Least area of triangles enclosing a circle with radius n

1) Examples for $n = 1,2,3,4,5$

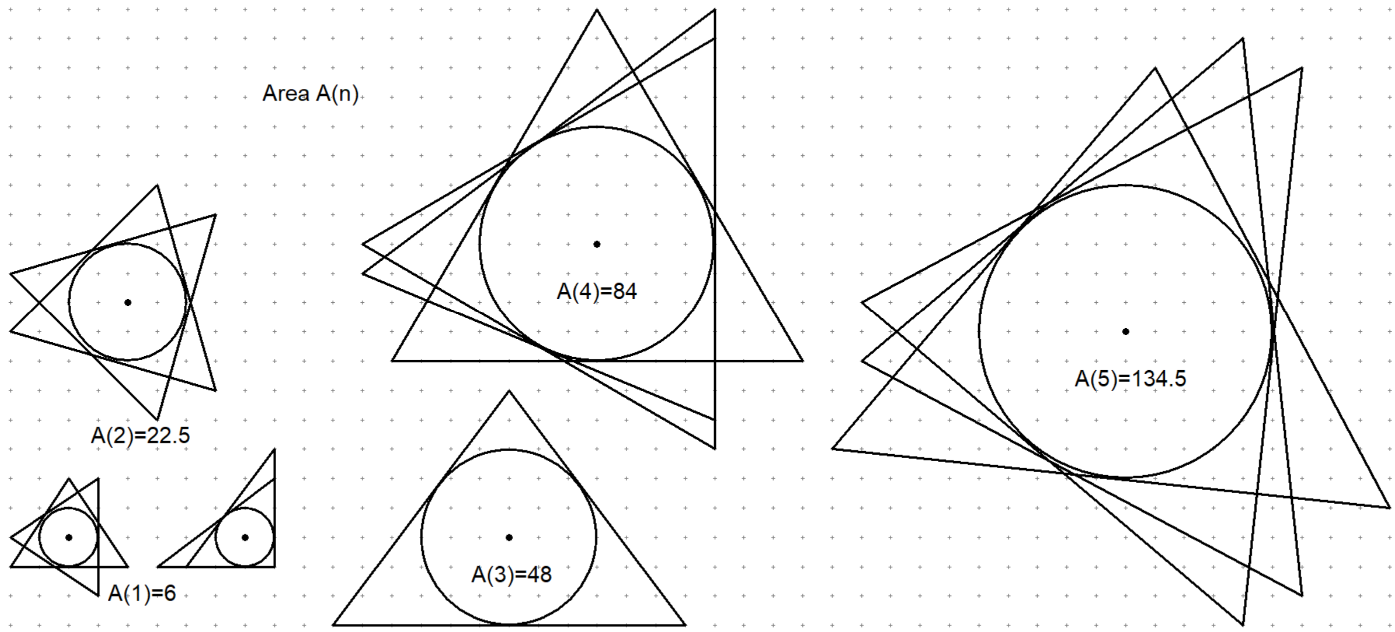

Fig. 1

2) Basic ideas of the algorithm

Definition: A triangle enclosing the circle with integer vertex coordinates is a grid triangle.
with three tangent edges is a tangent triangle.
without further restrictions is a general triangle.

Lemma:
The general triangle with a given vertex and the least area is the symmetrical tangent triangle whose given vertex is located on the symmetry axis. This is evident without proof.

Example n=5 (fig. 2)
$A_{01}B_0C_0$ is an equilateral tangent triangle (height 15 and area $75 \cdot \sqrt{3} \approx 129.9$). This is a lower limit for the least area of a grid triangle.
$A_{01}B_1C_1$ is a symmetrical grid triangle with the area 135.
There are two symmetrical tangent triangles $A_2B_2C_2$ and $A_3B_3C_3$ with the same area 135. Their heights are $h_2 = 8.05$ and $h_3 = 13.00$. The area of any symmetrical tangent triangle with height h and $h_2 < h < h_3$ is less than 135. The vertex on the symmetry axis of the triangle must be located within the figure $A_2D_2D_3A_3$. Because of the symmetry of the grid, the range can be reduced to $x \geq 0$ and $x \leq y$. As a consequence of the lemma, one vertex of the grid triangle with the least area must be located within the figure $A_2D_2D_3A_3$. The algorithm yields $A_4B_4C_4$ as one of the grid triangles with the least area 134.5.

