Some linear divisibility sequences of order 8

Peter Bala, Sep 25 2019

An integer sequence $(a_n)_{n\ge 1}$ is a *linear divisibility sequence of order k* if the sequence satisfies a linear recurrence of order k and if $a(n)$ divides $a(m)$ whenever n divides m and $a(n) \neq 0$. Examples include the 2-parameter family of Lucas sequences of the first kind of order two, the 2-parameter family of Lehmer sequences of order four and a 3-parameter family of fourth order linear divisibility sequences due to Williams and Guy [4]. We construct two families $U_n(P,Q,R)$ and $U_n^*(P,Q,R)$ of linear divisibility sequences of order 8, depending on three integer parameters P, Q and R. They are particular cases of a larger family of linear divisibility sequences of order 8 studied in [3].

Let

$$f(x) = 1 + Px + Qx^2 + Rx^3 + x^4 \tag{1}$$

be a monic quartic polynomial with integer coefficients. Let $\widetilde{f}(x)$ denote the reciprocal polynomial of $f(x)$ defined by

$$\begin{aligned}\widetilde{f}(x) &= x^4 f\left(\frac{1}{x}\right)\\ &= 1 + Rx + Qx^2 + Px^3 + x^4.\end{aligned}$$

We define two 3-parameter families of sequences $U_n \equiv U_n(P,Q,R)$ and $U_n^* \equiv U_n^*(P,Q,R)$ by means of the rational function expansions

$$x\frac{d}{dx}\left(\log\left(\frac{f(x)}{\widetilde{f}(x)}\right)\right) = \sum_{n\ge 1} U_n x^n \tag{2}$$

$$x\frac{d}{dx}\left(\log\left(\frac{f(x)}{\widetilde{f}(-x)}\right)\right) = \sum_{n\ge 1} U_n^* x^n. \tag{3}$$

Calculation gives the generating functions

$$\sum_{n\ge 1} U_n x^n = (P-R)x\frac{\left(1-(Q+3)x^2-2(P+R)x^3-(Q+3)x^4+x^6\right)}{\left(1+Px+Qx^2+Rx^3+x^4\right)\left(1+Rx+Qx^2+Px^3+x^4\right)} \tag{4}$$

$$\sum_{n\ge 1} U_n^* x^n = (P+R)x\frac{\left(1-(Q-3)x^2+2(P-R)x^3+(Q-3)x^4-x^6\right)}{\left(1+Px+Qx^2+Rx^3+x^4\right)\left(1-Rx+Qx^2-Px^3+x^4\right)}. \tag{5}$$

Theorem 1.

(i) The sequence $U_n(P,Q,R)$ is a linear divisibility sequence of order 8.

(ii) The sequence $U_n^*(P,Q,R)$ is a linear divisibility sequence of order 8.

Proof. (i) It is immediate from the rational generating function (4) that U_n is an integer sequence satisfying a linear recurrence of order 8. We show that U_n is a divisibility sequence. Let

$$f(x) = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)$$

be the factorisation of the quartic polynomial $f(x)$ over $\mathbb{C}$. The reciprocal polynomial then factors as

$$\widetilde{f}(x) = (1-x\alpha_1)(1-x\alpha_2)(1-x\alpha_3)(1-x\alpha_4).$$

From (2), the generating function for U_n is given by

$$\begin{aligned}\sum_{n\ge1} U_n x^n &= x\left(\frac{f'(x)}{f(x)} - \frac{\widetilde{f}\,'(x)}{\widetilde{f}(x)}\right)\\ &= x\sum_{i=1}^{4}\left\{-\frac{1}{\alpha_i\left(1-\dfrac{x}{\alpha_i}\right)} + \frac{\alpha_i}{1-\alpha_i x}\right\},\end{aligned} \tag{6}$$

where the prime $'$ indicates differentiation with respect to x.

Expanding the right side of (6) into geometric series yields

$$U_n = \alpha_1^n + \alpha_2^n + \alpha_3^n + \alpha_4^n - \frac{1}{\alpha_1^n} - \frac{1}{\alpha_2^n} - \frac{1}{\alpha_3^n} - \frac{1}{\alpha_4^n}. \tag{7}$$

We shall recast the sum (7) into the form of a product better suited to proving divisibility properties of the numbers U_n. It is straightforward to verify that if A, B, C and D are complex numbers such that $ABCD = 1$ then

$$A+B+C+D-\frac{1}{A}-\frac{1}{B}-\frac{1}{C}-\frac{1}{D} = (1-AB)(1-AC)(1-BC)D. \tag{8}$$

Equivalently,

$$A+B+C+D-\frac{1}{A}-\frac{1}{B}-\frac{1}{C}-\frac{1}{D} = -\frac{(1-AD)(1-BD)(1-CD)}{D}. \tag{9}$$

Multiplying (8) by (9) gives

$$\left(A+B+C+D-\frac{1}{A}-\frac{1}{B}-\frac{1}{C}-\frac{1}{D}\right)^2 = -(1-AB)(1-AC)(1-AD)(1-BC)(1-BD)(1-CD). \tag{10}$$

