A Short Proof of the Generalized Formula

Werner Schulte

November, 15 2020

There is a family of triangles $T(m; n, k) = 1 + m \cdot k \cdot (n - k)$ for $0 \leqq k \leqq n$ and some fixed integer m. The corresponding matrices inverse $M = T^{-1}$ are given by:
$M(m; n, n) = 1$ for $n \geqq 0$, and $M(m; n, n-1) = m \cdot (1 - n) - 1$ for $n > 0$, and

$$M(m; n, k) = (-1)^{n-k} \cdot (m \cdot k \cdot (k+1)+1) \cdot m \cdot \prod_{i=k+1}^{n-2} (m \cdot (i+1) - 1) \quad \text{for} \quad 0 \leqq k \leqq n-2.$$

Proof:

(1) Because of $\det(T(m; n, k)) = 1 \neq 0$ the matrix inverse $M = T^{-1}$ exists.

(2) Matrix product: $T * M = \sum_{r=k}^{n} T(m;n,r) \cdot M(m;r,k)$, because T and M are lower triangular matrices.

Case k = n: Entries of main diagonal.

$$\sum_{r=n}^{n} T(m;n,r) \cdot M(m;r,n) = T(m; n, n) \cdot M(m; n, n) = 1.$$

Case k = n-1: Entries of first subdiagonal.

$$\sum_{r=n-1}^{n} T(m;n,r) \cdot M(m;r,n-1) = T(m; n, n-1) \cdot M(m; n-1, n-1) + T(m; n, n) \cdot M(m; n, n-1)$$

$$= (1 + m \cdot (n-1)) \cdot 1 + 1 \cdot (m \cdot (1-n) - 1) = 0.$$

Case $k \leqq n-2$: Entries of lower subdiagonals.

$$0 = \sum_{r=k}^{n} (1 + m \cdot r \cdot (n-r)) \cdot M(m;r,k)$$

$$= (1 + m \cdot k \cdot (n-k)) \cdot 1 + (1 + m \cdot (k+1) \cdot (n-k-1)) \cdot (-m \cdot k - 1) +$$

$$\sum_{r=k+2}^{n} (1 + m \cdot r \cdot (n-r)) \cdot (-1)^{r-k} \cdot (m \cdot k \cdot (k+1)+1) \cdot m \cdot \prod_{i=k+1}^{r-2} (m \cdot (i+1) - 1)$$

$$= -m \cdot (n-k-1) \cdot (m \cdot k \cdot (k+1) + 1) +$$

$$\sum_{r=k+2}^{n} (1 + m \cdot r \cdot (n-r)) \cdot (-1)^{r-k} \cdot (m \cdot k \cdot (k+1)+1) \cdot m \cdot \prod_{i=k+1}^{r-2} (m \cdot (i+1) - 1).$$

For integers k, m holds: m · k · (k+1) + 1 ≠ 0. So we have an equivalent equation:

$$m\cdot(n-k-1)=\sum_{r=k+2}^{n}(-1)^{r-k}\cdot(m+m^2\cdot r\cdot n-m^2\cdot r^2)\cdot\prod_{i=k+1}^{r-2}(m\cdot(i+1)-1)$$

$$=\sum_{r=k+2}^{n}(-1)^{r-k}\cdot((m\cdot n+m-1)+(m\cdot r-1)\cdot((m\cdot n+m-2)-(m\cdot(r+1)-1)))\cdot\prod_{i=k+1}^{r-2}(m\cdot(i+1)-1)$$

$$=(m\cdot n+m-1)\cdot\sum_{r=k+2}^{n}(-1)^{r-k}\cdot\prod_{i=k+1}^{r-2}(m\cdot(i+1)-1)$$

$$+(m\cdot n+m-2)\cdot\sum_{r=k+2}^{n}(-1)^{r-k}\cdot\prod_{i=k+1}^{r-1}(m\cdot(i+1)-1)-\sum_{r=k+2}^{n}(-1)^{r-k}\cdot\prod_{i=k+1}^{r}(m\cdot(i+1)-1)$$

$$=(m\cdot n+m-1)\cdot\sum_{r=k}^{n-2}(-1)^{r-k}\cdot\prod_{i=k+1}^{r}(m\cdot(i+1)-1)$$

$$-((m\cdot n+m-1)-1)\cdot\sum_{r=k+1}^{n-1}(-1)^{r-k}\cdot\prod_{i=k+1}^{r}(m\cdot(i+1)-1)-\sum_{r=k+2}^{n}(-1)^{r-k}\cdot\prod_{i=k+1}^{r}(m\cdot(i+1)-1)$$

$$=(m\cdot n+m-1)\cdot\left(\sum_{r=k}^{n-2}(-1)^{r-k}\cdot\prod_{i=k+1}^{r}(m\cdot(i+1)-1)-\sum_{r=k+1}^{n-1}(-1)^{r-k}\cdot\prod_{i=k+1}^{r}(m\cdot(i+1)-1)\right)$$

$$+\sum_{r=k+1}^{n-1}(-1)^{r-k}\cdot\prod_{i=k+1}^{r}(m\cdot(i+1)-1)-\sum_{r=k+2}^{n}(-1)^{r-k}\cdot\prod_{i=k+1}^{r}(m\cdot(i+1)-1)$$

$$=(m\cdot n+m-1)\cdot\left(\prod_{i=k+1}^{k}(m\cdot(i+1)-1)+(-1)^{n-k}\cdot\prod_{i=k+1}^{n-1}(m\cdot(i+1)-1)\right)$$

$$+\left(-\prod_{i=k+1}^{k+1}(m\cdot(i+1)-1)-(-1)^{n-k}\cdot\prod_{i=k+1}^{n}(m\cdot(i+1)-1)\right)$$

$$=(m\cdot n+m-1)+(-1)^{n-k}\cdot\prod_{i=k+1}^{n}(m\cdot(i+1)-1)-(m\cdot(k+2)-1)-(-1)^{n-k}\cdot\prod_{i=k+1}^{n}(m\cdot(i+1)-1)$$

= m · (n - k - 1) with empty product 1. q.e.d.