On The OEIS Sequence A352178

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Abstract

Let S be a set of n distinct integers, and let $f(S)$ be the number of (unordered) pairs from S whose sum is a power of 2. Let $g(n)$ be the maximum of $f(S)$ over all S. $g(n)$ is the sequence A352178 on OEIS. $g(10)$ is known to be either 15 or 16. In this paper we prove that it cannot be 16, thereby proving it is 15. We also prove a few impossibility results for S, and use them to make conclusions about $g(11)$ and $g(12)$.

1 Introduction

Let S be a set of n distinct integers. Its ***associated graph*** G is the graph with nodes labelled by the integers of S, and two nodes are adjacent if their values sum to a power of 2. Let $f(S)$ be the number of edges in G, or equivalently, the number of (unordered) pairs from S whose sum is a power of 2. Let $g(n)$ be the maximum of $f(S)$ over all S. The goal is to determine $g(n)$ for all n, a problem posed by Dan Ullman and Stan Wagon. In this paper we determine $g(10)$, the smallest unknown value in A352178.

It is a result by M. S. Smith (in an email to Neil Sloane) that G cannot contain a 4-cycle. The maximum number of edges in 4-cycle-free (or "square-free") graph with 10 nodes is 16, and there are two such graphs [1], shown below.

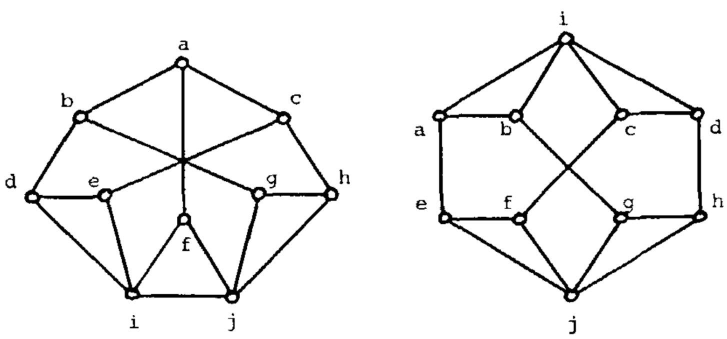


Figure 1: The two maximal square-free graphs on 10 nodes, from [1]. We shall simply call them the left and right graphs.

$g(10)$ is known to be either 15 or 16, and our goal is to prove the impossibility of 16, and therefore it must be 15.
As the two maximal square-free graphs on 10 vertices both contain a few 3-cycles (or "triangles"), we will focus on the properties of these 3-cycles formed by the integers of S.

2 3-Cycles

Let A, B, and C denote three integers in S that add up pairwise to powers of 2, i.e. $A+B=2^x$, $A+C=2^y$, and $B+C=2^z$. As A, B, and C are distinct, then so are x, y, and z. From these 3 equations we obtain

$$A = 2^{x-1} + 2^{y-1} - 2^{z-1}$$
$$B = 2^{x-1} - 2^{y-1} + 2^{z-1}$$
$$C = -2^{x-1} + 2^{y-1} + 2^{z-1}$$

We must have $x, y, z \geq 1$ for A, B, and C to be integers. Without loss of generality, we may assume that $x > y > z$, and hence $A > B > 0 > C$.
Making the substitution $\sigma = z - 1$, $\lambda = x - z$, and $\mu = y - z$, we get

$$A = 2^{\sigma}(2^{\lambda} + 2^{\mu} - 1)$$
$$B = 2^{\sigma}(2^{\lambda} - 2^{\mu} + 1)$$
$$C = 2^{\sigma}(-2^{\lambda} + 2^{\mu} + 1)$$

We shall call these the A-form, B-form, and C-form, respectively.

