FAIRLY 4-REGULAR GRAPHS A361135

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Abstract. These are illustrations of the 1, 2, 7, 21, 74 and 269 unlabeled undirected fairly 4-regular graphs on 2, 3, 4, 5, 6 and 7 nodes. Sequence A361135 of the OEIS counts the *connected* subset of these graphs.

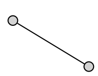
1. Introduction

Fairly 4-regular graphs on n nodes are graphs with 2 nodes of degree 1 and $n-2$ nodes of degree 4—nomenclature as in [5, 1]. (In some parts of the literature these are enumerated as graphs on $n-2$ nodes with two free legs, and occasionally these two legs are called fins (half-edges) and count as one node [2].) Alternatively one may remove the nodes of degree 1 and their incident edges and consider graphs with 2 nodes of degree 3 (formerly adjacent to the leafs) and $n-2$ nodes of degree 4, a bijection which obviously keeps all information intact. This text illustrates graphs which may have loops and multi-edges and any number of components. The connected graphs up to 6 nodes have been visualized by Kleinert et al [4].

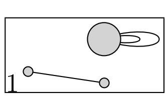
A result of the handshake lemma is that almost 4-regular graphs — with 1 node of degree 1 and n-1 nodes of degree 4 — do not exist.

The two nodes with degree 1 (leafs) are plotted with smaller radius than the nodes of degree 4. Graphs with more than one component are framed for visual clarity. Graphs are enumerated at the lower left edge starting at 1 (…sometimes this index may appear at the end of previous line).

2. 1 graph on 2 nodes

1 

3. 2 graphs (1 connected) on 3 nodes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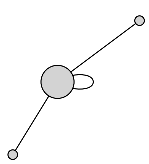
2 

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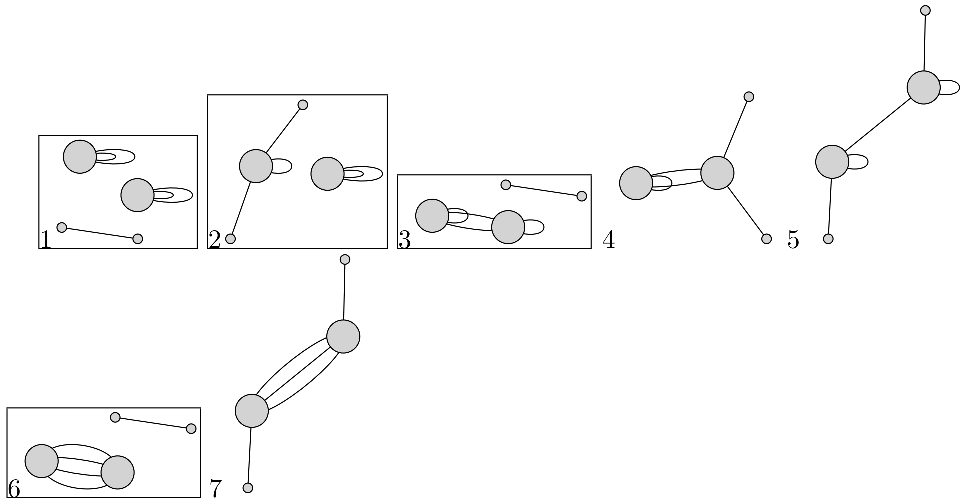
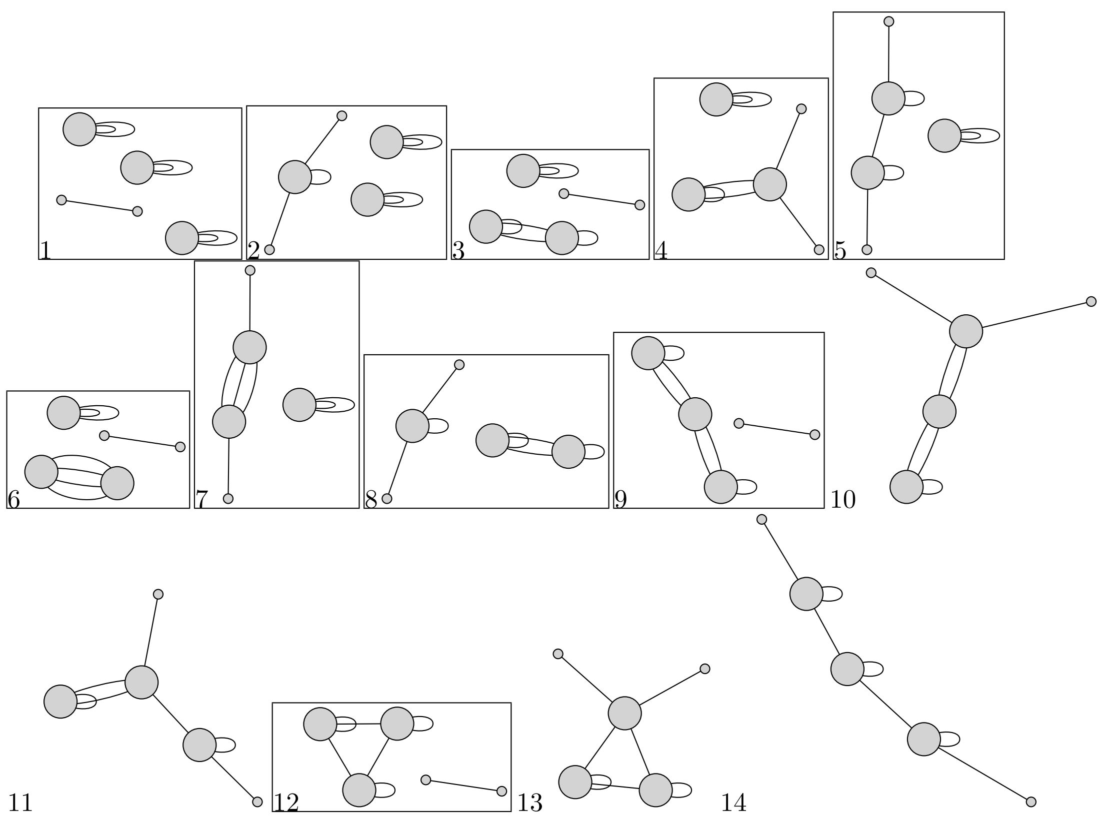
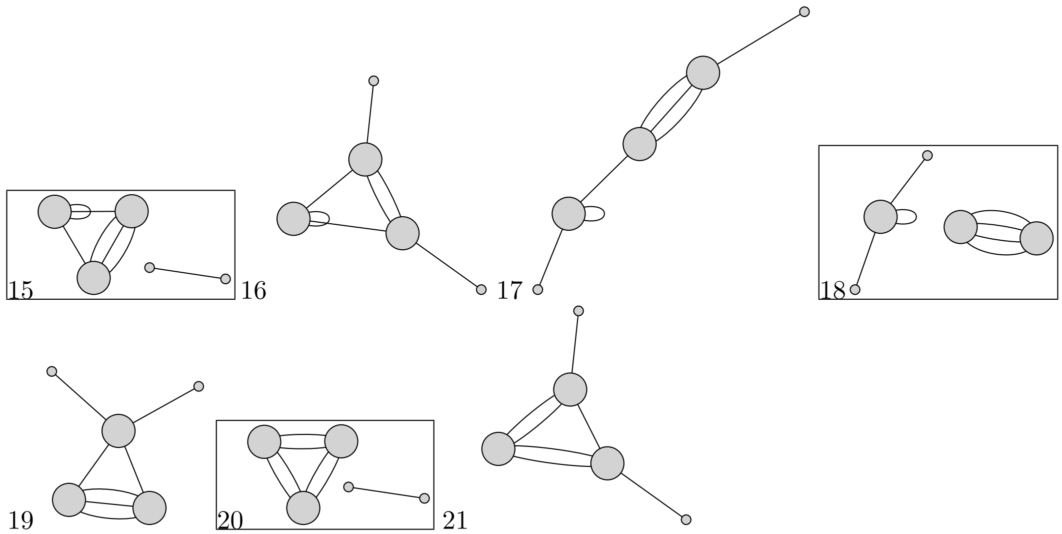
Key words and phrases. Graph Enumeration, Regular Graphs, phi(4).

