GCD sum theorems. Two Multivariable Cesáro Type Identities

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The purpose of these notes is to record a multivariable generalisation of Cesáro's identity (1) and a multivariable generalisation of its companion identity (2). These two results are probably in the literature, although I haven't been able to locate a reference; for the convenience of users of the OEIS I have written up the proofs. Using these results we give a pair of gcd summation identities in Section 4.

1. Introduction.

Let $f(n)$ be an arithmetical function. Cesáro gave the identity

$$\sum_{k=1}^{n} f\left(\gcd(k,n)\right) = \sum_{d|n} f(d)\phi\left(\frac{n}{d}\right), \tag{1}$$

where $\phi(n)$ denotes Euler's totient function. For a compact one-line proof see O. BORDELLES [1, Lemma 1].

A companion result to (1), which can be proved in a similar manner, is

$$\sum_{k=1}^{n} f\left(\frac{n}{\gcd(k,n)}\right) = \sum_{d|n} f(d)\phi\left(d\right). \tag{2}$$

A particularly interesting case of (1) is when f is a multiplicative function. Then the right-hand side of (1) is the Dirichlet convolution of two multiplicative functions and hence is also multiplicative. Examples in the OEIS include Pillai's arithmetical function A018804 ($f(n) = 1$), A069097 ($f(n) = n^2$), A343497 ($f(n) = n^3$), A343498 ($f(n) = n^4$), A343499 ($f(n) = n^5$), A029935 ($f(n) = \phi(n)$), A007434 (either $f(n) = \phi(n^2)$) or ($f(n) = n\phi(n)$) , A342534 ($f(n) = \phi(n)^2$), A007431 ($f(n) = mu(n)$), A063659 ($f(n) = mu(n)^2$), A008683 ($f(n) = n * mu(n)$), A078439 ($f(n) = n * mu(n)^2$), A300717 ($f(n) = mu(n) * \phi(n)$), A191356 ($f(n) = (-1)^{n+1}$), A332794 $\left(f(n) = (-1)^{n+1}n\right)$, A000203 ($f(n) = \tau(n)$), A060724 $\left(f(n) = \tau(n)^2\right)$, A344132 $\left(f(n) = \tau(n)^3\right)$, A344138 $\left(f(n) = \tau(n)^4\right)$, A344139 $\left(f(n) = \tau(n)^5\right)$, A060648 $\left(f(n) = \tau(n^2)\right)$, A344321 $\left(f(n) = \tau(n^3)\right)$, A344322 $\left(f(n) = \tau(n^4)\right)$, A038040 ($f(n) = \sigma(n)$), A064987 ($f(n) = \sigma_2(n)$), A328259 ($f(n) = \sigma_3(n)$), A281372 ($f(n) = \sigma_4(n)$), A341772 ($f(n) = J_2(n)$), A059376 ($f(n) = n * J_2(n)$) and A176345 ($f(n) = \text{rad}(n)$).

Particular cases in the OEIS of the companion identity (2) include A057660 $(f(n)=n)$, A068963 $\left(f(n)=n^2\right)$, A068970 $\left(f(n)=n^3\right)$, A368744 $\left(f(n)=(-1)^{n+1}\right)$, A332845 $\left(f(n)=(-1)^{\omega(n)}\right)$, A029939 $(f(n)=\phi(n))$, A338997 $\left(f(n)=\phi^2(n)\right)$, A342470 $\left(f(n)=\phi^3(n)\right)$, A276833 $(f(n)=mu(n))$, A007947 $\left(f(n)=mu^2(n)\right)$, A062949 $(f(n)=\tau(n))$ and A062952 $(f(n)=\sigma(n))$.

Our generalisations of (1) and (2) involve the Jordan totient functions. We recall some of the basic properties of these arithmetical functions.

2. Jordan totient functions. Let $[n]=\{1,2,...,n\}$. For a positive integer r, the Jordan totient function $J_r(n)$ gives the number of r-tuples $(k_1,k_2,...,k_r)$, such that each $k_i\in[n]$ and $\gcd(k_1,k_2,...,k_r,n)=1$:

$$J_r(n)=\sum_{\substack{k_i\in[n]\\ \gcd(k_1,k_2,..,k_r,n)=1}} 1.$$

In particular, $J_1(n)=\phi(n)$, so the Jordan totient functions generalise Euler's totient function.

