ASYMPTOTICS FOR A352178

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Abstract

We prove A352178$(n) = \frac{n}{2}\log_2(n) + \mathcal{O}(n)$.

1. Lower bound

Let $T(n) =$ A352178(n), the maximum over sets of size n of the number of pairs of elements from the set that sum to a power of two. For example, $T(3) = 3$ which is achieved by the set $\mathcal{A} = \{-1, 3, 5\}$.

In order to prove a lower bound for $T(n)$ we look at $\mathcal{A} = \left(-\frac{n}{2}, \frac{n}{2}\right] \cap \mathbb{Z}$. The number of pairs from $\mathcal{A}$ that sum to 1 is $n/2$. The number of pairs from $\mathcal{A}$ that sum to 2 is $n/2 - 1$. In general, for $2^k \leq n$, the number of pairs from $\mathcal{A}$ that sum to 2^k is $n/2 - 2^{k-1}$. For k such that $2^k > n$ there are no pairs that sum to 2^k. It follows that the number of pairs from $\mathcal{A}$ that sum to a power of two is

$$\sum_{0<k\leq\log_2(n)} \left(\frac{n}{2} - 2^{k-1}\right) = \frac{n}{2}\lfloor\log_2(n)\rfloor - \sum_{0<k\leq\log_2(n)} 2^{k-1} = \frac{n}{2}\log_2(n) + \mathcal{O}(n).$$

From the example above, one concludes that

$$T(n) \geq \frac{n}{2}\log_2(n) + \mathcal{O}(n).$$

2. Upper bound

We show $T(n) \leq \frac{n}{2}\log_2(n) + n$ (which shows that our example for the lower bound is asymptotically optimal).

We first show the following useful lemma:

Lemma 2.1. *Let $\mathcal{A} \subset \mathbb{Z}$ be some set of size n. The number of pairs of positive elements from $\mathcal{A}$ that sum to a power of two is at most n.*

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Proof. Denote $\mathcal{A}^+$ the set of positive elements from $\mathcal{A}$. If $\mathcal{A}^+$ is empty the statement is obviously true. Otherwise consider the largest element in $\mathcal{A}^+$. It can be a part of at most one pair from $\mathcal{A}^+$ for which the sum is a power of two. And so, we can remove this largest element and proceed by induction. □

This lemma shows that the number pairs of positive elements that sum to a power of two is going to be negligible. We now turn our attention to pairs that contain a positive and a negative element. We will call such a pair a mixed pair. A mixed pair that sums to a power of two will be called a good mixed pair.

We denote $T^*(n)$ the maximum, over all sets $\mathcal{A} \subset \mathbb{Z}$ of size n, of the number of good mixed pairs from $\mathcal{A}$. From our lemma, we see that $T(n) \leq T^*(n) + n$. And so we just need to prove the following

Lemma 2.2. $T^*(n) \leq \frac{n}{2}\log_2(n)$.

Proof. We prove this by induction. The statement is obviously true for $n = 1$ (since there are no pairs, so $T^*(1) = 0 = \frac{1}{2}\log_2(1)$).

Let $n \geq 2$ and assume that the statement is true for all $k < n$. Let $\mathcal{A} \subset \mathbb{Z}$ be some set of size n. If all elements of $\mathcal{A}$ are even then we can divide them by 2 without changing the number of good mixed pairs. If all elements from $\mathcal{A}$ are odd then we can add 1 to the positive elements, subtract 1 from the negative elements, and then divide all elements by 2. Once more, this does not change the number of good mixed pairs. Repeating this process if necessary, we can assume that our set $\mathcal{A}$ has both even and odd elements.

Denote $\mathcal{A}_e$ the set of even elements from $\mathcal{A}$, and $\mathcal{A}_o$ the set of odd elements from $\mathcal{A}$. Denote $|\mathcal{A}_e| = k, |\mathcal{A}_o| = n - k$, and we know that $0 < k < n$ since $\mathcal{A}$ has both even and odd elements. For every element from $a \in \mathcal{A}_e$, there is at most one $b \in \mathcal{A}_o$ such that (a, b) is a good mixed pair (namely $b = 1 - a$). The same is true for $b \in \mathcal{A}_o$. Such a b has at most one $a \in \mathcal{A}_e$ such that (a, b) is a good mixed pair. If follows that the number of good mixed pairs containing elements from both $\mathcal{A}_o$ and $\mathcal{A}_e$ is at most $\min(k, n - k)$.

The number of good mixed pairs with both elements from $\mathcal{A}_e$ is bounded by $T^*(k)$. Similarly, the number of good mixed pairs with both elements from $\mathcal{A}_o$ is bounded by $T^*(n - k)$.

Denote by G the number of good mixed pairs in $\mathcal{A}$. It follows that

$$G \leq T^*(k) + T^*(n - k) + \min(k, n - k).$$

We now use the induction hypothesis (which is possible since $k, n-k$ are both strictly less than n). We get that

$$G \le \frac{k}{2}\log_2(k) + \frac{n-k}{2}\log_2(n-k) + \min(k, n-k).$$

The last expression is maximized by $k = n/2$. To see this, note that we can assume, without loss of generality, that $1 \le k \le n/2$, and then we get the expression

$$\frac{k}{2}\log_2(k) + \frac{n-k}{2}\log_2(n-k) + k.$$

Looking at this as a function of k, we see that it decreases for $1 \le k \le n/5$ and then increases for $n/5 \le k \le n/2$. Thus, the maximum is attained in one of the end points $k = 1$ or $k = n/2$. Checking both, we find that $k = n/2$ is the maximum. It follows that

$$G \le \frac{n/2}{2}\log_2(n/2) + \frac{n/2}{2}\log_2(n/2) + n/2 = \frac{n}{2}\log_2(n).$$

We have shown that any set $\mathcal{A} \subset \mathbb{Z}$ of size n has at most $\frac{n}{2}\log_2(n)$ good mixed pairs, which proves

$$T^*(n) \le \frac{n}{2}\log_2(n)$$

as required.

□