4. 7 graphs (3 connected) on 4 nodes

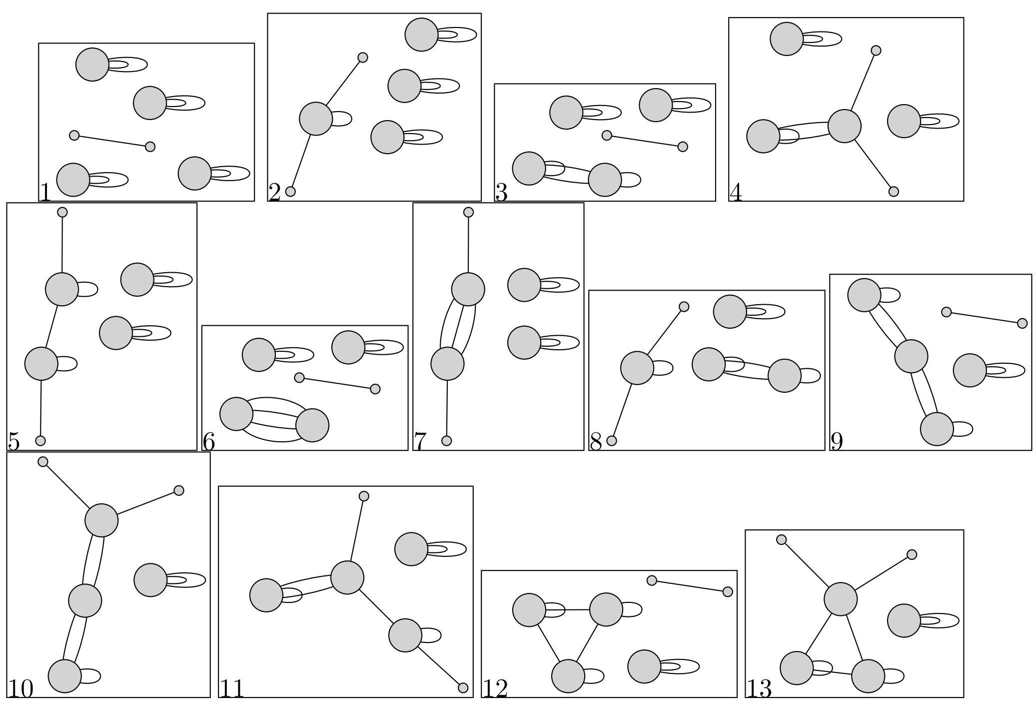


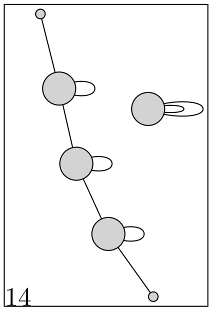
5. 21 graphs (8 connected) on 5 nodes





6. 74 graphs (30 connected) on 6 nodes

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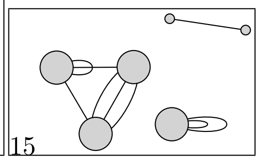
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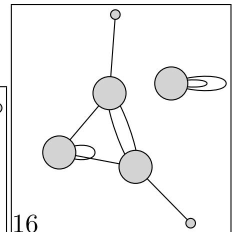
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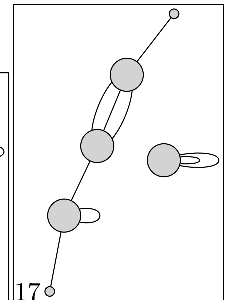
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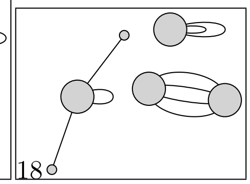
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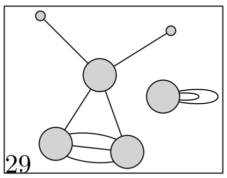
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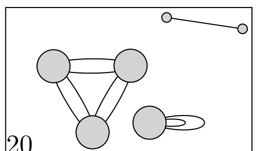
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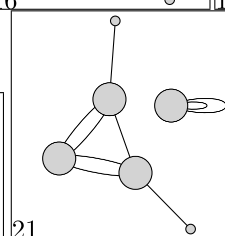
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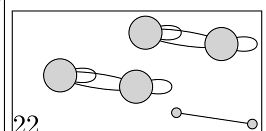
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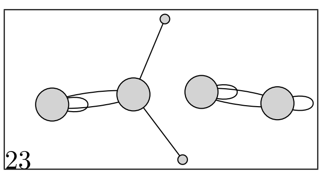
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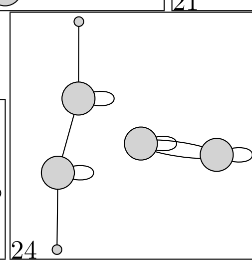
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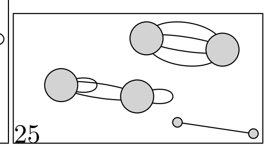
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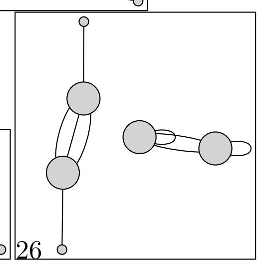
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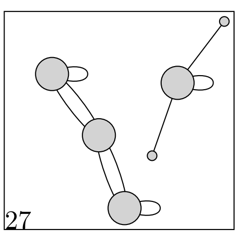
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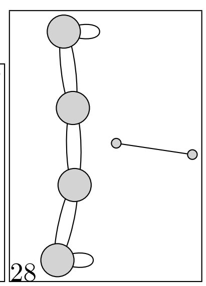
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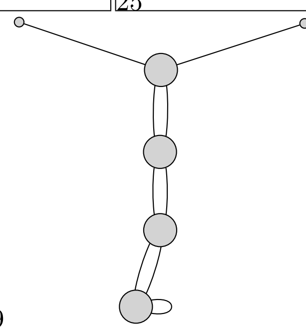
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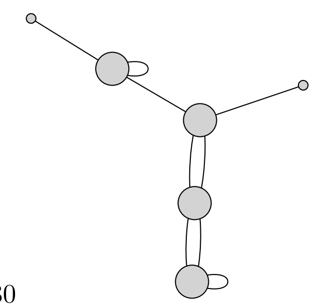
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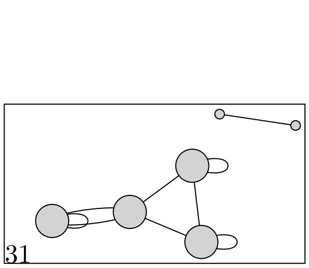
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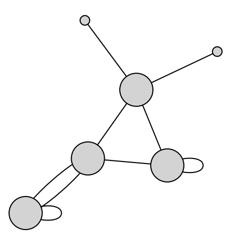
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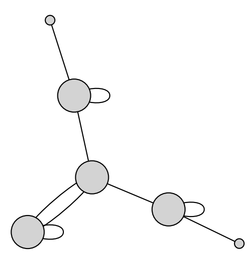
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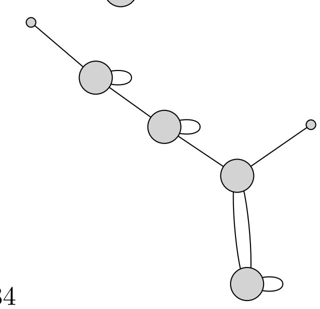
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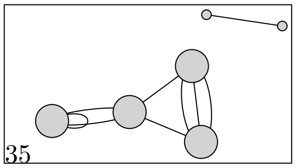

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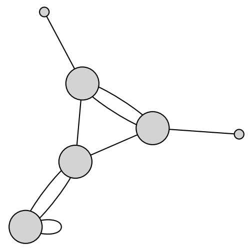

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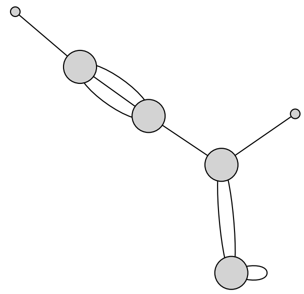

