OEIS A355448

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Abstract. Defining sequence [1, A355448] via the number of of divisors of n^2 and using modulo arithmetics of on the prime factors for the multiplicative $\tau(n^2)$ leads to the Dirichlet Convolution of two elementary characteristics function. This manuscript aims to proof the associated conjectured formula of mid-2022.

1. A355448 by definition

The terms of sequence A355448 are defined via

$$a(n) = \begin{cases} 1, & (6, \tau(n^2)) = 1 \\ 0, & (6, \tau(n^2)) > 1. \end{cases}, \tag{1}$$

where $(.,.)$ is the greatest common divisor. So this is a function which characterizes whether the number of divisors of n^2 is has a prime factor 2 or 3 on one hand or only prime factors $5, 7, 11, 13, \ldots$ on the other. With the simple formula $a(p^e) = 2e + 1$ for the multiplicative sequence

$$A048691(n) = \tau(n^2), \tag{2}$$

and the canonical prime factorization of n,

$$n \equiv p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots \tag{3}$$

we obviously have

$$\tau(n^2) = (2e_1 + 1)(2e_2 + 1)(2e_3 + 1) \cdots \tag{4}$$

So $a(n) = 1$ if that product $\prod_i (2e_1 + 1)$ has no prime factors 2 or 3, zero otherwise. Since all $2e_i + 1$ are odd we may rephrase this:

$a(n) = 1$ if that product $\prod_i (2e_1 + 1)$ has no prime factors 3.

$a(n) = 1$ if all $2e_1 + 1 \in \{1, 2\} \pmod 3$.

$a(n) = 1$ if all $2e_1 \in \{0, 1\} \pmod 3$.

Observing that the sequence $2n \pmod 3$ is the 3-periodic $0, 2, 1, 0, 2, 1 \ldots$ for $n = 0, 1, 2, 3 \ldots$ as in A080425, this may be rephrased

$$a(n) = \begin{cases} 1, & e_i \in \{0, 2\} \pmod 3 \forall i \\ 0, & \exists e_i \equiv 1 \pmod 3 \end{cases} \tag{5}$$

which means $a(n) = 1$ if all e_i are in A007494. $a(n) = 0$ if any e_i is in A016777.

Date: February 7, 2023.

2020 *Mathematics Subject Classification.* Primary 11A05, 11A07.

2. Convolution formula

The characteristic function A010057 of the cubes defines a sequence

$$A010057(n) = c(n) \tag{6}$$

where $c(n) = 1$ if all prime exponents are multiples of 3, which means $3 \mid e_i$ for all i.

The characteristic function A227291 of the squares of squarefree numbers defines a sequence

$$A227291(n) = s(n) \tag{7}$$

where $s(n) = 1$ if all prime exponents are 2, zero otherwise.

The divisors d of n have the canonical prime factorization $p_1^{e_1\downarrow} p_2^{e_2\downarrow} p_3^{e_3\downarrow} \cdots$ where $\downarrow$ indicates that e_i has been replaced by a non-negative number not larger than e_i.

A subset of these divisors $d|n$ will match the criterion $s(d) = 1$ and can be assembled by considering any sub-product of the p_i with exponents $e_i \geq 2$. The divisors will have factors p_i^2 at these places; the complementary divisors n/d will contain the factors $p_i^{e_i-2}$ at these places. The crucial observation is that this reduction to $e_i - 2$ in the exponents of the complementary divisors pushes these affected p_i from the class $\{0, 2\} \pmod 3$ in (5) to the class $\{1, 0\} \pmod 3$ and from the class $\{1\} \pmod 3$ in (5) to the class $\{2\} \pmod 3$.

The sub-cases are

- $a(n) = 0$ because at least one exponent is $1 \pmod 3$. The complementary divisors n/d of divisors in the set $s(d) = 1$ will have $c(n/d) = 0$, because at least one of the reduced exponents will be in $2 \pmod 3$ or $1 \pmod 3$ and n/d cannot be a cube. In detail:
  - If the divisor d affects at least one of the exponents in $1 \pmod 3$, this is pushed to $2 \pmod 3$ in n/d.
  - If the divisor d affects none of the exponents $1 \pmod 3$, this stays in n/d.

  Splitting the divisors into the sets of divisors in A062503, $s(d) = 1$, and in the complementary A000037, $s(d) = 0$, yields

$$\sum_{d|n} c(d/n)s(n) = \sum_{d \in A062503|n} c(d/n)s(d) + \sum_{d \notin A062503|n} c(d/n)s(d) = \sum_{d \in A062503|n} c(d/n) = 0. \tag{8}$$

  So we have shown that the Dirichlet Convolution of A010057 by A227291 gives the correct $a(n)$ for the cases $a(n) = 0$.
- $a(n) = 1$ because all exponents are $\{0, 2\} \pmod 3$. Selecting divisors d where $s(d) = 1$ will push the affected exponents in the complementary divisors n/d from $0 \pmod 3$ to $1 \pmod 3$ (non-cubes) and from $2 \pmod 3$ to $0 \pmod 3$ (cubes). The non-affected exponents stay in the class $\{0, 2\}$

(mod 3).

$$\sum_{d|n} c(d/n)s(n) = \sum_{d\in A062503|n} c(d/n)s(d) + \sum_{d\notin A062503|n} c(d/n)s(d) = \sum_{d\in A062503|n} c(d/n) \tag{9}$$

To obtain non-zero $c(d/n)$ all affected and non-affected exponents need to end up in $0 \pmod 3$. As we are decreasing with $\downarrow$ either by zero (non-affected exponents) or by 2 (affected), and as we are starting from exponents $\{0,2\} \pmod 3$ (because $a(n)=1$), the divisors d *must* collect *all* exponents in the set $2 \pmod 3$, because otherwise these would stay in $2 \pmod 3$ and yield non-cubes d/n. So to obtain non-zero $c(d/n)$ there is exactly one contributing d which is the product of all $p_i^{e_i}$ over $e_i \equiv 2 \pmod 3$. Since this d is unique and leads to a single term (equal to 1) in (9), we have shown that the Dirichlet Convolution of A010057 by A227291 gives the correct $a(n)$ for the cases $a(n)=1$.

Because the Dirichlet Generating Functions of A010057, $\zeta(3s)$ and A227291, $\zeta(2s)/\zeta(4s)$ are known, the Dirichlet Generating Function for $a(n)$ is just their product.

References

1. O. E. I. S. Foundation Inc., *The On-Line Encyclopedia Of Integer Sequences*, (2023), https://oeis.org/. MR 3822822

URL: `http://www.mpia.de/~mathar`

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