Notes on A333093

Peter Bala, March 2020

1. The o.g.f. of A333093

The terms of A333093 are defined by $a(n)$ = the n-th order Taylor polynomial (centred at 0) of $c(x)^n$ evaluated at $x = 1$. Here $c(x) = (1 - \mathrm{sqrt}(1-4x))/(2x)$ is the o.g.f. of the sequence of Catalan numbers A000108.

Let

$$c(x)^n = \sum_{k=0}^{n} c_{nk} x^{n-k} + \mathrm{O}\left(x^{n+1}\right)$$

so that

$$\sum_{k=0}^{n} c_{nk} x^{n-k}$$

is the the n-th order Taylor polynomial centred at 0 of $c(x)^n$.

The lower triangular array $H := (c_{nk})$ of coefficients of the Taylor polynomials begins

1				
1	1			
5	2	1		
28	9	3	1	
165	48	14	4	1

A333093 is thus the row sums of this array.

An application of [3, Theorem 4.1] shows that the array H is a Riordan array: it belongs to the Hitting time subgroup of the Riordan group and has the form

$$H = \left(1 + x\frac{h^{'}(x)}{h(x)}, xh(x)\right),$$

where

$$h(x) = \frac{1}{x}\mathrm{Rev}\left(\frac{x}{c(x)}\right). \tag{1}$$

Calculation gives

$$h(x) = 1 + x + 3x^2 + 12x^3 + 55x^4 + \cdots,$$

the generating function for A001764.

By the general theory of Riordan arrays, the bivariate generating function for the array H is equal to

$$\frac{1+x\dfrac{h'(x)}{h(x)}}{1-txh(x)}.$$

Setting $t=1$ gives the generating function for the sequence of row sums of H, that is, the generating function of the sequence A333093, as

$$\sum_{n\geq 0} a(n)x^n = \frac{1+x\dfrac{h'(x)}{h(x)}}{1-xh(x)}. \tag{2}$$

2. We claim that $a(n) = [x^n]\, G(x)^n$, where $G(x) = (1+x)c\left(\dfrac{x}{1+x}\right)$.

Proof. We need the following result - see [4, Exercise 5.56 (a), p. 98, and its solution on p. 146] or [1]: given an integer sequence $g(n)$ with $g(0)=1$ there exists a unique formal power series $G(x)$ with rational coefficients such that $g(n) = [x^n]G(x)^n$. The series $G(x)$ is given by the formula

$$G(x) = \frac{x}{\text{Rev }(xE(x))} \tag{3}$$

where

$$E(x) = \exp\left(\sum_{n\geq 1} g(n)\frac{x^n}{n}\right). \tag{4}$$

It follows from (4) that the generating function of the sequence $g(n)$ is given by

$$\sum_{n\geq 0} g(n)x^n = 1 + x\frac{d}{dx}\log(E(x)). \tag{5}$$

If we now now take $g(n) = a(n) = \text{A333093}(n)$ then by (5)

$$\begin{aligned} 1 + x\frac{d}{dx}\log\ (E(x)) &= \sum_{n\geq 0} a(n)x^n \\ &= \frac{1+x\dfrac{h'(x)}{h(x)}}{1-xh(x)} \end{aligned}$$

by (2).

Hence

$$
\begin{aligned}
\frac{d}{dx}(\log(E(x))) &= \frac{h^{'}(x)+h(x)^2}{h(x)(1-xh(x))} \\
&= \frac{d}{dx}\left(\log\left(\frac{h(x)}{1-xh(x)}\right)\right),
\end{aligned}
$$

from which we obtain

$$
E(x) = C\frac{h(x)}{1-xh(x)} \tag{6}
$$

for some constant C. By looking at the expansion of both sides of (6) one sees that $C = 1$.