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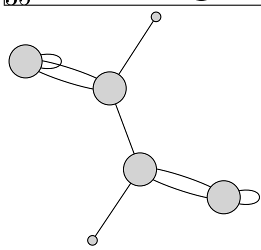

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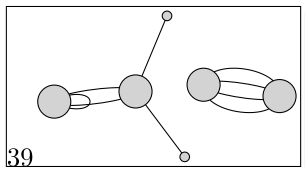

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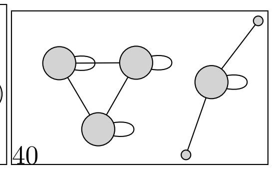

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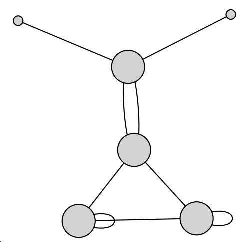

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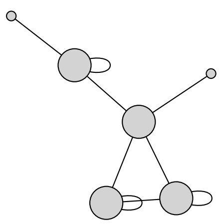

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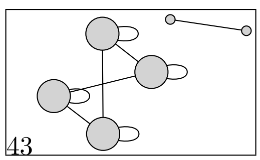

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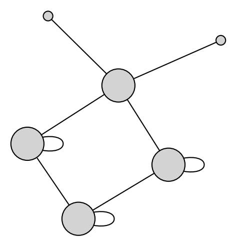

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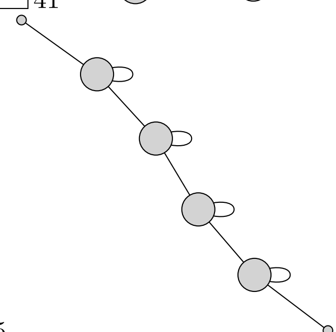

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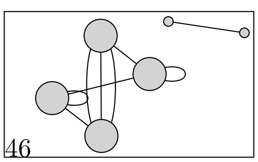

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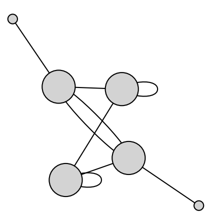

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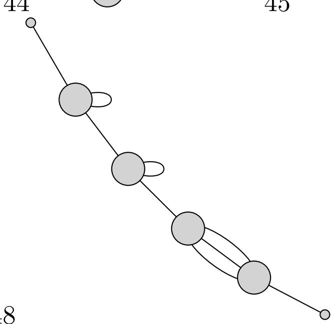

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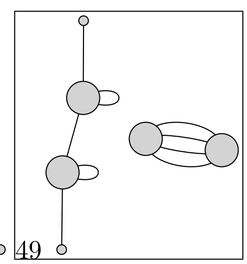

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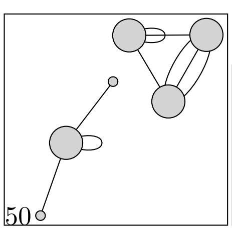

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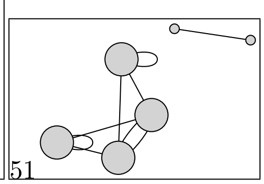

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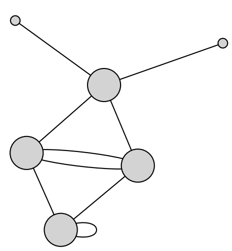

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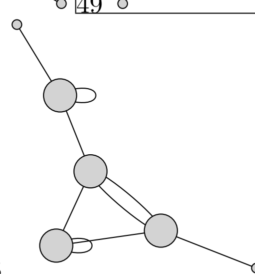

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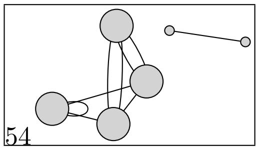
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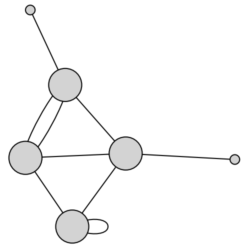
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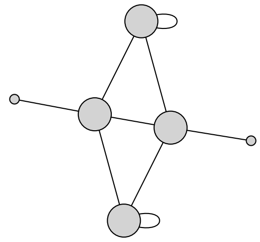
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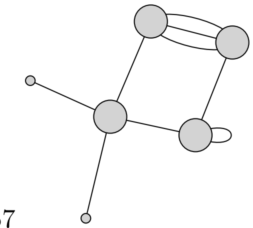
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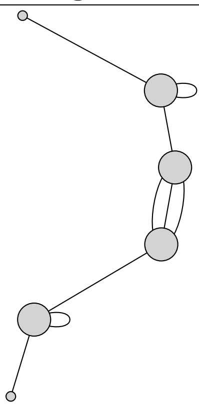
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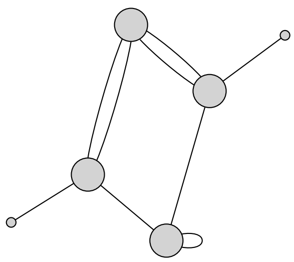
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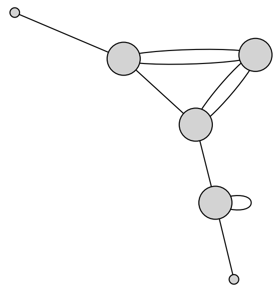
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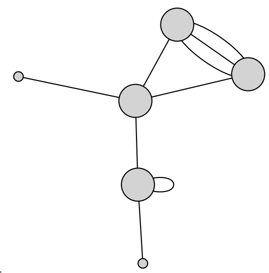
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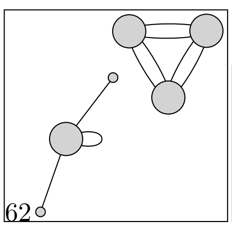
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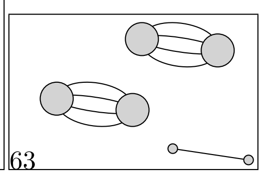
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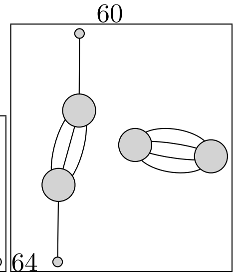
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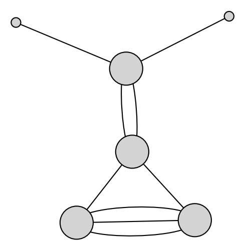
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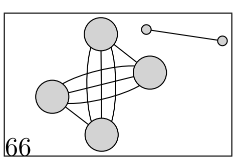
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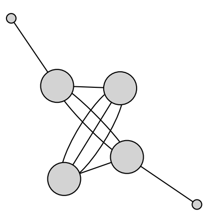
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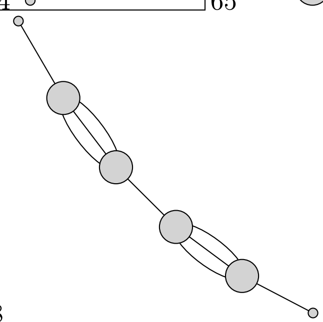
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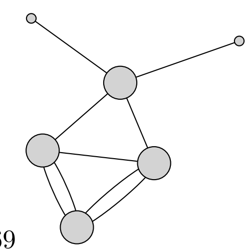
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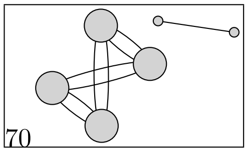
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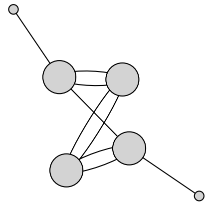
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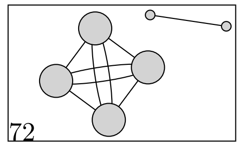
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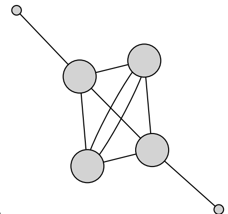
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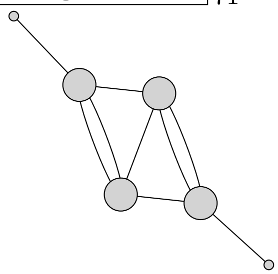
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7. 269 graphs on 7 nodes (118 connected)