Now α_1, α_2, α_3 and α_4 were defined as the zeros of the polynomial $f(x) = 1 + Px + Qx^2 + Rx^3 + x^4$. Hence $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$. Thus setting $A = \alpha_1^n$, $B = \alpha_2^n$, $C = \alpha_3^n$, $D = \alpha_4^n$ in (10) and comparing the result with (7) gives

$$U_n^2 = -\prod_{1\leq i<j\leq 4}\left(1-\alpha_i^n\alpha_j^n\right). \tag{11}$$

Thus, up to signs, the sequence U_n^2 is the Lehmer-Pierce sequence [1], [2] associated to the sextic polynomial $s(x) = \prod_{1\leq i<j\leq 4}(x-\alpha_i\alpha_j)$. A routine calculation gives

$$s(x) = x^6 - Qx^5 + (PR-1)x^4 + \left(P^2+R^2-2Q\right)x^3 + (PR-1)x^2 - Qx + 1.$$

Let n and m be positive integers and define

$$P(x) = \frac{1-x^{nm}}{1-x^n} = x^{n(m-1)} + x^{n(m-2)} + \cdots + x^n + 1$$

and put $S\left(x_1, ..., x_6\right) = \prod_{i=1}^{6} P\left(x_i\right)$, a symmetric polynomial function in the variables x_i. Then by (11), when $U_n \neq 0$, the quotient

$$\frac{U_{nm}^2}{U_n^2} = S\left(\beta_1, ..., \beta_6\right)$$

is a symmetric polynomial function of the roots β_i of the integral sextic equation $s(x) = 0$, and so is an integer by the fundamental theorem of symmetric polynomials. Therefore U_n^2 divides U_{nm}^2 and hence also U_n divides U_{nm}. This completes the proof that U_n is a divisibility sequence.

(ii) From the rational generating function (5) of the sequence U_n^*, we see that U_n^* is an integer sequence satisfying a linear recurrence of order 8. The proof that U_n^* is a divisibility sequence proceeds similarly to part (i).

From the generating function (3) we find

$$U_n^* = \begin{cases} \alpha_1^n + \alpha_2^n + \alpha_3^n + \alpha_4^n - \dfrac{1}{\alpha_1^n} - \dfrac{1}{\alpha_2^n} - \dfrac{1}{\alpha_3^n} - \dfrac{1}{\alpha_4^n} & n \text{ even} \\ -\left(\alpha_1^n + \alpha_2^n + \alpha_3^n + \alpha_4^n + \dfrac{1}{\alpha_1^n} + \dfrac{1}{\alpha_2^n} + \dfrac{1}{\alpha_3^n} + \dfrac{1}{\alpha_4^n}\right) & n \text{ odd.} \end{cases} \tag{12}$$

When n is even it follows exactly as in part (i) that

$$U_n^{*2} = -\prod_{1\leq i<j\leq 4}\left(1-\alpha_i^n\alpha_j^n\right). \tag{13}$$

Therefore, as in part (i), when n is even, U_n^{*2} divides U_{nm}^{*2} and hence also U_n^* divides U_{nm}^*.

To handle the case when n is odd we convert the sum expression for U_n^* in (12) into the form of a product by means of the following easily verified result: if A, B, C and D are complex numbers such that $ABCD = 1$ then

$$A + B + C + D + \frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D} = (1 + AB)(1 + AC)(1 + BC)D. \tag{14}$$

Equivalently,

$$A + B + C + D + \frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D} = \frac{(1 + AD)(1 + BD)(1 + CD)}{D}. \tag{15}$$

Multiplying (14) by (15) gives

$$\left(A + B + C + D + \frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D}\right)^2 = (1+AB)(1+AC)(1+AD)(1+BC)(1+BD)(1+CD). \tag{16}$$

Setting $A = \alpha_1^n$, $B = \alpha_2^n$, $C = \alpha_3^n$, $D = \alpha_4^n$ in (16) and then using (12) we find

$$U_n^{*2} = \prod_{1 \le i < j \le 4} \left(1 + \alpha_i^n \alpha_j^n\right) \quad \text{for } n \text{ odd.} \tag{17}$$

Using (17), we will prove that when n is odd U_n^* divides U_{nm}^* . There are two cases to consider depending on the parity of m. Firstly, suppose m is even. We define

$$P^*(x) = \frac{1 - x^{nm}}{1 + x^n} = x^{n(m-1)} - x^{n(m-2)} + \cdots + x^n - 1$$

and put $S^*(x_1, ..., x_6) = \prod_{i=1}^{6} P^*(x_i)$, a symmetric polynomial function in the variables x_i. Then by (13) and (17), when $U_n \neq 0$, the quotient

$$\frac{U_{nm}^{*2}}{U_n^{*2}} = S^*(\beta_1, ..., \beta_6)$$

is a symmetric polynomial function of the roots β_i of the integral sextic equation $s(x) = 0$, and so is an integer by the fundamental theorem of symmetric polynomials. Thus, when n is odd and m is even, U_n^{*2} divides U_{mn}^{*2} and hence also U_n^* divides U_{nm}^*. The remaining case when m is odd can be proven in a similar manner and is left to the reader. This completes the proof that U_n^* is a divisibility sequence. □

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