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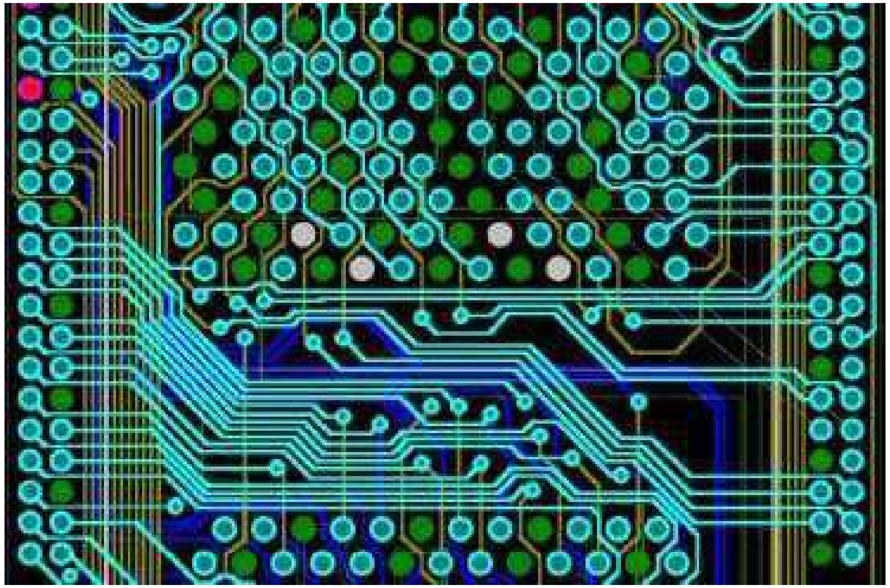
Counting i-paths



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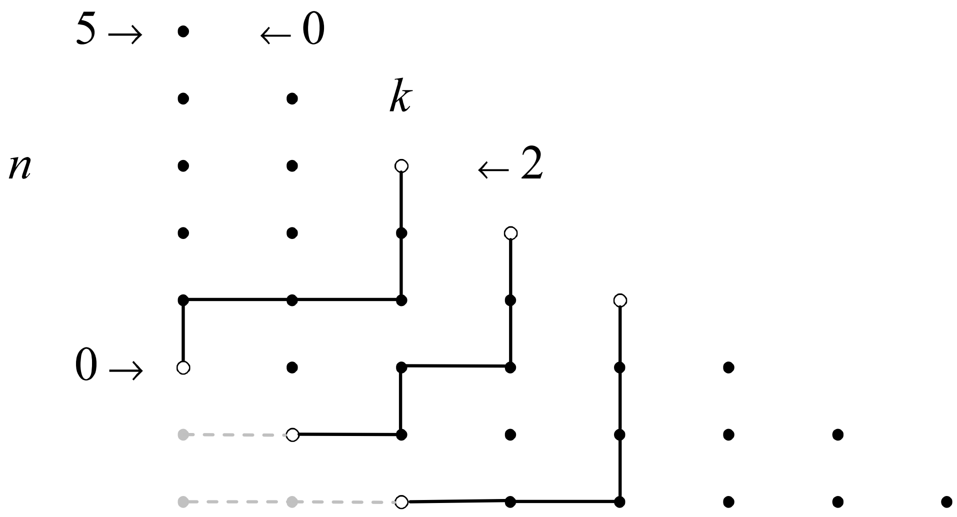
Abstract: https://www.mathstat.dal.ca/fibonacci/abstracts.pdf



One of the 3-paths counted in $p(\{0,1,2\},5,2)$

n indexes diagonals and k indexes columns

(beginning at points on the LH vertical gives the same number of 3-paths)

Notation

[i.e. monotonic non-decreasing]

