Δ	$\leq$	$3k+2$	$6k+6$
$3k+3$	$\leq$	Δ	$\leq$	$4k+3$	$6k+7$
$4k+4$	$\leq$	Δ	$\leq$	$5k+4$	$6k+8$
$5k+5$	$\leq$	Δ	$\leq$	$6k+6$	$6k+9$

Proof. For $\Delta = 0$, $\lceil\sqrt{12n-3}\rceil = 6k+3$. The rest of the table follows from the following abbreviated calculations (let $i = 1, \ldots, 6$):

$$\begin{array}{lrcrcl} & & & \lceil\sqrt{12n-3}\rceil & = & 6k+3+i \\ \iff & 6k+3+(i-1) & < & \sqrt{12n-3} & \leq & 6k+3+i \\ \iff & (i-1)k + \frac{i-1}{2} + \frac{(i-1)^2}{12} & < & \Delta & \leq & ik + \frac{i}{2} + \frac{i^2}{12}. \end{array}$$

□

Theorem 18. (**Minimal polyhex formula**) *The min perimeter of a polyhex with n hexagons is $2\lceil\sqrt{12n-3}\rceil$.*

Proof. Compare the tables in the Minimal polyhex intermediate perimeter theorem (Theorem 14) and the Minimal polyhex ceiling theorem (Theorem 17). □

10 Capacity generating function

Theorem 19. (**Even-perimeter capacity generating function**)

$$\sum_{q=0}^{\infty} \text{capacity}\,(2q)\, x^q = \sum_{q=0}^{\infty} \text{round}\left(\frac{q^2}{12}\right) x^q = \frac{x^3}{(1-x)(1-x^2)(1-x^3)}.$$

Proof. The first equality follows from the Maximal polyhex formula (Theorem 10). The second equality follows from [8]. □

Recall that capacity (p) was defined to be the max number of hexagons in a polyhex with perimeter p. However, there is no polyhex with odd perimeter or perimeter 2, 4, or 8, and so capacity (p) is undefined for these perimeters.

When we consider sequences a_n and the corresponding generating functions, it is nice to have a_n defined for all values of n. Let us make the following generalized definition: capacity (p) is the max number of hexagons in a polyhex with perimeter *at most* p. This definition agrees with the old one, except that now if p is odd, then capacity (p) = capacity $(p-1)$. Also, capacity (2) = capacity $(4) = 0$ and capacity $(8) = 1$.

Theorem 20. *(***Capacity generating function***)*

$$\sum_{p=0}^{\infty} \text{capacity}\,(p)\, x^p = \frac{x^6(1+x)}{(1-x^2)(1-x^4)(1-x^6)}.$$

Proof. Use the Even-perimeter capacity generating function (Theorem 19).

$$\begin{aligned}
\sum_{p=0}^{\infty} \text{capacity}\,(p)\, x^p &= \sum_{q=0}^{\infty} \text{capacity}\,(2q)\, x^{2q} + \sum_{q=0}^{\infty} \text{capacity}\,(2q+1)\, x^{2q+1} \\
&= \sum_{q=0}^{\infty} \text{capacity}\,(2q)\, x^{2q} + \sum_{q=0}^{\infty} \text{capacity}\,(2q)\, x^{2q+1} \\
&= \frac{x^6}{(1-x^2)(1-x^4)(1-x^6)} + \frac{x^7}{(1-x^2)(1-x^4)(1-x^6)} \\
&= \frac{x^6(1+x)}{(1-x^2)(1-x^4)(1-x^6)}.
\end{aligned}$$

□

Note that the Maximal polyhex formula (Theorem 10) is still valid for the perimeters 2, 4, and 8, so it is valid for all even perimeters. In fact, it is valid for odd perimeters, too, after a slight modification:

$$\begin{aligned}
\text{capacity}\,(p) &= \begin{cases} \text{round}\left(p^2/48\right) & p \text{ even} \\ \text{round}\left((p-1)^2/48\right) & p \text{ odd} \end{cases} \\
&= \text{round}\left(\frac{(2\lfloor p/2 \rfloor)^2}{48}\right) \\
&= \text{round}\left(\frac{\lfloor p/2 \rfloor^2}{12}\right).
\end{aligned}$$

11 Acknowledgements

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12 Appendix

The following table gives values of 2 sequences in the text. See Figures 10 and 11.

- minperim(n) is the min perimeter of an n-polyhex. Let $n \geq 1$. Then

$$\text{minperim}(n) = 2\lceil\sqrt{12n-3}\,\rceil.$$

- capacity (p) is the max number of hexagons in a polyhex with perimeter at most p (this is the generalized definition of capacity (p), discussed in the section about the capacity generating function). Included in the table are the HAD dimensions (h, a, d) of such a polyhex.

$$\begin{aligned} \text{capacity}\,(p) &= \text{round}\left(\frac{\lfloor p/2 \rfloor^2}{12}\right). \\ \sum_{p=0}^{\infty} \text{capacity}\,(p)\, x^p &= \frac{x^6(1+x)}{(1-x^2)(1-x^4)(1-x^6)}. \end{aligned}$$

The left side of the table, indexed by n, is independent of the right side of the table, indexed by p; the values between the 2 sides are not related. The (h, a, d) columns are to be used with only the capacity (p) column.

n	minperim(n)	p	capacity (p)	h	a	d
0	0	0	0	0	0	0
1	6	1	0	0	0	0
2	10	2	0	0	0	0
3	12	3	0	0	0	0
4	14	4	0	0	0	0
5	16	5	0	0	0	0
6	18	6	1	1	1	1
7	18	7	1	1	1	1
8	20	8	1	1	1	1
9	22	9	1	1	1	1
10	22	10	2	1	2	2
11	24	11	2	1	2	2
12	24	12	3	2	2	2
13	26	13	3	2	2	2
14	26	14	4	2	2	3
15	28	15	4	2	2	3
16	28	16	5	2	3	3