The function $J_r(n)$ is a multiplicative function of n since it can be expressed as the Dirichlet convolution of the multplicative functions n^r and the mobius function $\mu(n)$,

$$J_r(n)=\sum_{d|n} d^r mu\left(\frac{n}{d}\right).$$

It follows that the function J_r has the Dirichlet generating function (D.g.f.)

$$\sum_{n\geq 1}\frac{J_r(n)}{n^s}=\zeta(s-r)/\zeta(s),\quad \mathrm{Re}(s)>r+1, \tag{3}$$

where $\zeta(s)$ is the Riemann zeta function.

The value of the totient function on prime powers is given by

$$J_r\left(p^k\right)=p^{rk}-p^{r(k-1)}.$$

3. Generalised Ces$\acute{a}$ro identities.

The proof of Cesáro's identity (1) easily generalises to the following multivariable identity.

Theorem 1. *Let f be an arithmetical function. Then*

$$\sum_{k_i \in [n]} f\left(\gcd\left(k_1, k_2, \ldots, k_r, n\right)\right) = \sum_{d|n} f(d) J_r(n/d) \,.$$

Proof. We prove the theorem only in the case $r = 2$. The reader should have little trouble in extending the proof to the general case.

Let A_n denote the Cartesian product $[n]$ X $[n]$. For each positive integer d, a divisor of n, we define a subset A_d of A_n by

$$A_d = \{(i,j) : i,j \in [n] \text{ and } \gcd(i,j,n) = d\} \,. \tag{4}$$

Clearly, A_n is the disjoint union of the subsets A_d taken over all the divisors of n:

$$A_n = \sqcup_{d|n} A_d \,.$$

The function f takes the constant value $f(d)$ on A_d. Hence

$$\sum_{i,j \in [n]} f\left(\gcd\left(i,j,n\right)\right) = \sum_{d|n} f(d) |A_d| \,. \tag{5}$$

We determine the cardinality $|A_d|$ of the set A_d.

For each pair $(i,j) \in A_d$ both i and j are divisible by d, say $i = dy$ and $j = dz$. Now $\gcd(i,j,n) = \gcd(dy,dz,n) = d$ if and only if $\gcd(y,z,n/d) = 1$. Furthermore, $1 \le dy \le n$ and $1 \le dz \le n$ if and only if $1 \le y \le n/d$ and $1 \le z \le n/d$.

Therefore, from (4),

$$A_d = \{(dy,dz) : 1 \le y \le n/d, 1 \le z \le n/d \text{ and } \gcd(y,z,n/d) = 1\} \,.$$

It follows from the definition of the Jordan totient function J_2 that the cardinality of A_d is $J_2(n/d)$.

Hence, by (5),

$$\sum_{k_i \in [n]} f\left(\gcd\left(k_1, k_2, n\right)\right) = \sum_{d|n} f(d) J_2\left(\frac{n}{d}\right) \,,$$

completing the proof of the Theorem in the case $r = 2$. ■

Corollary 1. If $f(n)$ is a multiplicative function then the gcd sum $\sum_{k_i \in [n]} f\left(\gcd\left(k_1, k_2, \ldots, k_r, n\right)\right)$ is also a multiplicative function of n. ■

Examples 3.1. A129194 $\left(r=2, f(n)=(-1)^{n+1}\right)$, A341772 $(r=2, f(n)=\phi(n))$, A321322 $(r=2, f(n)=\mu(n))$, A158949 $\left(r=2, f(n)=\tau\left(n^2\right)\right)$, A001158 $(r=3, f(n)=\tau(n))$, A001159 $(r=4, f(n)=\tau(n))$, A001160 $(r=5, f(n)=\tau(n))$, A281372 $(r=4, f(n)=\sigma_1(n))$, A282097 $(r=3, f(n)=\sigma_2(n))$.

Next we give a multivariable extension of the companion identity (2) to Cesáro's identity.

Theorem 2. *Let f be an arithmetical function. Then*

$$\sum_{k_i \in [n]} f\left(\frac{n}{\gcd(k_1, k_2, \ldots, k_r, n)}\right) = \sum_{d|n} f(d) J_r(d).$$

Proof. As in Theorem1 we prove the theorem only in the case $r=2$ (this particular case of the theorem has been observed by Werner Schulte - see his comment in A350156). The extension of the theorem to the general case is straightforward.