On 7 nodes only the connected graphs are illustrated:

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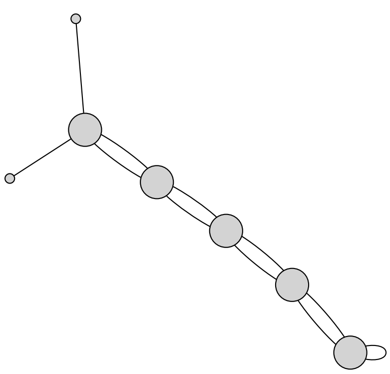
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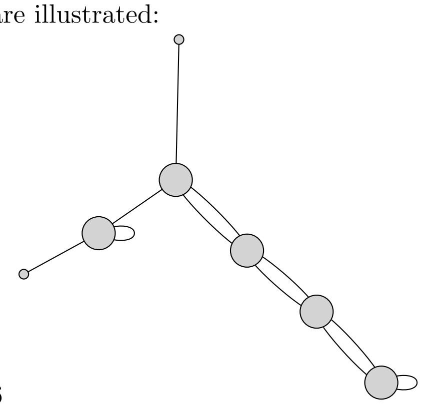
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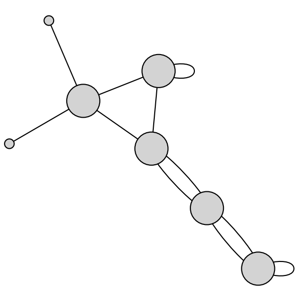
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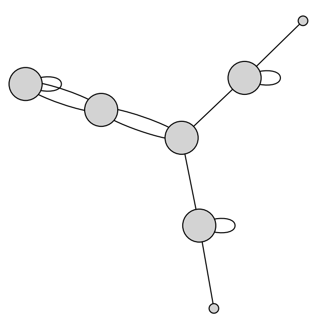
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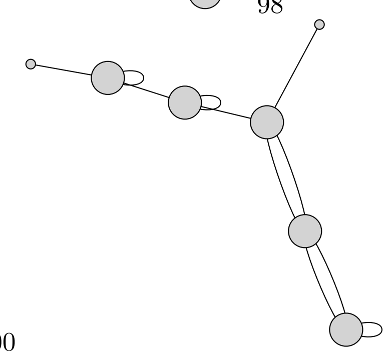

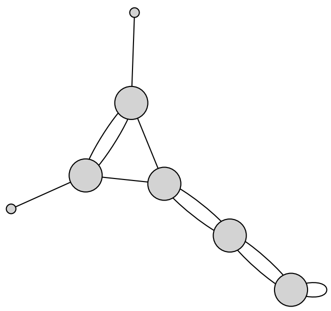

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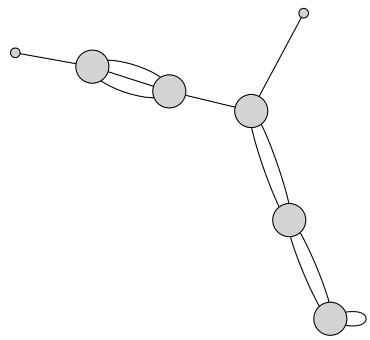

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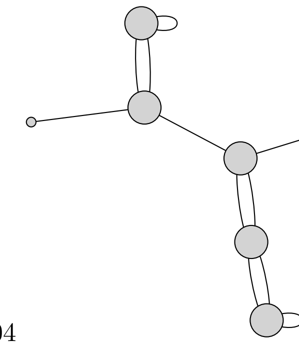

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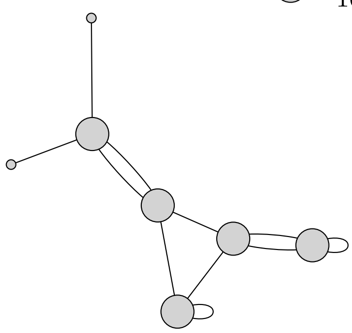

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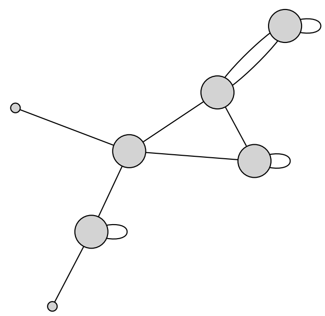

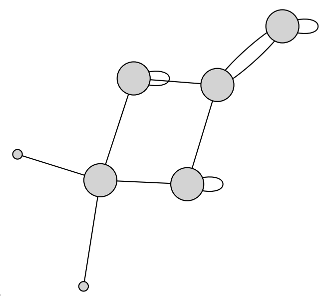

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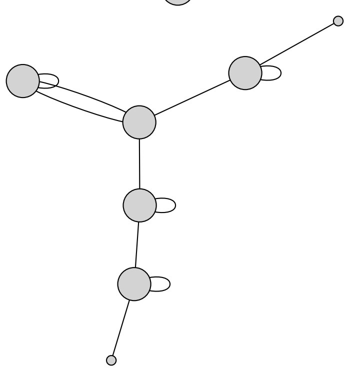

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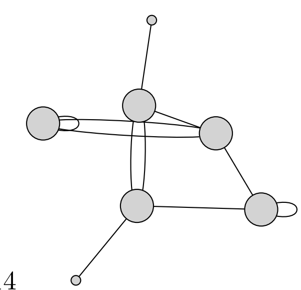

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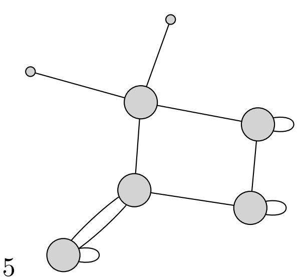

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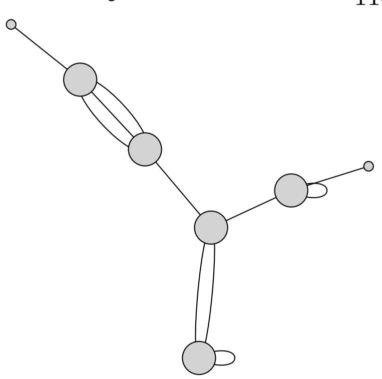

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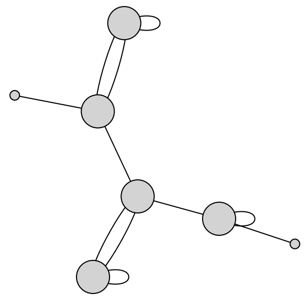

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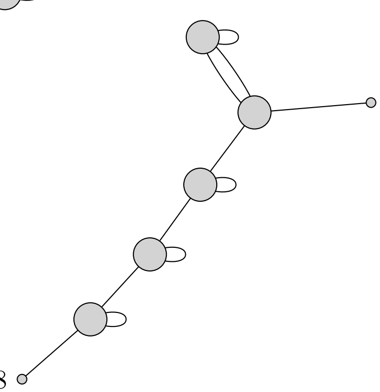

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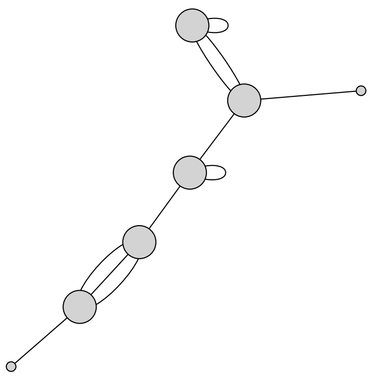

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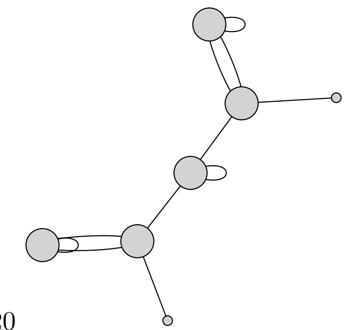

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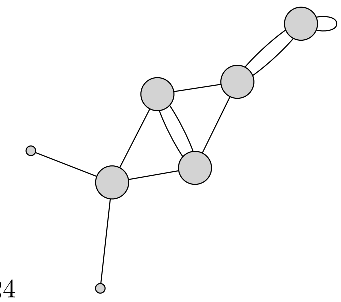