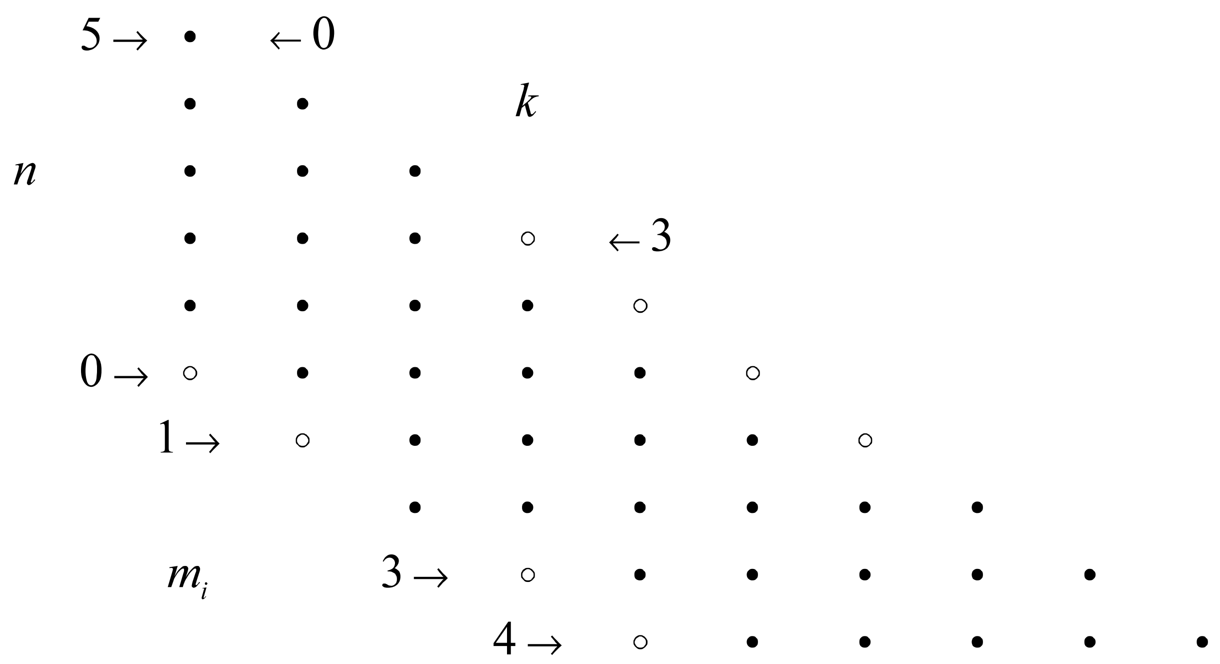
i-path: i non-intersecting paths on a lattice using only vertical and horizontal moves, beginning (LHS) on i points on a descending diagonal and ending (RHS) on i consecutive points on a descending diagonal

$\mathbf{m}_i = \{0, m_1, m_2, ..., m_{i-1}\}$: positions on 0^{th} descending diagonal of initial points of an i-path

$p(\mathbf{m}_i, n, k)$: number of i-paths of length $n \geq 0$ from $\mathbf{m}_i$ to i consecutive points on the n^{th} descending diagonal beginning at column k ($0 \leq k \leq n$)

$a(\mathbf{m}_i, n)$: total number of i-paths of length $n \geq 0$ from $\mathbf{m}_i$ to i consecutive points on the n^{th} descending diagonal

$$a(\mathbf{m}_i, n) = \sum_{k=0}^{n} p(\mathbf{m}_i, n, k)$$



Start and end points (∘) for the 4-paths counted by $p(\{0,1,3,4\},5,3)$

1-paths

Arrows show movement of initial LH point

•1					
•1	•5				
•1	•4	•10			
•1	•3	•6	•10		
•1	•2	•3	•4	•5	
○1	•1	•1	•1	•1	•1

$p(\{0\},n,k)$

$$p(\{0\},n,k) \quad = \quad \overset{\uparrow}{p(\{0\},n-1,k)} \quad + \quad \overset{\rightarrow}{p(\{0\},n-1,k-1)}$$

$$= \binom{n}{k}$$

$$\begin{aligned} a(\{0\},n) &= a(\{0\},n-1) + a(\{0\},n-1) \\ &= 2a(\{0\},n-1) \\ &= \{1,2,4,8,16,32,...\} \end{aligned}$$

2-paths

•1						
•1	•15					
•1	•10	•50				
•1	•6	•20	•50			
•1	•3	•6	•10	•15		
○1	•1	•1	•1	•1	•1	
	○	•	•	•	•	•