For each positive integer d, a divisor of n, we define the codivisor $d' = n/d$. We define the subset $A_{d'}$ of the Cartesian product $A_n = [n] \mathrm{X} [n]$ by

$$A_{d'} = \{(i,j) : i,j \in [n] \text{ and } \gcd(i,j,n) = d'\}.$$

Clearly, A_n is the disjoint union of the subsets $A_{d'}$ taken over all the divisors d of n:

$$A_n = \sqcup_{d|n} A_{d'}.$$

One checks that the function f takes the constant value $f(d)$ on A'_d. Hence

$$\sum_{i,j \in [n]} f\left(\frac{n}{\gcd(i,j,n)}\right) = \sum_{d|n} f(d) |A_{d'}|. \tag{6}$$

In Theorem 1 we showed that $|A_{d'}|$ is equal to $J_2(n/d') = J_2(d)$.

Hence by (6)

$$\sum_{i,j \in [n]} f\left(\frac{n}{\gcd(i,j,n)}\right) = \sum_{d|n} f(d) J_2(d),$$

completing the proof of the Theorem in the case $r=2$. ■

Examples 3.2. A084218 $(r=2, f(n)=n^2)$ and A078615 $(r=2, f(n)=\mu(n)^2)$.

4. Two GCD sum identities.

One easy consequence of Theorem 1 is the following pretty identity for gcd sums.

Theorem 3. *For positive integers i and j,*

$$\sum_{k's\in[n]} \gcd\left(k_1,k_2,...,k_i,n\right)^j = \sum_{k's\in[n]} \gcd\left(k_1,k_2,...,k_j,n\right)^i. \tag{7}$$

Proof. We show that the arithmetic functions on both sides of (7) have the same D.g.f.'s on a region of the complex plane. The theorem then follows by the uniqueness of the coefficients of a Dirichlet series convergent in an open domain of $\mathbb{C}$.

By Theorem 1, the left-hand side of (7) is equal to the divisor sum

$$\sum_{d|n} d^j J_i(n/d), \tag{8}$$

the Dirichlet convolution $n^j \bigstar J_i$. The D.g.f. of n^j is $\zeta(s-j)$ and hence by (3) the D.g.f. of the left-hand side of (7) is $\zeta(s-j)\dfrac{\zeta(s-i)}{\zeta(s)}$, convergent in the half-plane $\mathrm{Re}(s) > \max(i,j)+1$.

Again by Theorem 1, the right-hand side of (7) is equal to the divisor sum

$$\sum_{d|n} d^i J_j(n/d),$$

the Dirichlet convolution $N^i \bigstar J_j$, with D.g.f. $\zeta(s-i)\dfrac{\zeta(s-j)}{\zeta(s)}$, the same as the D.g.f. of the left-hand side of (7). ■

Examples 4.1. A069097 ($i=2, j=1$), A343497 ($i=3, j=1$), A343498 ($i=4, j=1$) and A368743 ($i=2, j=3$).

We conclude with a second identity for gcd sums involving the sum of divisors function σ_k, defined by

$$\sigma_k(n) = \sum_{d|n} d^k.$$

The function σ_k is a multiplicative function of n with D.g.f. $\zeta(s)\zeta(s-k)$, convergent for $\mathrm{Re}(s) > k+1$ when $k \geq 0$.

Theorem 4. *For positive integers i and j,*

$$\sum_{k's \in [n]} \sigma_i \left(\gcd(k_1, k_2, ..., k_j) \right) = \sum_{k's \in [n]} \sigma_j \left(\gcd(k_1, k_2, ..., k_i) \right). \tag{9}$$

Sketchproof. Exactly similar to the proof of Theorem 3. One uses Theorem 1 to express the left-hand and right-hand sides of (9) as Dirichlet convolutions and then show that their corresponding D.g.f.'s are equal on a half-plane of $\mathbb{C}$. ■

Examples 4.2. A064987 ($i = 1, j = 2$), A328259 ($i = 1, j = 3$), A281372 ($i = 1, j = 4$) and A282097 ($i = 2, j = 3$).

References

[1] OLIVIER BORDELLES, A Multidimensional Cesaro Type Identity and Applications, J. Int. Seq. 18 (2015) # 15.3.7.

[2] WIKIPEDIA, Divisor function

[3] WIKIPEDIA, Jordan's totient function