Integers sequences A328348 and A328350 to A328356

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December 10, 2019

Abstract

The sequence A328348 is the building block for the calculation of the sums of positive integers whose decimal notation only uses two distinct, non-zero digits.

The sequences A328350 to A328356 are also building blocks for the calculation of the sums of positive integers, in the case where more than two distinct, non-zero digits are used .

1 Sums of whole numbers whose decimal notation only uses two distinct, non-zero digits

Let α and β be two distinct non-zero digits (i.e. $(\alpha, \beta) \in \{1, 2, \ldots, 9\}^2$, with $\alpha \neq \beta$), let $n \in \mathbb{N}$, and let $E_{\alpha,\beta,n}$ be the set of whole numbers whose decimal notation only uses the α and β digits and that have at most n such digits. Formally,

$$E_{\alpha,\beta,n} = \big\{x \in \mathbb{N} \mid \exists p \in [\![0; n-1]\!],\ \exists (a_0, \ldots, a_p) \in \{\alpha, \beta\}^{p+1},\ x = \sum_{i=0}^{p} a_i 10^i \big\}$$

We define

$$S_{\alpha,\beta,n} = \sum_{x \in E_{\alpha,\beta,n}} x$$

We state that

$$S_{\alpha,\beta,n} = (\alpha + \beta) \times u_{2,n}$$

where

$$\forall n \in \mathbb{N},\ u_{2,n} = \frac{10.20^n - 19.2^n + 9}{171}$$

The sequence $\left(u_{2,n}\right)_{n\in\mathbb{N}} = (0, 1, 23, 467, 9355, 187131, 3742683, 74853787, 1497075995, \ldots)$ corresponds to the A328348 OEIS sequence.

For example, with $\{\alpha, \beta\} = \{1, 2\}$ and $n = 3$, we have

$$E_{1,2,3} = \{1, 2, 11, 12, 21, 22, 111, 112, 121, 122, 211, 212, 221, 222\}$$

and

$$S_{1,2,3} = \sum_{x \in E_{1,2,3}} x = (1+2) \times u_3 = 1401.$$

Proof : by definition, $S_{\alpha,\beta,n} = \sum_{p=0}^{n-1} \sum_{(a_0,\dots,a_p)\in\{\alpha,\beta\}^{p+1}} \sum_{i=0}^{p} a_i 10^i$. For a given $p \in [\![0;n-1]\!]$, a given $\epsilon \in \{\alpha,\beta\}$ and a given $j \in [\![0;p]\!]$, $\mathrm{Card}\big(\{(a_0,\dots,a_{j-1},\epsilon,a_{j+1},\dots,a_p) \mid (a_i)_{i\neq j} \in \{\alpha,\beta\}^p\big) = 2^p$: it means that the digit α appears 2^p times in first position ($j=0$), 2^p times in second position ($j=1$),etc. Therefore, $\forall p \in [\![0;n-1]\!]$, $\sum_{(a_0,\dots,a_p)\in\{\alpha,\beta\}^{p+1}} \sum_{i=0}^{p} a_i 10^i = \alpha \times 2^p \times \sum_{i=0}^{p} 10^i + \beta \times 2^p \times \sum_{i=0}^{p} 10^i = (\alpha+\beta)2^p \sum_{i=0}^{p} 10^i = (\alpha+\beta)\dfrac{10.20^p - 2^p}{9}$. Thus $S_{\alpha,\beta,n} = \frac{\alpha+\beta}{9} \sum_{p=0}^{n-1} (10.20^p - 2^p) = \dfrac{\alpha+\beta}{9 \times 19}(10.20^n - 19.2^n + 9)$. $\square$

Remark : it can be noted that, in the expression of $S_{\alpha,\beta,n}$, only the sum $\alpha+\beta$ matters and not the individual values of α and β. As a consequence,as soon as $\alpha+\beta = \alpha'+\beta'$, we have $S_{\alpha,\beta,n} = S_{\alpha',\beta',n}$. For example, for all positive integer n, $S_{1,7,n} = S_{2,6,n} = S_{3,5,n}$.