$p(\{0,1\},n,k)$

$$p(\{0,m_1\},n,k) = \overset{\uparrow\uparrow}{p(\{0,m_1\},n-1,k)} + \overset{\rightarrow\uparrow}{p(\{0,m_1-1\},n-1,k-1)}$$
$$+ \overset{\uparrow\rightarrow}{p(\{0,m_1+1\},n-1,k)} + \overset{\rightarrow\rightarrow}{p(\{0,m_1\},n-1,k-1)}$$

$$p(\{0,1\},n,k) = p(\{0,1\},n-1,k)$$
$$+p(\{0,2\},n-1,k)+p(\{0,1\},n-1,k-1)$$

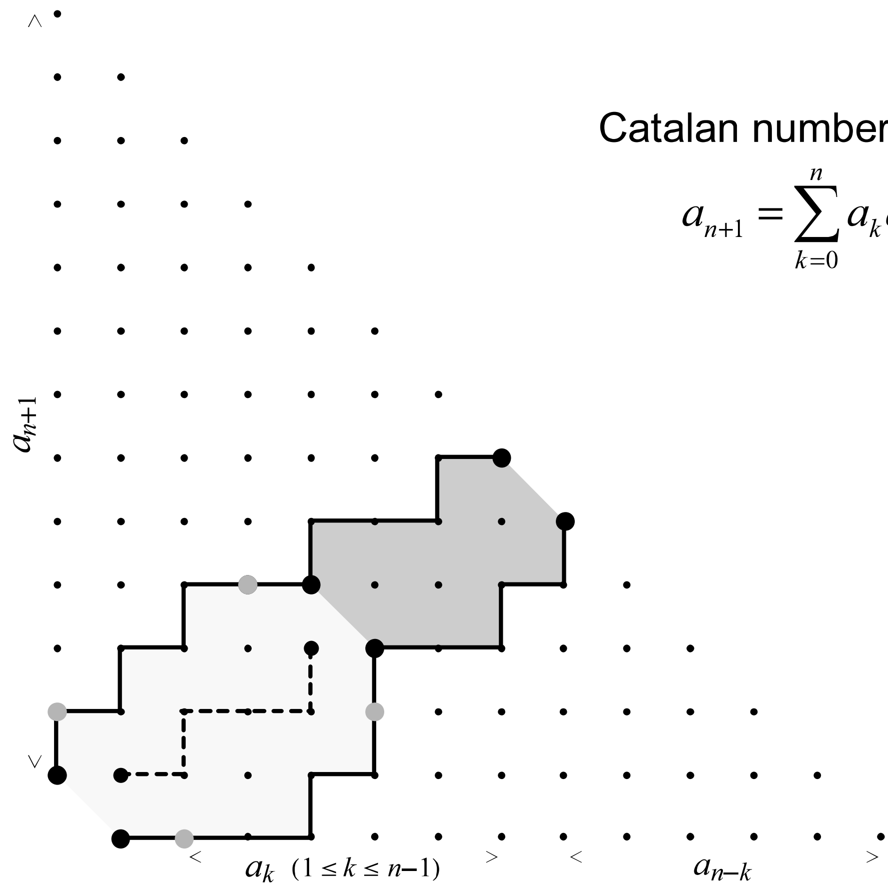
$$a(\{0,m_1\},n)=2a(\{0,m_1\},n-1)$$
$$+a(\{0,m_1+1\},n-1)+a(\{0,m_1-1\},n-1)$$

$$a(\{0,1\},n)=2a(\{0,1\},n-1)+a(\{0,2\},n-1)$$
$$=\{1,2,5,14,42,132,...\}$$

(effectively given by Shapiro 1976 and shown to be Catalan)

Catalan numbers satisfy

$$a_{n+1} = \sum_{k=0}^{n} a_k a_{n-k}$$



$$a(\{0,1\},n) = 2a(\{0,1\},n-1) + a(\{0,2\},n-1)$$
$$= \{1,2,5,14,42,132,...\} \quad (\text{Catalan, A000108})$$

$$a(\{0,2\},n) = 2a(\{0,2\},n-1) + a(\{0,1\},n-1) + a(\{0,3\},n-1)$$
$$= \{0,1,4,14,48,165,572,...\} \quad (4^{\text{th}} \text{ convolution of Catalan, A002057})$$

$$a(\{0,3\},n) = \{0,0,1, 6, 27, 110, 429, 1638,...\} \quad (6^{\text{th}} \text{ convolution of Catalan, A003517})$$

$$a(\{0,4\},n) = \{0, 0, 0, 1, 8, 44, 208, 910,...\} \quad (8^{\text{th}} \text{ convolution of Catalan, A003518})$$