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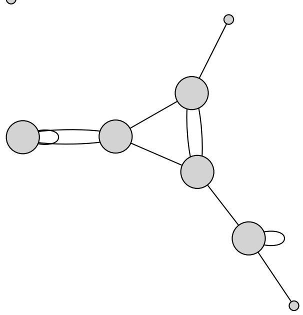

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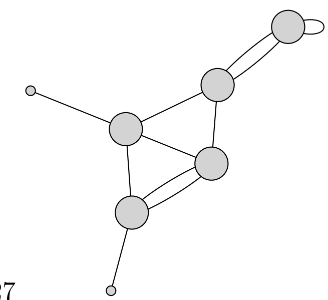

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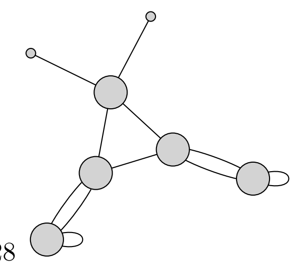

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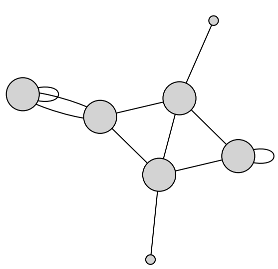

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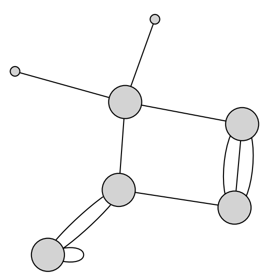

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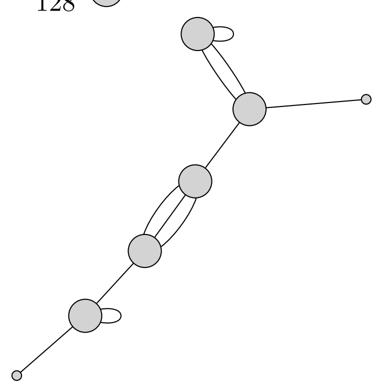

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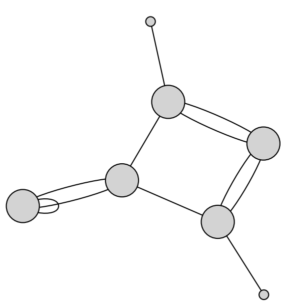

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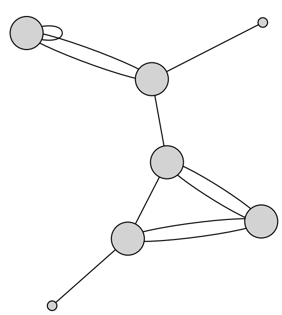

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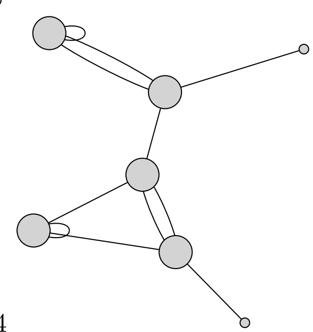

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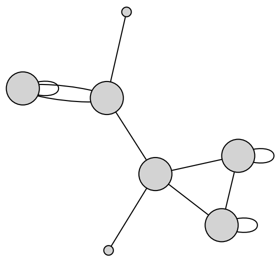

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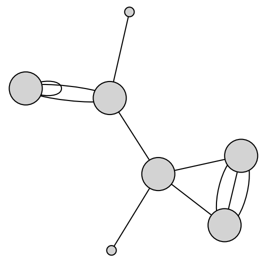

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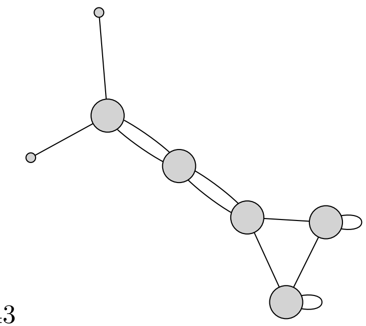

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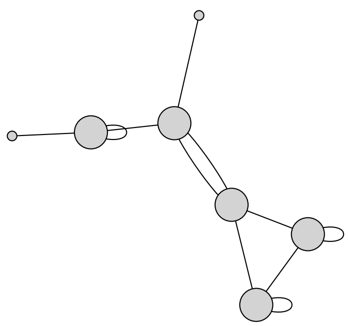

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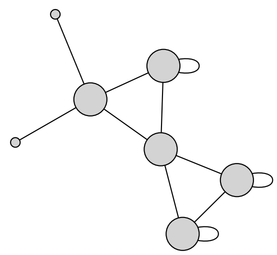

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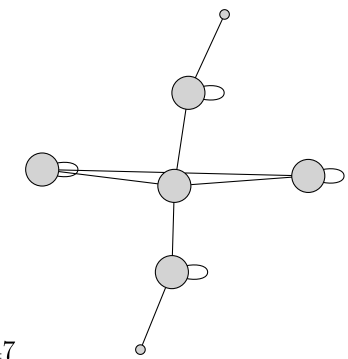

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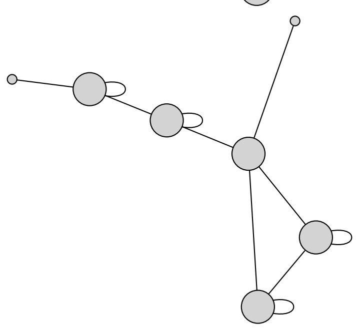

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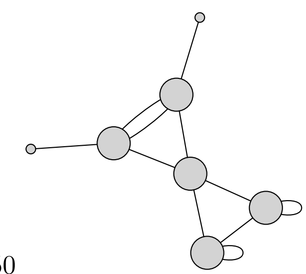

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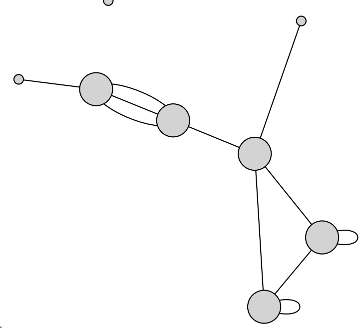

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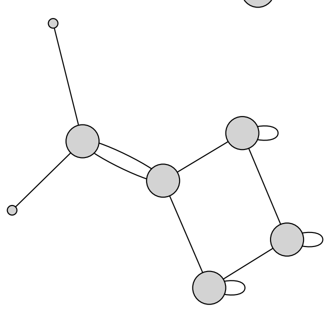

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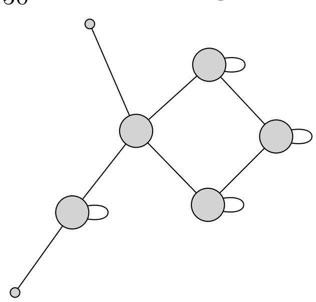

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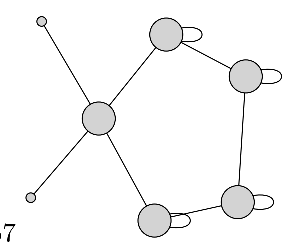

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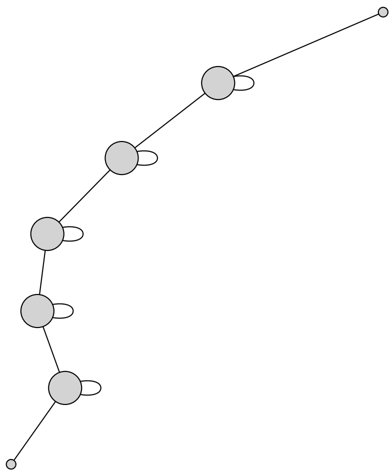

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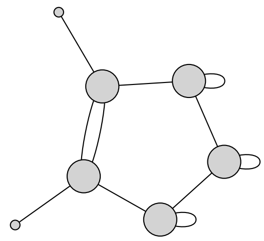