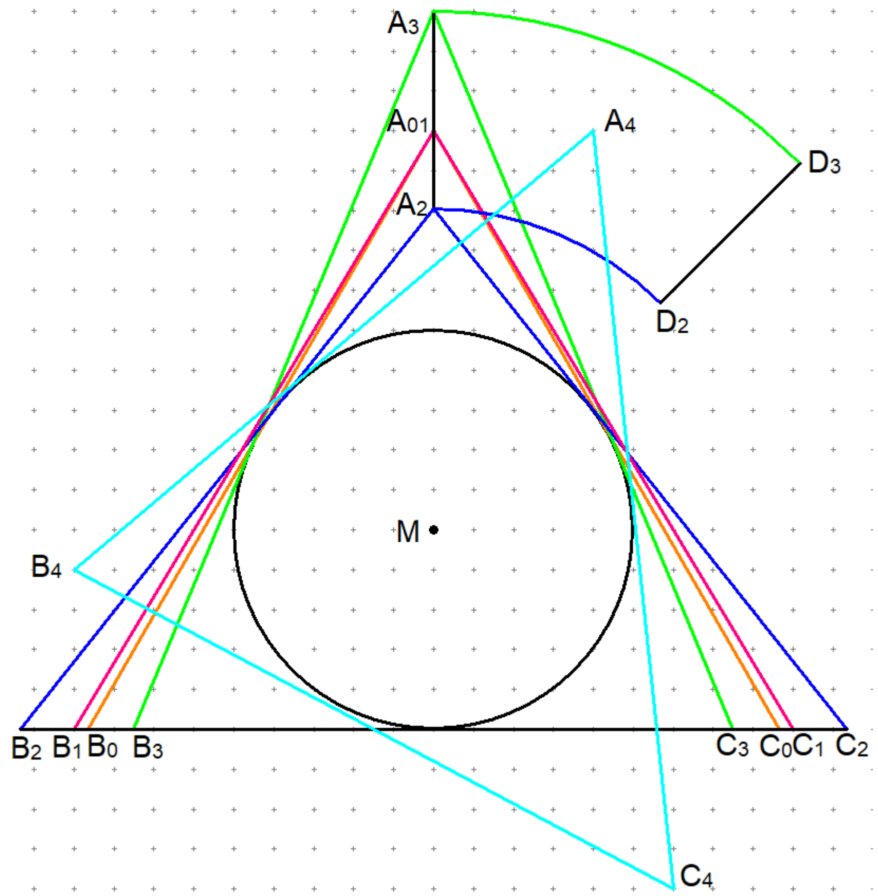

Fig. 2

3) Details of the algorithm for $n = 5$

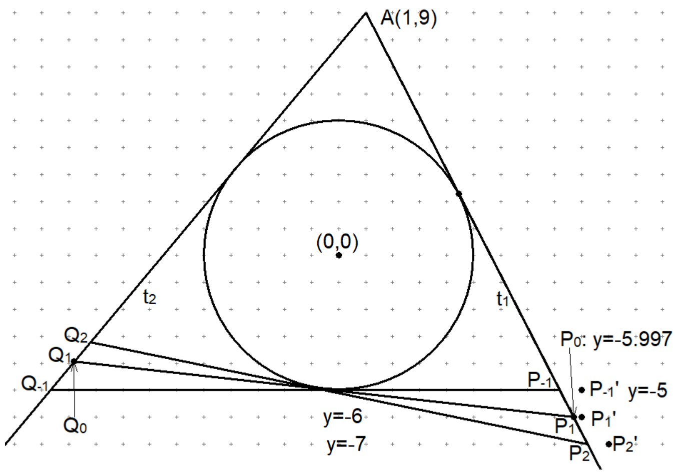

Fig. 3a

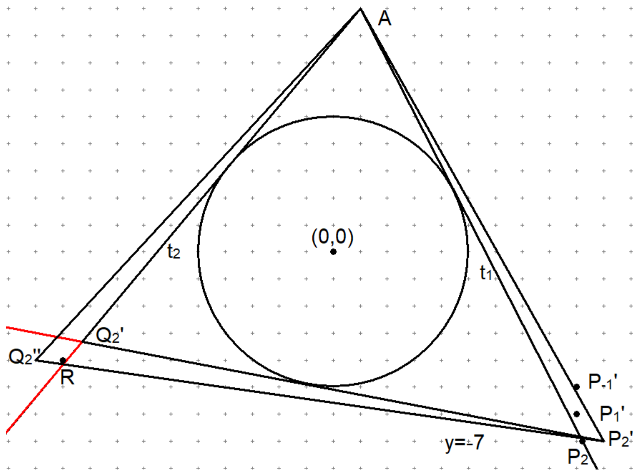

Fig. 3b

a) Select a vertex within $A_2D_2D_3A_3$ in fig. 2, for example $A(1,9)$ in fig. 3a.

b) Generate the symmetrical tangent triangle AP_0Q_0 (clockwise, see annotation below).
Its area is necessarily less than 135.

c) Generate another "neighbored" tangent triangle $AP_kQ_k, k = -1,1,2$ for example, such that y in $P_k(x, y)$ is an integer. Note that the difference between AP_0Q_0 and AP_1Q_1 is extremely small ("by accident").

d) Select $P_k'(x', y)$ with $x' = \lceil x \rceil$ (fig. 3a).

e) The procedure is continued with $P_2'(10, -7)$, for example. Q_2' is the intersection of t_2 and the tangent through P_2' (fig. 3b).

f) Check the areas $|AP_2Q_2| = 131.7$ (fig. 3a) and $|AP_2'Q_2'| = 137.7 > 135$ (fig. 3b). The area $|AP_2'Q_2''|$ of any grid triangle is even greater. Therefore, no minimum grid triangle with vertices A and P_2' exists. So try another point P_k and stop the search (above and below the point P_0) when $|AP_kQ_k| \geq 135$ and $|AP_k'Q_k'| \geq 135$.

g) For demonstrating the algorithm in fig. 3b, let us assume $|AP_2'Q_2'| < 135$. Any point Q_2'' generating a grid triangle AP_2Q_2'' is located within the (red) angle between the tangents through P_2' and A, for example $Q_2''(-11, -4)$. Obviously, $Q_2'' = R(-10, -4)$ would be a better choice generating a smaller area. The general way of finding the best choice is evident and will not be discussed here.

h) By selecting P_1 instead of P_2, the triangle $(1,9)(9, -6)$ $(-10, -4)$ with the least area 134.5 can be generated. In fig. 3b however, this would be a bad demonstration of the algorithm because the difference between the tangent triangle and the grid triangle is very small.

Annotation:
The tangent t_1 is always steep (slope $|m_1| > 1.39$), whereas t_2 has a wide range of slopes $(0.15 < m_2 < 3.2)$. Starting with t_2 instead of t_1 (anticlockwise) would require case distinctions in 4c) such that either x or y is an integer in $P_k(x, y)$.