(note the $n = 4$ terms of the sequences above)

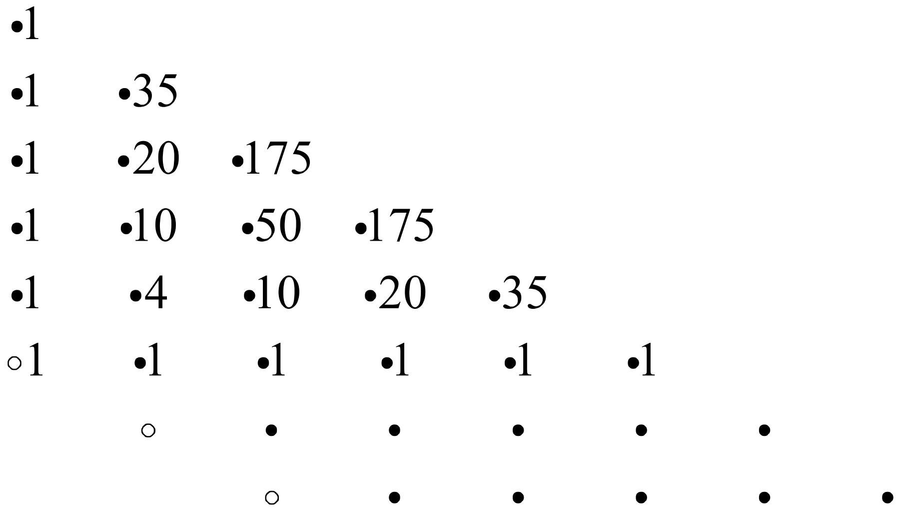
$$a(\{0,m_1\},n)=a(\{0,m_1-1\},n-1)+2a(\{0,m_1\},n-1)+a(\{0,m_1+1\},n-1)$$

$$\begin{pmatrix} a(\{0,1\},n) \\ a(\{0,2\},n) \\ \\ \vdots \\ \\ a(\{0,n-1\},n) \\ a(\{0,n\},n) \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & & \cdots & & 0 \\ 1 & 2 & 1 & 0 & & & 0 \\ 0 & 1 & 2 & 1 & 0 & & 0 \\ & & \cdots & & & & \vdots \\ \vdots & & 0 & 1 & 2 & 1 & 0 \\ & & & 0 & 1 & 2 & 1 \\ 0 & & \cdots & & 0 & 1 & 2 \end{pmatrix}_{n\times n} \begin{pmatrix} a(\{0,1\},n-1) \\ a(\{0,2\},n-1) \\ \\ \vdots \\ \\ a(\{0,n-1\},n-1) \\ a(\{0,n\},n-1)=1 \end{pmatrix}$$

$$\begin{pmatrix} a(\{0,1\},4) \\ a(\{0,2\},4) \\ a(\{0,3\},4) \\ a(\{0,4\},4) \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 42 \\ 48 \\ 27 \\ 8 \end{pmatrix}$$

Similar recurrence and matrix relationships can be developed for higher order paths

3-paths



$p(\{0,1,2\},n,k)$

$$\begin{array}{ccccc}
 & & \uparrow\uparrow\uparrow & & \rightarrow\uparrow\uparrow \\
p(\{0,m_1,m_2\},n,k) & = & p(\{0,m_1,m_2\},n-1,k) & + & p(\{0,m_1-1,m_2-1\},n-1,k-1) \\
 & & \uparrow\rightarrow\uparrow & & \rightarrow\rightarrow\uparrow \\
 & + & p(\{0,m_1+1,m_2\},n-1,k) & + & p(\{0,m_1,m_2-1\},n-1,k-1) \\
 & & \uparrow\uparrow\rightarrow & & \rightarrow\uparrow\rightarrow \\
 & + & p(\{0,m_1,m_2+1\},n-1,k) & + & p(\{0,m_1-1,m_2\},n-1,k-1) \\
 & & \uparrow\rightarrow\rightarrow & & \rightarrow\rightarrow\rightarrow \\
 & + & p(\{0,m_1+1,m_2+1\},n-1,k) & + & p(\{0,m_1,m_2\},n-1,k-1)
\end{array}$$