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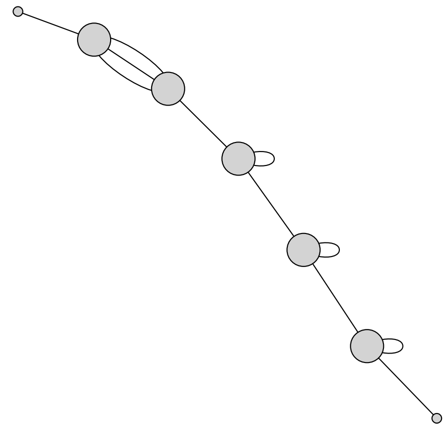

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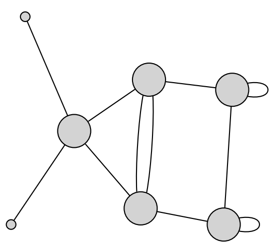

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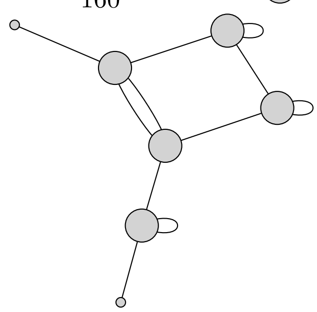

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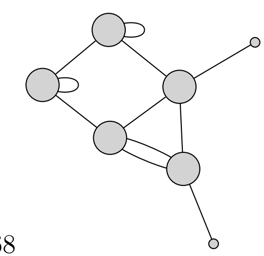

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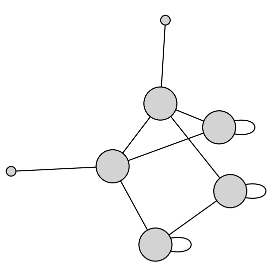

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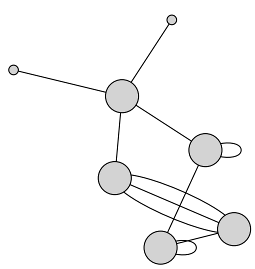

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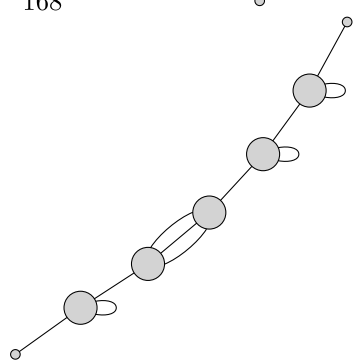

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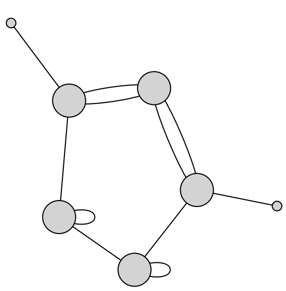

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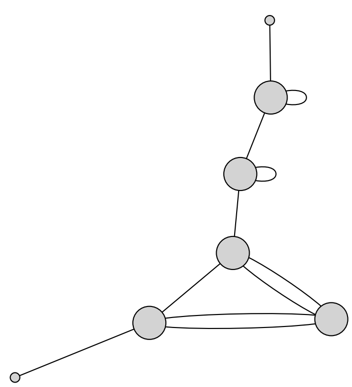

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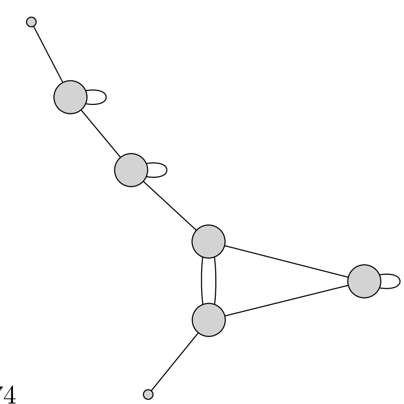

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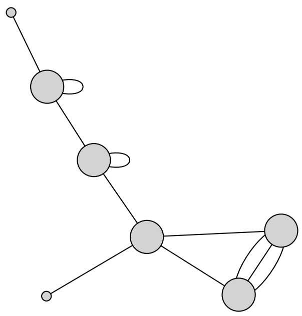

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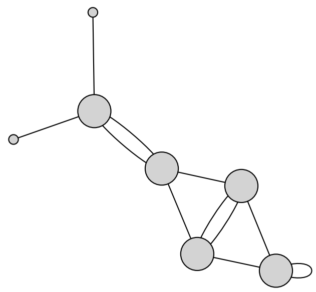

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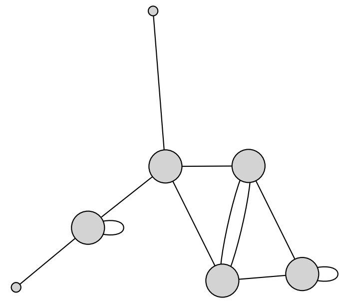

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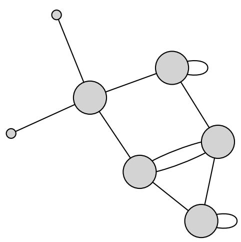

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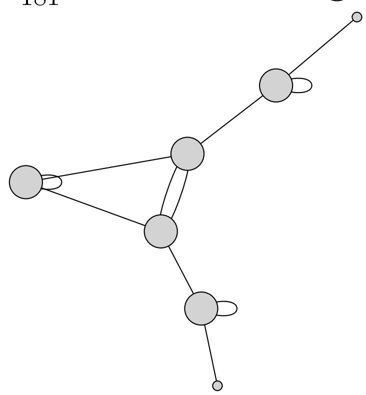

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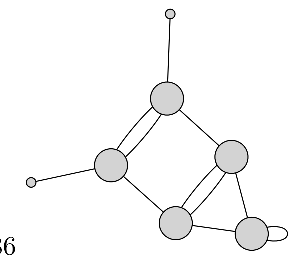

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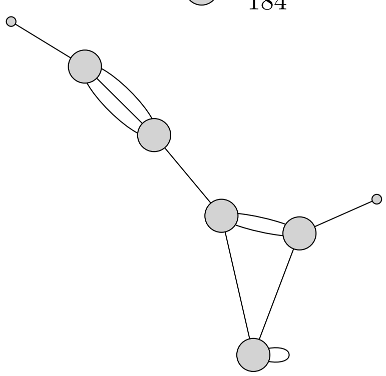

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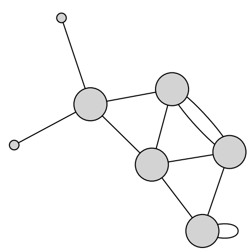

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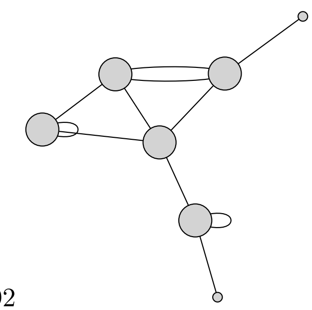

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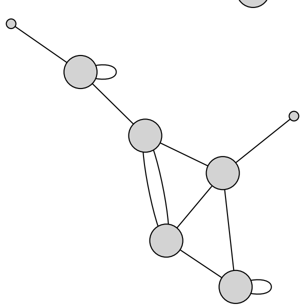

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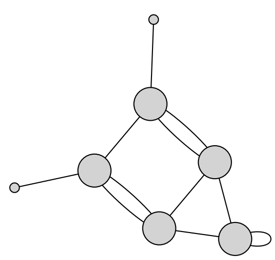

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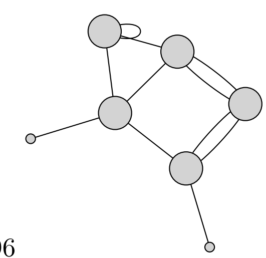

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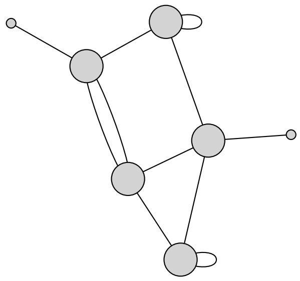

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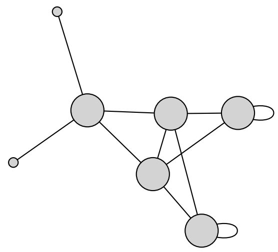

