When $[x,\ y,\ z]$ is a row, $f(a,b) = xab + y(a+b) + z$ is associative.
For each triple, the corresponding $f(a,b)$ has a unique identity element (I), meaning
$f(a,I) = f(I,a) = a$, for all a. $I = -\frac{z}{y}$. $f(a,b)$ also has a unique zero element (call it θ),
meaning $f(a,\theta) = f(\theta,a) = \theta$, for all a. $\theta = -\frac{y}{x}$.

$f(a,b)$, defined by each row, also has a distributive rule when the generalized zero is taken into account. This means that if we define a "partition" of b by $b = b_1 + b_2 - \theta$, then
$f(a,b) = f(a,\ b_1 + b_2 - \theta) = f(a,b_1) + f(a,b_2) - \theta$ for all a and b, and all "partitions" of b.
Notice that when $\theta = 0$, we have the usual distributive rule. However, in that case we would need $y = 0$ which corresponds to $f(a,b) = xab$ or $f(a,b) = z$, neither of which is allowed.

Another way to write $f(a,b)$ for a row is to first compute I and θ from x, y and z. Then

$$f(a,b) = \frac{ab - \theta(a+b) + I\theta}{I - \theta}$$ or equivalently

$$f(a,b) = \frac{ab - \theta(a+b) + \theta^2}{I - \theta} + \theta.$$

Notice that I cannot equal θ because of $I - \theta$ in the denominator. Also notice that when $I = 1$ and $\theta = 0$, $f(a,b) = ab$, but multiplication is not represented in the table since the corresponding row would be $[1,\ 0,\ 0]$, which is not allowed.

If (i) two rows are $[x_1,\ y_1,\ z_1]$ and $[x_2,\ y_2,\ z_2]$

(ii) $I_1 = -\frac{z_1}{y_1}$, $I_2 = -\frac{z_2}{y_2}$, $\theta_1 = -\frac{y_1}{x_1}$, $\theta_2 = -\frac{y_2}{x_2}$

(iii) $\frac{I_1}{\theta_2} + \frac{I_2}{\theta_1} = 2$

then $[x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2]$ is a row.

Consequently,
if (i) $f_1(a,b) = x_1 ab + y_1(a+b) + z_1$ is associative and
$f_2(a,b) = x_2 ab + y_2(a+b) + z_2$ is associative

(ii) $I_1 = -\frac{z_1}{y_1}$, $I_2 = -\frac{z_2}{y_2}$, $\theta_1 = -\frac{y_1}{x_1}$, $\theta_2 = -\frac{y_2}{x_2}$

(iii) $\frac{I_1}{\theta_2} + \frac{I_2}{\theta_1} = 2$

then $f_1(a,b) + f_2(a,b) = (x_1 + x_2)ab + (y_1 + y_2)(a+b) + z_1 + z_2$ is associative.

Proof:
Given $[x_1,\ y_1,\ z_1]$ and $[x_2,\ y_2,\ z_2]$ are rows, the following algebraic manipulations show that $[x_1+x_2,\ y_1+y_2,\ z_1+z_2]$ is a row.

Say $\dfrac{I_1}{\theta_2} + \dfrac{I_2}{\theta_1} = 2.$

$$\frac{-\dfrac{z_1}{y_1}}{-\dfrac{y_2}{x_2}} + \frac{-\dfrac{z_2}{y_2}}{-\dfrac{y_1}{x_1}} = 2$$

$\dfrac{x_2 z_1}{y_1 y_2} + \dfrac{x_1 z_2}{y_1 y_2} = 2$ [Multiply by $y_1 y_2$ and move to the right.]

$0 = 2y_1 y_2 - x_1 z_2 - x_2 z_1$ [Add to the right side $y_1^2 - y_1 - x_1 z_1$ and $y_2^2 - y_2 - x_2 z_2$, which both equal 0.]

$0 = 2y_1 y_2 - x_1 z_2 - x_2 z_1 + y_1^2 - y_1 - x_1 z_1 + y_2^2 - y_2 - x_2 z_2$ [Rearrange.]