3 The 3 Forms

If an integer has a representation in one of the above forms (with $\sigma \geq 0$, $\lambda > \mu > 0$), it is not difficult to see that it is unique, via parity arguments.
Therefore if a number is in two triangles (which happens in both maximal graphs), but uses the same form for both triangles, then the other two numbers are the same in both triangles by the uniqueness of σ, λ and μ, which contradicts that the numbers in S are distinct. We also note that a number cannot have a C-form and a different form, due to them having different signs.
The only possibility left to have a number be in two different triangles is for it to be representable in A-form and B-form.
A by-product of the previous discussion is the following proposition:

Proposition 1. *An associated graph cannot have three different triangles sharing a single vertex.*

In each form, we can factor out 2^{μ} and deduce from there the binary representations of these forms as regular expressions: $10^{+}1^{+}0^{*}$ for A-forms, $1^{+}0^{*}10^{*}$ for B-forms, and $-1^{*}01^{+}0^{*}$ for C-forms. Hence we have that for a number to be both representable in A-form and B-form, it must be in the form $10^{+}10^{*}$, which is to say it equals $2^{\sigma}(2^{\alpha}+1)$ for some $\alpha \geq 2$.
Seen as an A-form, we can conclude $\mu = 1$, $\lambda = \alpha$. On the other hand, as a B-form, we conclude $\lambda = \mu + 1 = \alpha + 1$. We will substitute these values in for the triangles.

4 Impossibility of The Left Graph

Let us turn our attention to the bottom middle triangle in Figure 2. The vertices i and j are both in two separate triangles, hence they are of the form $2^{\sigma}(2^{\alpha}+1)$ and $2^{\sigma}(2^{\beta}+1)$. However, as they are adjacent, this implies that their sum is a power of 2, which is impossible, since $\alpha, \beta \geq 2$, and as such the second-to-last bit in $(2^{\alpha}+1)+(2^{\beta}+1)$ is 1 (different from the leading bit).

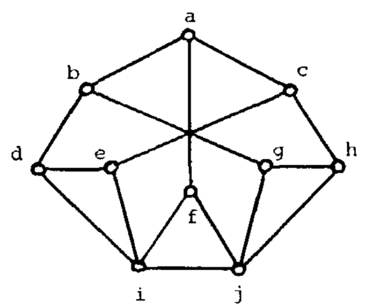


Figure 2: The left graph [1].

From the above, we arrive at the following proposition:

Proposition 2. *If a graph contains three triangles T_1, T_2, and T_3 such that T_1 shares a node with T_2 and a different node with T_3, then it is not the associated graph of a set S.*

5 Impossibility of The Right Graph

For the right graph, we point out the two subgraphs where each has two triangles sharing a node; the subgraph of $\{i, a, b, c, d\}$, and the subgraph of $\{j, e, f, g, h\}$. In such a subgraph, the σ must be the same for all 5 nodes, due to the shared node and the uniqueness of σ.