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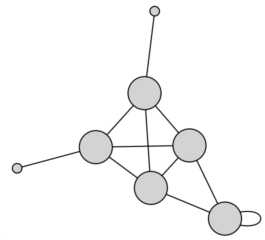

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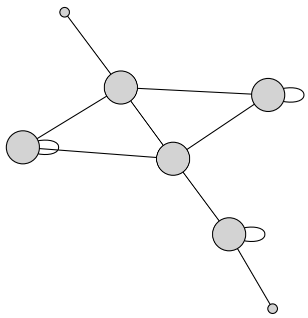

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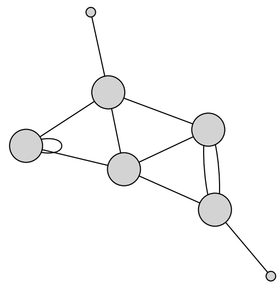

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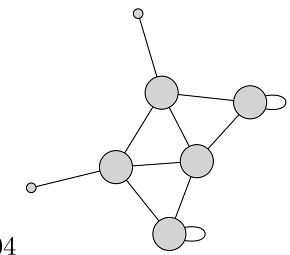

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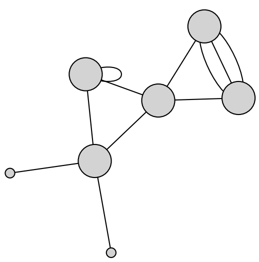

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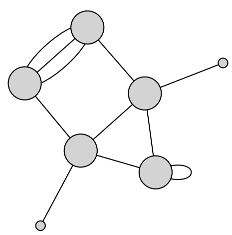

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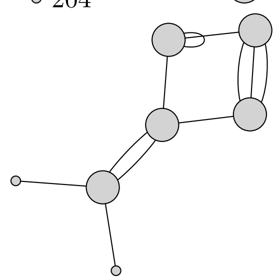

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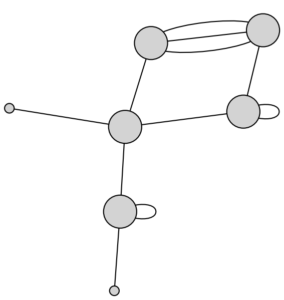

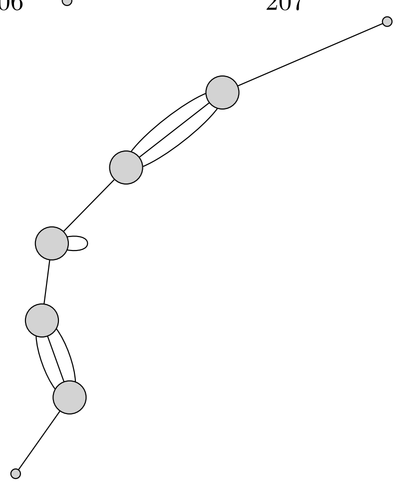

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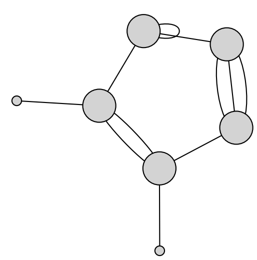

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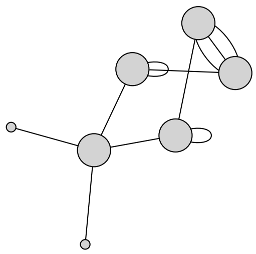

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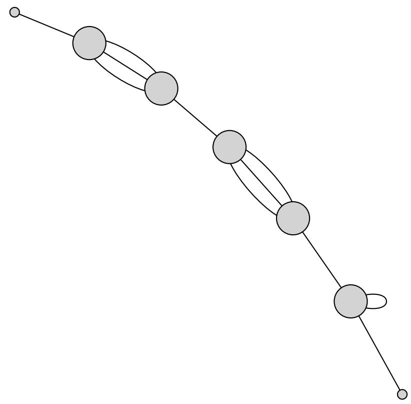

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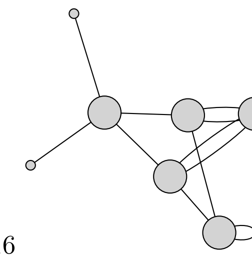

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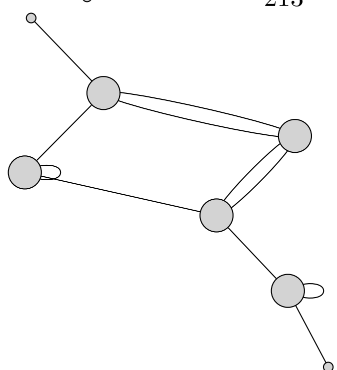

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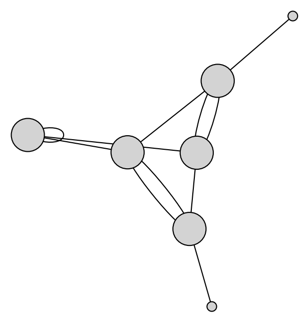

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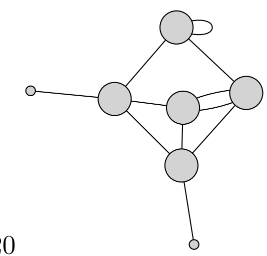

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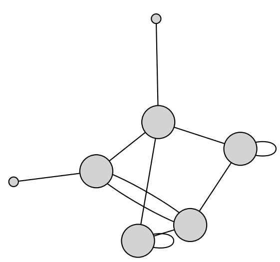

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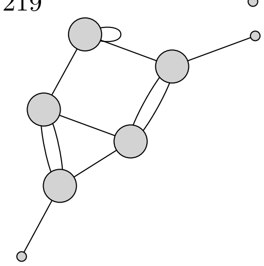

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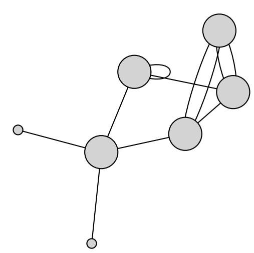

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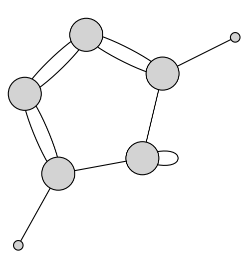

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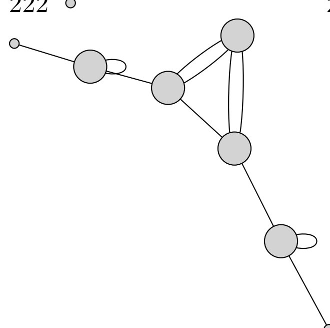

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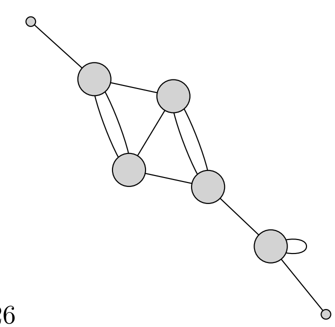

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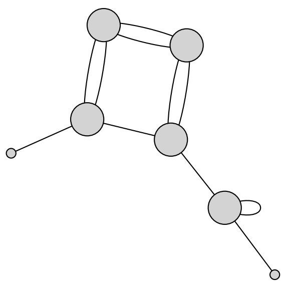

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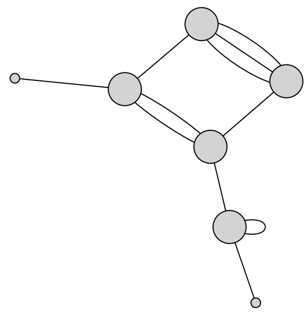

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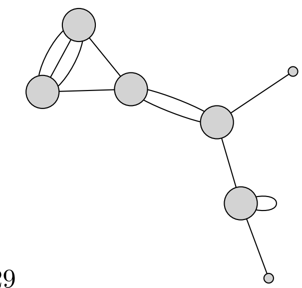

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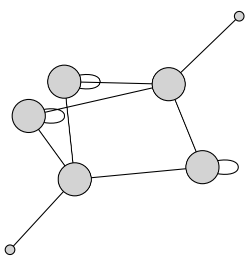