$0 = y_1^2 + 2y_1 y_2 + y_2^2 - y_1 - y_2 - x_1 z_1 - x_1 z_2 - x_2 z_1 - x_2 z_2$
$0 = (y_1+y_2)^2 - (y_1+y_2) - (x_1+x_2)(z_1+z_2)$ QED.

The idea of summing rows to get another row can be extended.

If (i) three rows are $[x_1, y_1, z_1]$, $[x_2, y_2, z_2]$, and $[x_3, y_3, z_3]$

(ii) I and θ are defined as above

(iii) $$\left(\frac{\dfrac{I_1}{\theta_2} + \dfrac{I_2}{\theta_1} - 2}{y_3}\right) + \left(\frac{\dfrac{I_1}{\theta_3} + \dfrac{I_3}{\theta_1} - 2}{y_2}\right) + \left(\frac{\dfrac{I_2}{\theta_3} + \dfrac{I_3}{\theta_2} - 2}{y_1}\right) = 0$$

then $[x_1+x_2+x_3,\ y_1+y_2+y_3,\ z_1+z_2+z_3]$ is a row.

Generalizing, when summing n rows to another row, the criterion involves the sum of binomial(n,2) versions of $\dfrac{I_i}{\theta_j} + \dfrac{I_j}{\theta_i} - 2$ as i and j go from 1 to n and i < j. Furthermore, each of these expressions is divided by the product of the y values from rows other than i and j. There are binomial(n,n–2) = binomial(n,2) such products.

Formally this is:

If $[x_1,\ y_1,\ z_1],\ \ldots,\ [x_n,\ y_n, z_n]$ are rows and

$$\sum_{1 \le i < j \le n} \frac{\frac{I_i}{\theta_j} + \frac{I_j}{\theta_i} - 2}{\prod_{k=1..n}^{k \ne i,j} y_k} = 0,$$

then $\left[\sum_{m=1}^{n} x_m,\ \sum_{m=1}^{n} y_m,\ \sum_{m=1}^{n} z_m\right]$ is a row.

All of the above is still true when x, y, z, I and θ are complex numbers with $y \neq 1$ and $x, y, z \neq 0$.

If $[x,\ y,\ z]$ is not a row, compute $K = \frac{y}{y^2 - xz}$. Then $K[x,\ y,\ z]$ is a row if it is an integer triple. Note that if $[x,\ y,\ z]$ were a row, K = 1. Furthermore, if $[nx,\ ny,\ nz]$ is not a row,

compute $K' = \frac{ny}{(ny)^2 - (nx)(nz)} = \frac{y}{n(y^2 - xz)} = \frac{K}{n}$. Then $K'[nx,\ ny,\ nz] = K[x,\ y,\ z]$ as before. When $K[x, y, z]$ is not a triple for not having integer values, we still have

$(Ky)^2 - Ky - (Kx)(Kz) = 0.$

Examples of the distributive rule:

$[x, y, z] = [1, 2, 2]$

$f(a, b) = ab + 2(a+b) + 2$

$$\theta = -\frac{y}{x} = -2$$

$f(5, 7) = 35 + 2(5+7) + 2 = 61$ which equals

$f(5, 3 + 2 - (-2)) = f(5, 3) + f(5, 2) - (-2) = (15+16+2) + (10+14+2) + 2 = 61.$

$f(5, 8) = 40 + 2(5+8) + 2 = 68$ which equals

$f(5, 4 + 2 - (-2)) = f(5, 4) + f(5, 2) - (-2) = (20+18+2) + (10+14+2) + 2 = 68.$

Examples of rows that sum to another row:

$[1, 7, 42] + [2, 8, 28] = [3, 15, 70]$ because

$$\frac{I_1}{\theta_2} + \frac{I_2}{\theta_1} = \frac{-\frac{42}{7}}{-\frac{8}{2}} + \frac{-\frac{28}{8}}{-\frac{7}{1}} = 2.$$

$[42, 7, 1] + [28, 8, 2] = [70, 15, 3]$ because

$$\frac{I_1}{\theta_2} + \frac{I_2}{\theta_1} = \frac{-\frac{1}{7}}{-\frac{8}{28}} + \frac{-\frac{2}{8}}{-\frac{7}{42}} = 2.$$