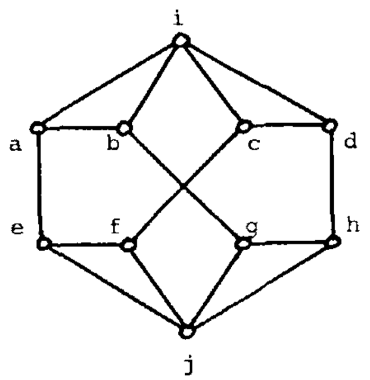
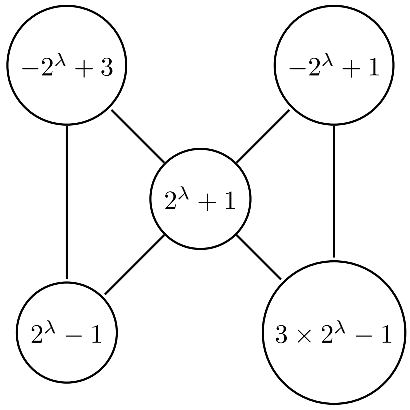
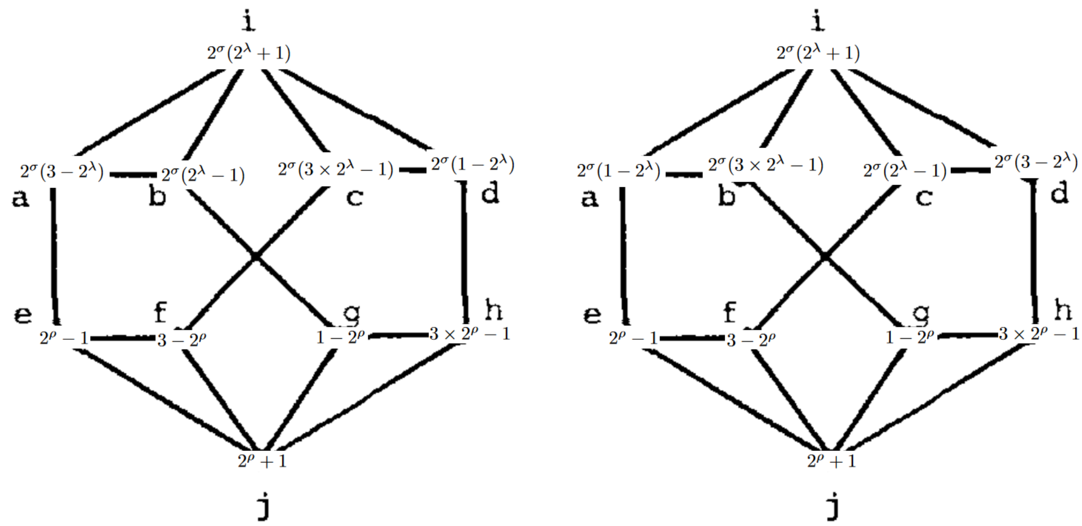


Figure 3: The right graph [1].

Setting aside the 2^σ factors, the below figure shows the values of each of the nodes for the given λ, where the center node is considered in one triangle as an A-form, and considered as a B-form in the other.



The reader can verify that indeed, each adjacent pair adds to a power of 2. Assume that the highest power of 2 dividing the integers in the top half of Figure 3 is 2^σ, and for the bottom half it is 2^τ. Without loss of generality, assume $\sigma \geq \tau$. Then we may divide the integers of the set S by 2^τ, and the numbers that were (un)adjacent will remain (un)adjacent in the new graph. So we may assume that the bottom half has odd numbers.
There is 1 negative number among each of the pairs $\{a, b\}$, $\{c, d\}$, $\{e, f\}$, and $\{g, h\}$. Each of these negative numbers is adjacent to a number from the other half. However, they cannot be adjacent to each other, as the sum of two negatives cannot be a power of 2. So they can only be adjacent to positive numbers. From here, we have only two possible configurations, based on which negative connects to which positive, shown below



We will split into two cases: $\sigma = 0$, and $\sigma > 0$. The former case essentially has only one configuration, as the two configurations only differ by swapping λ and ρ. The latter case can only have 1 as a possible sum for even (top half) and odd (bottom half) numbers. We remind the reader that $\lambda, \rho \geq 2$, which we'll use in the coming subsections.

5.1 Case: $\sigma = 0$

As $2^\lambda - 1$ and $1 - 2^\rho$ are adjacent, this implies $2^\lambda - 2^\rho$ is a power of 2, so the only possibility is $\lambda = \rho + 1$. But $-2^\lambda + 3$ and $2^\rho - 1$ are adjacent. Summing them and substituting $\lambda = \rho + 1$, we get $2 - 2^\rho$, necessarily a power of 2. But this implies $\rho = 0$, which is false, as $\rho \geq 2$.