The sequence $(u_{2,n})_{n\in\mathbb{N}}$ satisfies the following recurrence formula :

$$u_{0,2} = 0,\ u_{1,2} = 1 \text{ and } \forall n \geqslant 1,\ u_{n+1,2} = 21u_{n,2} - 20u_{n-1,2} + 2^n$$

2 Generalisation : sums of whole numbers whose decimal notation only uses k distinct, non-zero digits

Instead of selecting only two distinct digits, lets take $k \in [\![2;9]\!]$ distinct, non-zero digits and consider the set $E_{\alpha_1,\dots,\alpha_k,n}$ of whole numbers whose decimal notation only uses $\alpha_1,\dots,\alpha_k$ as digits and that have at most n such digits.
We define

$$S_{\alpha_1,\dots,\alpha_k,n} = \sum_{x\in E_{\alpha_1,\dots,\alpha_k,n}} x$$

With a proof similar to the above, it can be established that

$$S_{\alpha_1,\dots,\alpha_k,n} = \left(\sum_{i=1}^{k} \alpha_i\right) \times u_{k,n}$$

where

$$\forall n \in \mathbb{N},\ u_{k,n} = \frac{10(k-1)(10k)^n - (10k-1)k^n + 9}{9(k-1)(10k-1)}$$

The sequences appearing in the above formula are :

- for $k=3$, $\left(u_{3,n}\right)_{n\in\mathbb{N}} = (0, 1, 34, 1033, 31030, 931021, 27930994, 837930913, \ldots)$ corresponds to the A328350 OEIS sequence.
- for $k=4$, $\left(u_{4,n}\right)_{n\in\mathbb{N}} = (0, 1, 45, 1821, 72925, 2917341, 116695005, 4667805661, \ldots)$ corresponds to the A328351 OEIS sequence.
- for $k=5$, $\left(u_{5,n}\right)_{n\in\mathbb{N}} = (0, 1, 56, 2831, 141706, 7086081, 354307956, 17715417331, \ldots)$ corresponds to the A328352 OEIS sequence.
- for $k=6$, , $\left(u_{6,n}\right)_{n\in\mathbb{N}} = (0, 1, 67, 4063, 244039, 14643895, 878643031, 52718637847, \ldots)$ corresponds to the A328353 OEIS sequence.
- for $k=7$, $\left(u_{7,n}\right)_{n\in\mathbb{N}} = (0, 1, 78, 5517, 386590, 27064101, 1894506678, 132615604717, \ldots)$ corresponds to the A328354 OEIS sequence.
- for $k=8$, $\left(u_{8,n}\right)_{n\in\mathbb{N}} = (0, 1, 89, 7193, 576025, 46086681, 3686971929, 294958053913, \ldots)$ corresponds to the A328355 OEIS sequence.
- for $k=9$, $\left(u_{9,n}\right)_{n\in\mathbb{N}} = (0, 1, 100, 9091, 819010, 73718281, 6634711720, 597124652671, \ldots)$; since, in that case, there is only one possible sum of digits (namely $1+2+\cdots+9=45$), we have recorded the sequence $\left(45\times u_{9,n}\right)_{n\in\mathbb{N}} = (0, 45, 4500, 409095, 36855450, 3317322645, \ldots)$ in OEIS under the reference A328356.

For example, with $k=5$, $\{\alpha_1,\ldots,\alpha_5\}=\{1,2,3,4,5\}$ and $n=2$, we have

$$E_{1,2,3,4,5,2} = \{1, 2, 3, 4, 5, 11, 12, 13, 14, 15, 21, 22, 23, 24, 25, \ldots, 54, 55\}$$

and

$$S_{1,2,3,4,5,2} = (1+2+3+4+5)\times u_{5,2} = 15\times 56 = 840$$

The sequence $(u_{k,n})_{n\in\mathbb{N}}$ satisfies the following recurrence formula :

$u_{0,k}=0,\ u_{1,k}=1$ and $\forall n\geqslant 1,\ u_{n+1,k}=(10k+1)\times u_{n,k}-10k\times u_{n-1,k}+k^n$

The generating function of the sequence $(u_{k,n})_{n\in\mathbb{N}}$ is

$$f(z) := \sum_{n=0}^{+\infty} u_{k,n} z^n = \frac{z}{1-(11k+1)x+(10k^2+11k)z^2-10k^2z^3}$$

where $|z| < \frac{1}{10k}$