$[2, 8, 28] + [3, 18, 102] = [5, 26, 130]$ because

$$\frac{I_1}{\theta_2} + \frac{I_2}{\theta_1} = \frac{-\frac{28}{8}}{-\frac{18}{3}} + \frac{-\frac{102}{18}}{-\frac{8}{2}} = 2.$$

$[28, 8, 2] + [102, 18, 3] = [130, 26, 5]$ because

$$\frac{I_1}{\theta_2} + \frac{I_2}{\theta_1} = \frac{-\frac{2}{8}}{-\frac{18}{102}} + \frac{-\frac{3}{18}}{-\frac{8}{28}} = 2.$$

Examples of three rows that sum to a row:

$[1,\ 2,\ 2] + [1,\ 2,\ 2] + [1,\ 5,\ 20] = [3,\ 9,\ 24]$ because

$$\left(\frac{\frac{I_1}{\theta_2}+\frac{I_2}{\theta_1}-2}{y_3}\right)+\left(\frac{\frac{I_1}{\theta_3}+\frac{I_3}{\theta_1}-2}{y_2}\right)+\left(\frac{\frac{I_2}{\theta_3}+\frac{I_3}{\theta_2}-2}{y_1}\right)$$

$$=\left(\frac{\frac{-\frac{2}{2}}{-\frac{2}{1}}+\frac{-\frac{2}{2}}{-\frac{2}{1}}-2}{5}\right)+\left(\frac{\frac{-\frac{2}{2}}{-\frac{5}{1}}+\frac{-\frac{20}{5}}{-\frac{2}{1}}-2}{2}\right)+\left(\frac{\frac{-\frac{2}{2}}{-\frac{5}{1}}+\frac{-\frac{20}{5}}{-\frac{2}{1}}-2}{2}\right)=0.$$

In this example no two of the rows sum to another row.

$[1,\ 7,\ 42] + [2,\ 8,\ 28] + [3,\ 10,\ 30] = [6,\ 25,\ 100]$ because

$$\left(\frac{\frac{I_1}{\theta_2}+\frac{I_2}{\theta_1}-2}{y_3}\right)+\left(\frac{\frac{I_1}{\theta_3}+\frac{I_3}{\theta_1}-2}{y_2}\right)+\left(\frac{\frac{I_2}{\theta_3}+\frac{I_3}{\theta_2}-2}{y_1}\right)$$

$$=\left(\frac{\frac{-\frac{42}{7}}{-\frac{8}{2}}+\frac{-\frac{28}{8}}{-\frac{7}{1}}-2}{10}\right)+\left(\frac{\frac{-\frac{42}{7}}{-\frac{10}{3}}+\frac{-\frac{30}{10}}{-\frac{7}{1}}-2}{8}\right)+\left(\frac{\frac{-\frac{28}{8}}{-\frac{10}{3}}+\frac{-\frac{30}{10}}{-\frac{8}{2}}-2}{7}\right)=0.$$

In this example $[1,\ 7,\ 42] + [2,\ 8,\ 28] = [3,\ 15,\ 70]$, another row.

For n=4,

$$\text{if}\quad \left(\frac{\frac{I_1}{\theta_2}+\frac{I_2}{\theta_1}-2}{y_3y_4}\right)+\left(\frac{\frac{I_1}{\theta_3}+\frac{I_3}{\theta_1}-2}{y_2y_4}\right)+\left(\frac{\frac{I_1}{\theta_4}+\frac{I_4}{\theta_1}-2}{y_2y_3}\right)$$

$$+\left(\frac{\frac{I_2}{\theta_3}+\frac{I_3}{\theta_2}-2}{y_1y_4}\right)+\left(\frac{\frac{I_2}{\theta_4}+\frac{I_4}{\theta_2}-2}{y_1y_3}\right)+\left(\frac{\frac{I_3}{\theta_4}+\frac{I_4}{\theta_3}-2}{y_1y_2}\right)=0,$$

then $[x_1+x_2+x_3+x_4,\ y_1+y_2+y_3+y_4,\ z_1+z_2+z_3+z_4]$ is a row.

$[x,\ y,\ z] = [3,\ 2,\ 1]$ is not a row, but $\frac{y}{y^2-xz}[x,y,z] = \frac{2}{4-3}[3,\ 2,\ 1] = [6,\ 4,\ 2]$ is a row.