$$p(\{0,1,2\},n,k) = p(\{0,1,2\},n-1,k) + p(\{0,1,3\},n-1,k) + p(\{0,2,3\},n-1,k) + p(\{0,1,2\},n-1,k-1)$$

$$a(\{0,1,2\},n) = 2a(\{0,1,2\},n-1) + a(\{0,1,3\},n-1) + a(\{0,2,3\},n-1) = \{1,2,6,22,92,422,...\}$$

(Dulucq & Guibert 1998 established the correspondence between 3-paths and Baxter numbers A001181 by use of two bijections)

Consider the rotated triangles for paths beginning on consecutive points (n, k now count down and across respectively)

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
1	5	10	10	5	1

$$\binom{n}{k}_1$$

1					
1	1				
1	3	1			
1	6	6	1		
1	10	20	10	1	
1	15	50	50	15	1

$$\binom{n}{k}_2$$

1					
1	1				
1	4	1			
1	10	10	1		
1	20	50	20	1	
1	35	175	175	35	1

$$\binom{n}{k}_3$$

These appear in Fielder & Alford 1988 as triangles derived from successive columns of Pascal's triangles. They name the row sums of higher order triangles Hoggatt sequences (A005362+), the connection with i-paths does not appear to have been known.

Gessel & Viennot 1985 (also Benjamin & Cameron 2006) give

$$\binom{n}{k}_i = \begin{vmatrix} \binom{n}{k} & \binom{n}{k+1} & \cdots \\ \binom{n+1}{k} & \ddots & \\ \vdots & & \binom{n+i-1}{k+i+1} \end{vmatrix}_{i\times i}$$

$$= \prod_{j=1}^{k} \frac{\binom{n+i-j}{i}}{\binom{j+i-1}{i}}$$

This product generalizes the binomial coefficient formula as follows:

$$\binom{n}{k} = \frac{n\times(n-1)\times...\times(n-k+1)}{k\times(k-1)\times...\times 1} = \frac{\binom{n+1-1}{1}\binom{n+1-2}{1}...\binom{n+1-k}{1}}{\binom{k}{1}\binom{k-1}{1}...\binom{1}{1}} = \binom{n}{k}_1$$

$$\binom{n}{k}_i = \prod_{j=1}^{k}\frac{\binom{n+i-j}{i}}{\binom{j+i-1}{i}} = \frac{\binom{n+i-1}{i}\binom{n+i-2}{i}...\binom{n+i-k}{i}}{\binom{k+i-1}{i}\binom{k+i-2}{i}...\binom{i}{i}}$$

As a short hand, $\binom{n}{k}_i = \frac{{}_k(n)^i}{{}_k(k)^i}$ where ${}_k(n)^i$ is the product of terms "n: k left, i up" and ${}_0(n)^i \stackrel{\Delta}{=} 1$

and which also neatly generalizes the Pochhammer symbol $(n)_i = n(n+1)...(n+i-1) = {}_1(n)^i$

For example, $\binom{n}{k}_i = \frac{{}_k(n)^i}{{}_k(k)^i}$ gives

$$\binom{5}{3} = \binom{5}{3}_1 = \frac{{}_3(5)^1}{{}_3(3)^1} = \frac{\begin{pmatrix}3 & 4 & 5\end{pmatrix}}{\begin{pmatrix}1 & 2 & 3\end{pmatrix}} = 10$$

$$\binom{5}{4}_3 = \frac{{}_4(5)^3}{{}_4(4)^3} = \frac{\begin{pmatrix}4 & 5 & 6 & 7\\ 3 & 4 & 5 & 6\\ 2 & 3 & 4 & 5\end{pmatrix}}{\begin{pmatrix}3 & 4 & 5 & 6\\ 2 & 3 & 4 & 5\\ 1 & 2 & 3 & 4\end{pmatrix}} = 35$$ and $$\binom{5}{2}_4 = \frac{{}_2(5)^4}{{}_2(2)^4} = \frac{\begin{pmatrix}7 & 8\\ 6 & 7\\ 5 & 6\\ 4 & 5\end{pmatrix}}{\begin{pmatrix}4 & 5\\ 3 & 4\\ 2 & 3\\ 1 & 2\end{pmatrix}} = 490$$

