A continued fraction representation for A346441

Peter Bala, Feb 2024

A346441 is defined to be the real constant

Sum_{k = 0..n} (-1)^k/(3*k)! = 0.83471 94685 77210 ….

Bowman and Mc Laughlin [Corollary 10, p. 341] give the following generalized continued fraction representation

1/(1 + 1/(5 + 6/(119 + ... + (3*n-3)*(3*n-4)*(3*n-5)/
(((3*n)*(3*n-1)*(3*n-2) - 1) + ...)))))

for this constant as the particular case of a more general result. We sketch a quick proof of this result.

For n >= 0, define an integer sequence by

a(n) = (3*n)! * Sum_{k = 0..n} (-1)^k/(3*k)!.

The first few terms of the sequence are [1, 5, 601, 302903, 399831961,...].
It is straightforward to show that a(n) satisfies the first-order recurrence

a(n) = (3*n)(3*n-1)*(3*n-2)*a(n-1) + (-1)^n

leading easily to the second-order recurrence

a(n) = ((3*n)*(3*n-1)*(3*n-2) - 1)*a(n-1) + (3*n-3)*(3*n-4)*(3*n-5)*a(n-2)

with the initial conditions a(0) = 1 and a(1) = 5.

One checks that b(n) := (3*n)! also satisfies the same second-order recurrence with the initial conditions b(0) = 1 and b(1) = 6.

Then by the fundamental recurrence formulas for the partial numerators and denominators of a generalized continued fraction [Wikipedia] we have the finite continued fraction representation

a(n)/b(n) = Sum_{k = 0..n} (-1)^k/(3*k)!

= 1/(1 + 1/(5 + ... + (3*n-3)*(3*n-4)*(3*n-5)/(((3*n)*(3*n-1)*(3*n-2) – 1)))))).

Letting n -> oo gives Bowman and Mc Laughlin's continued fraction for the constant A346441.

The same method can be used to find continued fraction representations for other constants, for instance, Lord Brouncker's continued fraction

4/Pi = 1 + 1^2/(2 + 3^2/(2 + 5^2/(2 + ...))).

See A024199.

For other examples see A051396, A051397, A074790, A143819, A143820, A143821, A275651, A332890, A334378, A334379 and A346440.

References

Bowman and J. Mc Laughlin, Polynomial continued fractions, arXiv:1812.08251 [math.NT], Acta Arith. 103 (2002), no. 4, 329-342.

Wikipedia, Generalized continued fraction