5.2 Case: $\sigma > 0$

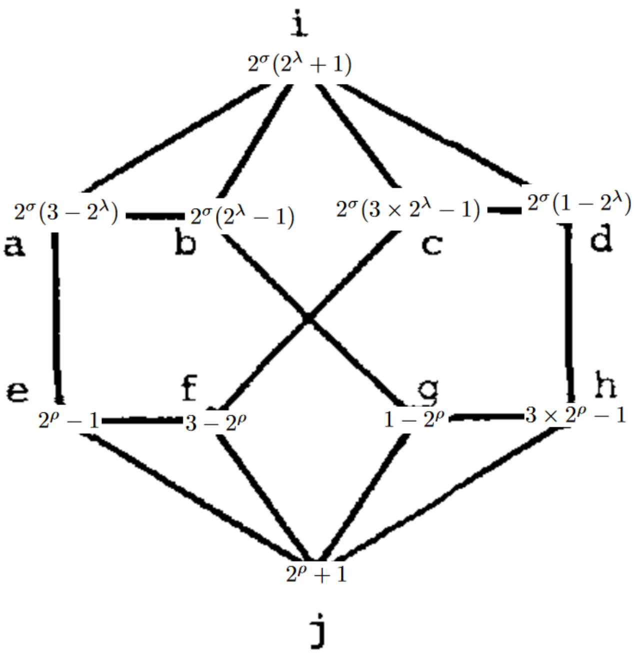
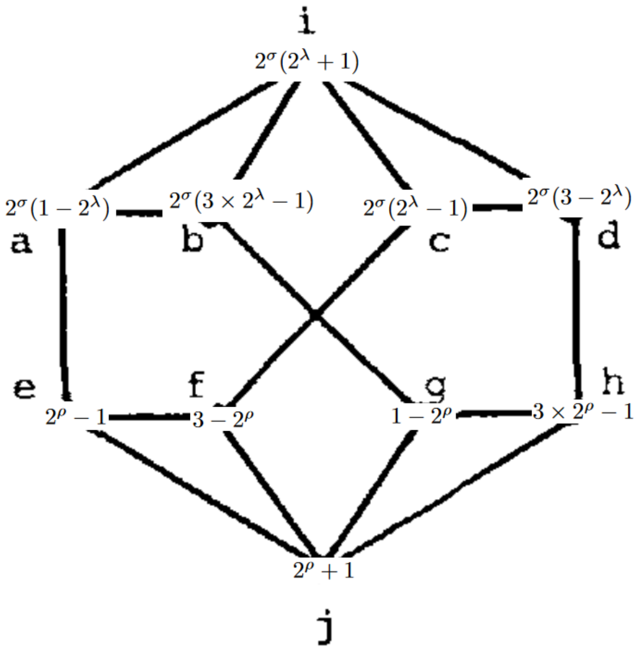


Figure 4: The first configuration.

Nodes b and g are adjacent in this configuration, therefore:

$$2^\sigma(2^\lambda - 1) + (1 - 2^\rho) = 1 \implies 2^\sigma(2^\lambda - 1) = 2^\rho \implies 2^\lambda - 1 = 1 \implies \lambda = 1$$

a contradiction, as $\lambda \geq 2$.



Nodes b and g are adjacent in this configuration, therefore:

$$2^\sigma(3\times2^\lambda-1)+(1-2^\rho)=1 \implies 2^\sigma(3\times2^\lambda-1)=2^\rho \implies 3\times2^\lambda-1=1 \implies 3\times2^\lambda=2$$

i.e. 3 divides 2, a contradiction.

6 Conclusion

There were only two possible square-free graphs on 10 nodes with 16 edges, and we've shown that they are both impossible to have as associated graphs. And 15 edges is possible, using $\{-7,-5,-3,-1,3,5,7,9,11,23\}$. Therefore $g(10)=15$. We've also shown some impossibility results for associated graphs. These can be used on the four 11-node graphs in [1], though we could not find a source with the other square-free seven graphs to study them. We can also use the results on the three 12-node graphs in [1], proving $g(12)<21$, and so it is either 19 or 20, as per the bounds in the OEIS page of A352178.

References

[1] C. R. J. Clapham; A. Flockhart; J. Sheehan. “Graphs without four-cycles”. In: *Journal of Graph Theory* 13.1 (1989), pp. 29–47.