# Proofs of Conjectures 2 and 3 for Sequence A320137

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## Notations

n = $2^m \times s$ with m ≥ 0 and s > 1 odd.

SRS(n) denotes the symmetric representation of sigma(n), i.e., the list of the sizes of its parts or the area between the (n-1)-st and n-th Dyck path of the geometric representation of SRS(n).

row(n) = $\left\lfloor \left(\sqrt{8n+1} - 1\right)/2 \right\rfloor$ is the length of the n-th row of many of the irregular triangles associated with SRS(n), such as A235791, A237048 and A249223.

The two largest odd divisors of s, when they exist, that are at most as large as row(s) are denoted by e < d ≤ row(s).

w(n) is the width(SRS(n)) at the diagonal, i.e., the rightmost entry in the n-th row of the irregular triangle of A249223.

## Claims

Lemma 1:

Suppose that n has a single middle divisor f then $f = 2^{\lfloor \frac{m}{2} \rfloor} \times s^{\frac{1}{2}}$ and n = $f^2$ or n = $2 \times f^2$.

Proof:

Let f be the single middle divisor of n. Then:

(1) If n = e × f with e < f, then $e < \sqrt{\frac{n}{2}}$ so that $e \times f < \sqrt{\frac{n}{2}} \times f < \sqrt{\frac{n}{2}} \times \sqrt{2 \times n} = n$.

(2) If n = f × e with f < e and $e > \sqrt{2 \times n}$ then $f \times e > f \times \sqrt{2 \times n} \geq \sqrt{\frac{n}{2}} \times \sqrt{2 \times n} = n$.

(3) If n = f × e with f < e and $e = \sqrt{2 \times n}$ then $f = \frac{n}{\sqrt{2 \times n}} = \sqrt{\frac{n}{2}}$.

Therefore n = $f^2$ or n = $2 \times f^2$.

Note that for m even $f = 2^{\frac{m}{2}} \times s^{\frac{1}{2}}$, i.e., n = $f^2$ and $\sqrt{\frac{n}{2}} < f < \sqrt{2 \times n}$, and for m odd $f = 2^{\frac{m-1}{2}} \times s^{\frac{1}{2}}$, i.e., n = $2 \times f^2 = f \times (2 \times f) = \sqrt{\frac{n}{2}} \times \sqrt{2 \times n}$.

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Lemma 2:

If w(n) = 1 then n has a single middle divisor.

Proof:

By assumption the number of 1's at odd indices, representing odd divisors r of s with r ≤ row(n), is one larger than the number of 1's at even indices, representing odd divisors r of s with r > row(n), in row n of triangle A237048. If d is the largest divisor of s with d ≤ row(n) then s = $d^2$ and f = $2^{\lfloor \frac{m}{2} \rfloor} \times s^{\frac{1}{2}}$ is a middle divisor. For the next smaller odd divisor e < d, which exists since s is a square, its odd cofactor g = $\frac{s}{e}$ is represented by a 1 at index $2^{m+1} \times e \leq$ row(n) < $\sqrt{2 \times n}$ since otherwise w(n) > 1. Now $2^m \times e$ | n and $2^m$ < $2^m \times e < \sqrt{\frac{n}{2}}$ . Therefore, for any divisor h of n with h ≤ row(n) and an odd factor less than d the inequality h < $\sqrt{\frac{n}{2}}$ holds, and for any divisor h > row(n) and an odd factor greater than d the inequality h > $\sqrt{2 \times n}$ holds, in other words, f is the only middle divisor.

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Lemma 3:

If n has a single middle divisor then w(n) = 1.

Proof:

If n is a power of 2 then $2^{\lfloor \frac{m}{2} \rfloor}$ is its only middle divisor and SRS($2^m$) consists of a single part of width 1 since the only 1 in row $2^m$ of the triangle of A237048 occurs in position 1 and thus row $2^m$ of widths in the triangle A249223 consists of 1's only, i.e., w(n) = 1.

If s ≥ 3 then s is a square by Lemma 1 and the single middle divisor is f = $2^{\lfloor \frac{m}{2} \rfloor} \times s^{\frac{1}{2}}$. Therefore, s has an odd number of (odd) divisors so that w(s) > 0. Let e < $s^{\frac{1}{2}}$ be an odd divisor of s and suppose that $2^{m+1} \times$ e > $\sqrt{2 \times n}$ then for some 0 < k ≤ m divisor $2^k \times$ e ≠ f satisfies $\sqrt{\frac{n}{2}} \leq 2^k \times$ e < $\sqrt{2 \times n}$ and would be another middle divisor. Therefore, every 1 at an odd index is matched by a 1 at an even index in row n of triangle A237048, except for the middle divisor f, so that w(n) =1.

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