**Proofs of two observations from Sequence A337978** (*Ya-Ping Lu, Jan 10, 2021*)

**NAME of Sequence A337978**

a(n) = n + pi(n) - pi(n + pi(n)), where pi(n) is the prime count of n (n>=1).

**DATA of Sequence A337978**

1, 1, 2, 3, 4, 5, 6, 7, 7, 8, 10, 10, 11, 12, 13, 14, 15, 16, 18, 19, 19, 20, 21, 22, 23, 24, 25, 25, 27, 28, 29, 29, 30, 31, 32, 32, 34, 35, 36, 37, 38, 39, 41, 42, 42, 43, 44, 45, 46, 47, 48, 48, 50, 51, 51, 52, 52, 53, 55, 56, 57, 58, 59, 60, 60, 61, 63

***Observation 1: A337978 is a non-decreasing sequence.***

***Proof***: By the definition of the sequence,

$$a_n - a_{n-1} = [\pi(n) - \pi(n-1)] - [\pi\big(n + \pi(n)\big) - \pi\big(n - 1 + \pi(n-1)\big)] + 1 \quad (1)$$

The first term on the right side of Equation 1, $\pi(n) - \pi(n-1)$, is equal to 1 (if n is prime) or 0 (if n is composite), or

$$\pi(n) - \pi(n-1) = isprime(n) \qquad isprime(n) = 1\ if\ n\ is\ prime, 0\ otherwise \quad (2)$$

Since

$$1 + \pi(n) - \pi(n-1) = [n + \pi(n)] - [(n-1) + \pi(n-1)] \quad (3)$$

which means the two terms on the right side of Equation 3, $n + \pi(n)$ and $(n-1) + \pi(n-1)$, differ by 2 (if n is prime) or by 1 (if n is composite), and the prime counts between these two terms differ by 1 or 0:

$$0 \le \pi[n + \pi(n)] - \pi[(n-1) + \pi(n-1)] \le 1 \quad (4)$$

From Equations 1, 2, and 4, we have $a_n - a_{n-1} \ge 0$, or

$$a_n \ge a_{n-1}$$

Which proves that the sequence is non-decreasing.

***Observation 2: a(n) < n for n >= 2.***

***Proof:*** By definition,

$$a_n = n + \pi(n) - \pi\big(n + \pi(n)\big) \tag{5}$$

According to Corollary 1 by Lu and Deng [1], there is at least one prime number in the range of $(n,\ n + \pi(n)]$ for $n \geq 2$. Thus,

$$\pi\big(n + \pi(n)\big) - \pi(n)\ \geq 1. \tag{6}$$

From Equations 5 and 6, we have

$$a_n \leq n - 1 < n.$$

for $n \geq 2$.

**Reference**

[1] Ya-Ping Lu and Shu-Fang Deng, An upper bound for the prime gap, arXiv:2007.15282 [math.GM], 2020.