Thus

$$
E(x) = \frac{h(x)}{1-xh(x)}. \tag{7}
$$

It now follows from (3) and (7) that $a(n) = [x^n]G(x)^n$ where

$$
\begin{aligned}
G(x) &= \frac{x}{\mathrm{Rev}\left(\frac{xh(x)}{1-xh(x)}\right)} \qquad (8)\\
&= \frac{x}{\mathrm{Rev}\left(\frac{\mathrm{Rev}\left(\frac{x}{c(x)}\right)}{1-\mathrm{Rev}\left(\frac{x}{c(x)}\right)}\right)} \qquad (9)
\end{aligned}
$$

by (1).

In order to identify the power series on the right side of (9) we require some facts about the binomial transform of power series. Recall that the action of the binomial transform Bin on a power series $F(x) = 1 + f_1x + f_2x^2 + \cdots$ is defined by

$$
\mathrm{Bin}(F(x)) = \frac{1}{1-x}F\left(\frac{x}{1-x}\right)
$$

with inverse transform

$$
\mathrm{Bin}^{-1}(F(x)) = \frac{1}{1+x}F\left(\frac{x}{1+x}\right).
$$

Let Rev denote the series reversion operator. The operator Bin^{-1} can be expresed in terms of the operator Rev using [2, Proposition 3 with $m = 1$] :

$$\mathrm{Bin}^{-1}(F(x)) = \frac{1}{x}\mathrm{Rev}\left(\frac{\mathrm{Rev}\ (xF(x))}{1-\mathrm{Rev}\ (xF(x))}\right). \tag{10}$$

Take $F(x) = 1/c(x)$ in (10) to find that

$$\frac{1}{(1+x)c\left(\dfrac{x}{1+x}\right)} = \frac{1}{x}\mathrm{Rev}\left(\frac{\mathrm{Rev}\ \left(\dfrac{x}{c(x)}\right)}{1-\mathrm{Rev}\ \left(\dfrac{x}{c(x)}\right)}\right). \tag{11}$$

Comparing (11) with (9) gives

$$G(x) = (1+x)c\left(\frac{x}{1+x}\right). \ \square$$

3. Congruences

We see from (7) that the power series $E(x) = \exp\left(\sum_{n\geq 1} a(n)\dfrac{x^n}{n}\right)$ equals the power series $h(x)/(1-h(x)) = 1+2x+6x^2+23x^3+102x^4+\cdots$ (see A098746). The integrality of the coefficients of this last series is equivalent to the Gauss congruences holding for the sequence $a(n)$:

$$a\left(np^k\right) \equiv a(np^{k-1}) \pmod{p^k} \tag{12}$$

for all prime p and positive integers n and k [4, Exercise 5.2 (a), p. 72, and its solution on p. 104]. Calculation suggests that the stronger supercongruences

$$a\left(np^k\right) \equiv a(np^{k-1}) \pmod{p^{3k}} \tag{13}$$

hold for prime $p \geq 5$.

4. Generalisations It may be possible to replace the power series $G(x) = (1+x)c\left(\dfrac{x}{1+x}\right)$ with other power series to obtain further sequences of the form $[x^n]\, G(x)^n$, which satisfy the supercongruences (13). For example, $G(x) = (1+x)S\left(\dfrac{x}{1+x}\right)$, where

$$S(x) = \left(1-x-\mathrm{sqrt}\left(1-6x+x^2\right)\right)/(2x)$$

is the generating function of the sequence of large Schröder numbers A006318. See A333090.

References

[1] P. Bala Representing a sequence as [x{̂}n] G(x){̂}n, uploaded to A066398

[2] P. Bala Notes on logarithmic differentiation, the binomial transform and series reversion uploaded to A100100

[3] P. Peart and W.-J. Woan A divisibility property for a subgroup of Riordan matrices Discrete Applied Mathematics, Vol. 98, Issue 3, Jan 2000, 255-263.

[4] R. P. Stanley Enumerative Combinatorics, Volume 2, Cambridge University Press, 1999