<u>4) Formulas (used in fig. 3ab)</u>

a) Slope of the tangent t_i: $m_i = \frac{x_0 \cdot y_0 \pm n \cdot \sqrt{x_0^2 + {y_0}^2 - n^2}}{x_0^2 - n^2}, i = 1,2$ for $x_0 \neq n$

For $x_0 = n$: $m_2 = \frac{{y_0}^2 - n^2}{2 \cdot n \cdot y_0}$. m_1 is not defined: Set $P_k' = P_k$ in fig. 3a.

b) $P_2(x, y)$ in fig. 3a: $y = -n \cdot \frac{(r+n)}{y_0}$ for $x_0 = n$

$y = \frac{((x_0 \cdot y_0 - m_1 \cdot (n \cdot r + x_0^2))}{m_1 \cdot y_0 + x_0}$ for $x_0 \neq n$ with $r = \sqrt{x_0^2 + {y_0}^2}$

c) Intersection of t_1 and $y = -k$: $x_1 = x_0 - \frac{y_0 + k}{m_1}$

d) Area of the triangle $(x_0, y_0)(x_1, y_1)(x_2, y_2)$:

$$½ \cdot \big((x_2 - x_0) \cdot (y_1 - y_0) - (y_2 - y_0) \cdot (x_1 - x_0)\big)$$

5) Appendix

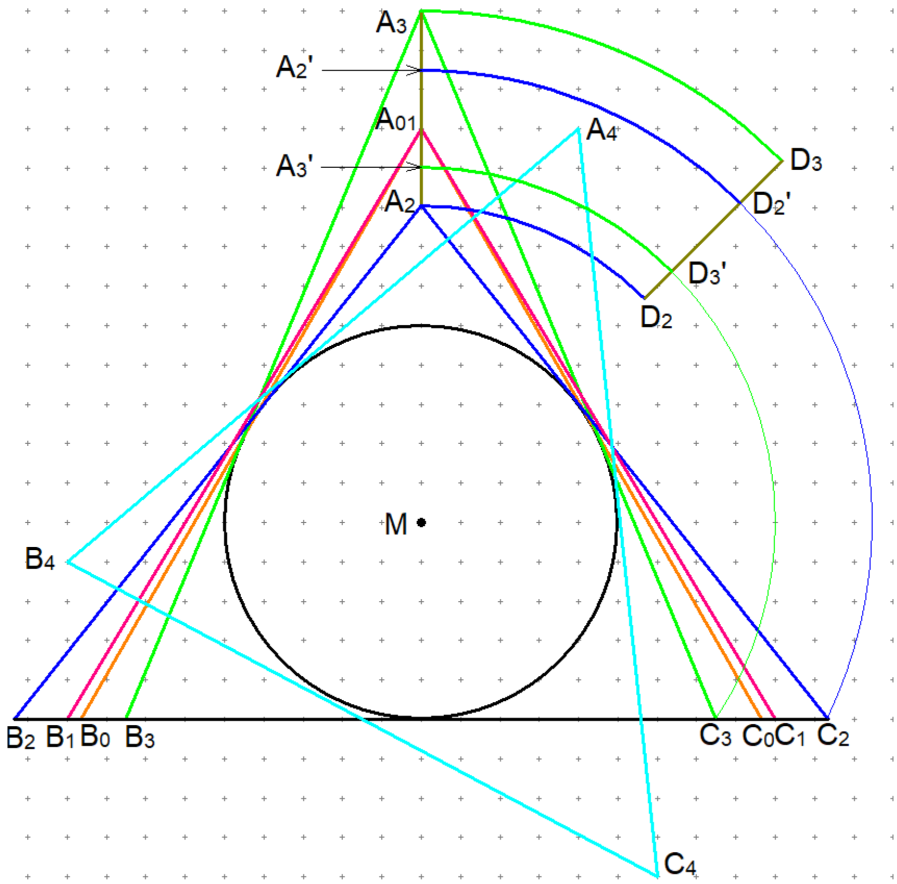

Fig. 4

Fig. 4 is an extension of fig. 2. The circular arcs show $|MC_2| - |MC_3| < |MA_3| - |MA_2|$: The distances of baseline vertices from M vary less than those of vertices on the symmetry axis.

Conjecture: There is a minimum grid triangle with a vertex located in $A_3'D_3'D_2'A_2'$.

The conjecture has been tested for $n \leq 200$.