The product

$$\binom{n}{k}_i = \prod_{j=1}^{k} \frac{\binom{n+i-j}{i}}{\binom{j+i-1}{i}}$$

also gives

$$\binom{n+1}{k}_i = \prod_{j=1}^{k} \frac{\binom{n+1+i-j}{i}}{\binom{j+i-1}{i}} = \frac{C_i^{n+i}}{C_i^{n+i-k}} \binom{n}{k}_i \quad \text{and} \quad \binom{n}{k+1}_i = \prod_{j=1}^{k+1} \frac{\binom{n+i-j}{i}}{\binom{j+i-1}{i}} = \frac{C_i^{n+i-k-1}}{C_i^{k+i}} \binom{n}{k}_i$$

which allow transitions in the n and k dimensions of the triangles.

To find how to move in the i direction we first consider row sums of the triangles:

i	**Row sum** $a(\{0,1,2,...,i-1\},n)$	**Sequence**	**Mathematica** HypergeometricPFQ[**a**,**b**,z]
1	1,2,4,8,16,32,64,…	2^n	[{-n}, {}, -1]
2	1,2,5,14,42,132,429,…	Catalan A000108	[{-1-n, -n}, {2}, 1]
3	1,2,6,22,92,422,2074,…	Baxter A001181	[{-2-n, -1-n, -n}, {2, 3}, -1]
4	1,2,7,32,177,1122,7898,…	Hoggatt A005362	[{-3-n, -2-n, -1-n, -n}, {2, 3, 4}, 1]
5	1,2,8,44,310,2606,25202,…	Hoggatt A005363	[{-4-n, -3-n, -2-n, -1-n, -n}, {2, 3, 4, 5}, -1]
6	1,2,9,58,506,5462,70266,…	Hoggatt A005364	[{-5-n, -4-n, -3-n, -2-n, -1-n, -n}, {2, 3, 4, 5, 6}, 1]

HypergeometricPFQ[**a**,**b**,z] is: $${}_pF_q(\mathbf{a};\mathbf{b};z)=\sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k}{(b_1)_k...(b_q)_k}\frac{z^k}{k!}$$

The hypergeometric versions of the i-path sums become

$$a(\{0,1,2,...,i-1\},n)=\sum_{k=0}^{n}\frac{(-n)_k...(-n-i+1)_k}{(1)_k...(i)_k}(-1)^{ik}=\sum_{k=0}^{n}\frac{C_k^nC_k^{n+1}...C_k^{n+i-1}}{C_0^kC_1^{k+1}...C_{i-1}^{k+i-1}}$$

and

$$\begin{pmatrix} n \\ k \end{pmatrix}_i=\frac{C_k^nC_k^{n+1}...C_k^{n+i-1}}{C_0^kC_1^{k+1}...C_{i-1}^{k+i-1}}=\prod_{j=0}^{i-1}\frac{C_k^{n+j}}{C_j^{k+j}}$$

so

$$\begin{pmatrix} n \\ k \end{pmatrix}_{i+1}=\frac{C_k^{n+i}}{C_i^{k+i}}\begin{pmatrix} n \\ k \end{pmatrix}_i$$

which thus allows transition in the i-dimension.

Combining the three transition formulae

$$\binom{n+1}{k}_i = \frac{C_i^{n+i}}{C_i^{n+i-k}} \binom{n}{k}_i \qquad \binom{n}{k+1}_i = \frac{C_i^{n+i-k-1}}{C_i^{k+i}} \binom{n}{k}_i \qquad \binom{n}{k}_{i+1} = \frac{C_k^{n+i}}{C_i^{k+i}} \binom{n}{k}_i$$

gives the form

$$C_i^{n+i} C_k^{n+i} \binom{n}{k}_i = C_i^{k+i} C_k^{n+i} \binom{n+1}{k+1}_i = C_i^{n+i} C_k^{k+i} \binom{n}{k}_{i+1}$$

Further research:

Combinatorial interpretations of these formulae in terms of the i-paths, diagonal sums, generating functions, development of the matrix approach…