A supercongruence for A333593

Peter Bala, March 2020

We prove the supercongruence A333593$(p) \equiv 0 \pmod{p^3}$ holds for all prime $p \geq 5$.

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The terms of A333593 are defined by means of the binomial sum

$$a(n) = \sum_{k=0}^{n} (-1)^{n+k} \binom{n+k-1}{k}^2. \qquad (1)$$

Conjecture. We conjecture that the supercongruences

$$a\left(mp^k\right) \equiv a\left(mp^{k-1}\right) \pmod{p^{3k}} \qquad (2)$$

hold for prime $p \geq 5$ and all positive integers m and k.

We are able to prove the following particular case.

Proposition 1. *The supercongruence $a(p) \equiv 0 \pmod{p^3}$ holds for prime $p \geq 5$.*

Proof. Let $p \geq 5$ be a prime. We rewrite the binomial sum representation (1) for $a(p)$ by separating out the first $(k = 0)$ and last $(k = p)$ summand and adding together the k-th and $(p-k)$-th summands for $1 \leq k \leq \frac{p-1}{2}$ to obtain

$$a(p) = (-1)^p + (-1)^{2p} \binom{2p-1}{p}^2 + \sum_{k=1}^{\frac{p-1}{2}} \left((-1)^{p+k} \binom{p+k-1}{k}^2 + (-1)^k \binom{2p-k-1}{p-k}^2 \right). \qquad (3)$$

Now by Wolstenholme's theorem [Mes'11, p. 3]

$$\binom{2p-1}{p} = \frac{1}{2}\binom{2p}{p} \equiv 1 \pmod{p^3}. \qquad (4)$$

Hence by (3) and (4)

$$a(p) \equiv \sum_{k=1}^{\frac{p-1}{2}} \left((-1)^{p+k} \binom{p+k-1}{k}^2 + (-1)^k \binom{2p-k-1}{p-k}^2 \right) \pmod{p^3}. \qquad (5)$$

To establish the Proposition we show that each summand on the right-hand side of (5) is divisible by p^3.

One easily checks that the summand

$$(-1)^{p+k}\binom{p+k-1}{k}^2+(-1)^k\binom{2p-k-1}{p-k}^2=(-1)^k\frac{(p+k-1)!^2AB}{k!^2(p-1)!^2(p-k)!^2}, \tag{6}$$

where the integer factors A and B are given by

$$A=\frac{k!(2p-k-1)!}{(p+k-1)!}+(p-k)!$$

$$B=\frac{k!(2p-k-1)!}{(p+k-1)!}-(p-k)!.$$

Clearly, the first factor $(p+k-1)!^2$ in the numerator of the right-hand side of (6) is divisible by p^2 since $k \geq 1$. We show that the second factor A in the numerator is divisible by p.

Set $r = p-2k \geq 1$. Then we have

$$\begin{aligned}
A=\frac{k!(2p-k-1)!}{(p+k-1)!}+(p-k)! &= k!(2p-k-1)(2p-k-2)\cdots(2p-k-r)+(p-k)! \\
&\equiv ((-1)^r k!(k+1)(k+2)\cdots(k+r)+(p-k)!) \pmod{p} \\
&\equiv (-(k+r)!+(p-k)!) \pmod{p} \\
&\equiv (-(p-k)!+(p-k)!) \pmod{p} \\
&\equiv 0 \pmod{p}.
\end{aligned}$$

Thus the numerator of the right-hand side of (6) is divisible by $p^2 \times p = p^3$. Clearly, the denominator of the right-hand side of (6) has no factor equal to p for k in the summation range $1 \ldots \frac{p-1}{2}$. We have therefore shown that each summand $(-1)^{p+k}\binom{p+k-1}{k}^2+(-1)^k\binom{2p-k-1}{p-k}^2$ in (5) is divisible by p^3 for $1 \leq k \leq \frac{p-1}{2}$, completing the proof of the Proposition. □

Remarks.

1) Calculation suggests that the above approach of adding pairs of terms to get divisibility by powers of the prime p may extend to proving the more general supercongruences (2).

2) The approach used in Proposition 1 can be used to establish the corresponding supercongruences for A002003, A103885, A119259 and A156894.

A generalisation.

We define a two parameter family of sequences $a_{(r,s)}(n)$ by

$$a_{(r,s)}(n) = \sum_{k=0}^{rn} (-1)^{n+k} \binom{sn+k-1}{k}^2 \quad r \in \mathbb{N}, s \in \mathbb{Z}. \tag{7}$$

so that $a_{(1,1)}(n) =$ A333593. We conjecture that the supercongruences

$$a_{(r,s)}\left(mp^k\right) \equiv a_{(r,s)}\left(mp^{k-1}\right) \pmod{p^{3k}} \tag{8}$$

hold for all prime $p \geq 5$ and $r \in \mathbb{N}$ and $s \in \mathbb{Z}$.

Table of values $a_{(r,s)}(n)$

r = 1

	$n=1$	2	3	4	5	6	7
$s=1$	0	6	72	910	12000	163086	2266544
$s=2$	3	85	2730	95733	3542253	135822622	5341253970
$s=3$	8	406	25280	1736670	126295008	9531366796	738304396544
$s=4$	15	1233	126555	14375761	1730506890	216286373925	27753139999530
$s=5$	24	2926	448224	76083126	13691975024	2558872044190	491030572585248

Table of values $(-1)^n a_{(r,s)}(n)$

r = 2

	$n=1$	2	3	4	5
$s=1$	1	15	496	17655	659001
$s=2$	6	910	163086	32043726	6651400756
$s=3$	28	13146	7581385	4835694370	3268833458528
$s=4$	85	95733	135822622	214041146101	358021699058835
$s=5$	201	465751	1376042592	4525666068735	15811226744434576

References

[Mes'11] R. Mestrovic, Wolstenholme's theorem: Its Generalizations and Extensions in the last hundred and fifty years (1862-2011), arXiv:1111.3057 [math.NT], 2011.