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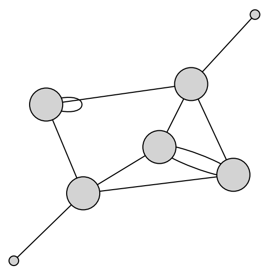

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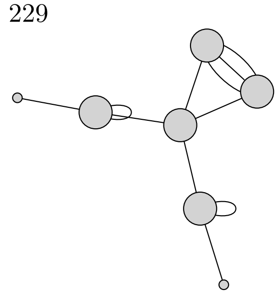

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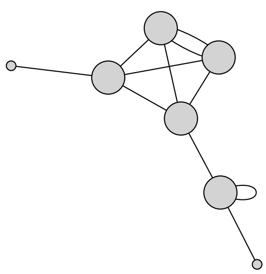

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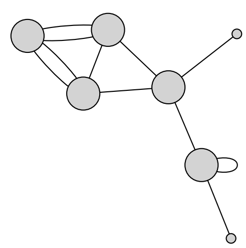

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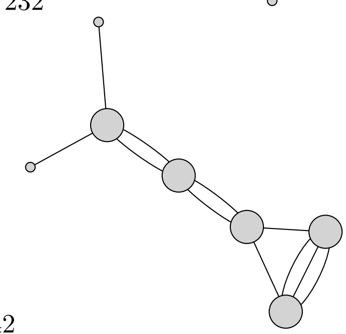

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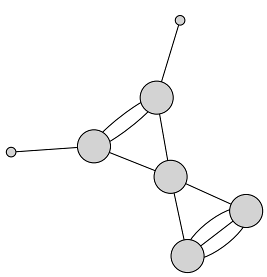

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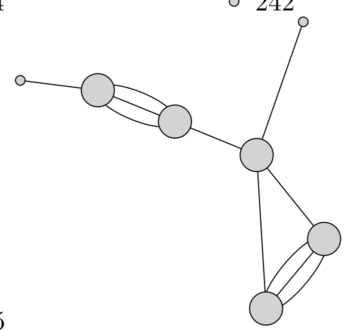

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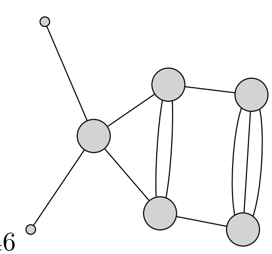

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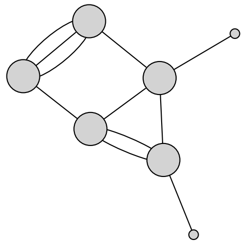

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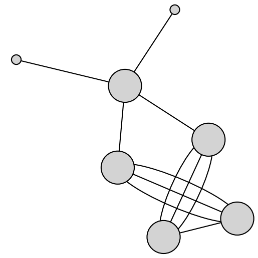

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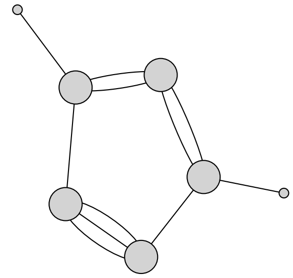

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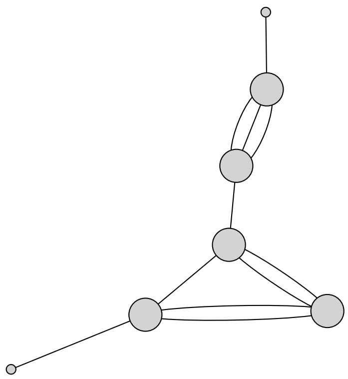

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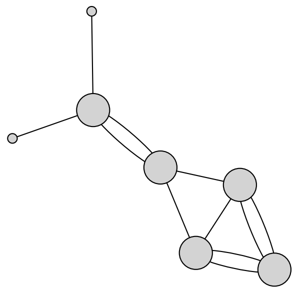

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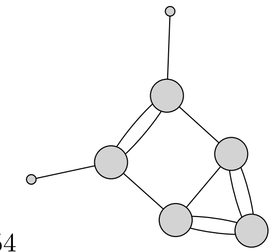

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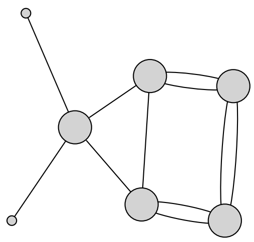

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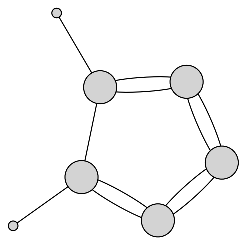

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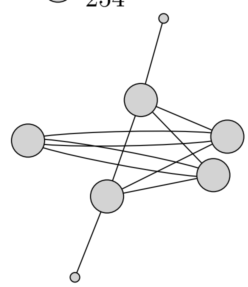

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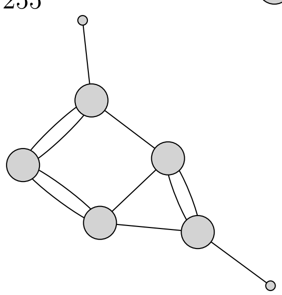

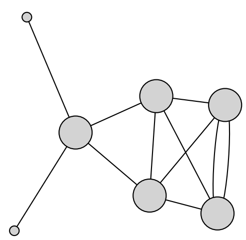

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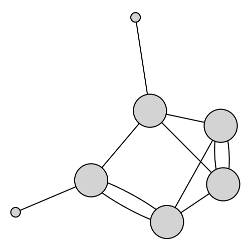

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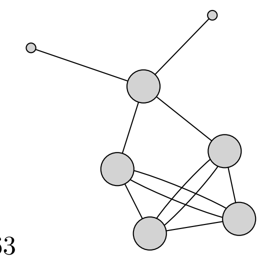

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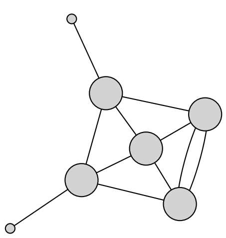

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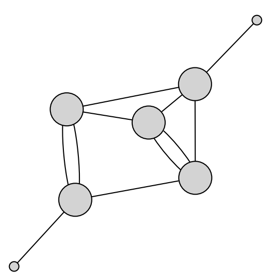

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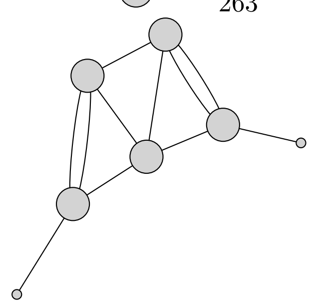

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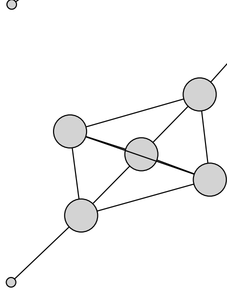

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8. Summary

The generating function (GF) for the 4-regular graphs of any number of components (allowing multi-edges and loops) is [3, A129429]

$$R(x) = 1 + x + 3x^2 + 7x^3 + 20x^4 + 56x^5 + 187x^6 + \cdots \tag{1}$$

Remark 1. *The 1, 3, 7 ... graphs are subgraphs of those graphs in Sections 2–7 where the simple graph on 2 nodes is a component and that component then deleted.*

The GF for the fairly cubic graphs illustrated above is

$$F(x) = 1 + x^2 + 2x^3 + 7x^4 + 21x^5 + 74x^6 + 269x^7 + \cdots \tag{2}$$

The GF for the subset of *connected* fairly cubic graphs illustrated above is [3, A361135]

$$F^{(c)}(x) = 1 + x^2 + x^3 + 3x^4 + 8x^5 + 30x^6 + 118x^7 \cdots \tag{3}$$

Finally

$$F(x) - 1 = R(x)[F^{(c)}(x) - 1] \tag{4}$$

relates the fairly 4-regular graphs to the 4-regular graphs and the connected fairly 4-regular graphs, where the subtraction of 1 means the component with the leaf nodes must not be empty.

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