THE WYTHOFF AND THE ZECKENDORF REPRESENTATIONS OF NUMBERS ARE EQUIVALENT

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1. INTRODUCTION AND SYNOPSIS

The quintessence of many application of Fibonacci numbers is the binary substitution sequence $1 \rightarrow 10$, $0 \rightarrow 1$. The infinite sequence generated this way is self-similar and quasiperiodic. See refs. [7, 16] for details on this rabbit or golden sequences. It is intimately related to Wythoff's complementary sequences which cover the natural numbers ($\lfloor \cdot \rfloor$ is the greatest integer function)

$$A(m) := \lfloor m\varphi \rfloor,\ \ B(m) := \lfloor m\varphi^2 \rfloor,\ m \in \boldsymbol{N},\ \varphi^2 = \varphi + 1,\ \varphi > 0. \tag{1.1}$$

$A(0) := 0 =: B(0)$. The relationship is

$$A(1) = 1,\ A(n) = n + \sum_{k=1}^{n-1} h(k),\ n = 2,\ldots,\ B(n) = n + A(n), \tag{1.2}$$

where $h(k)$ is the k-th entry of the half-sided infinite substitution sequence $\{1,0,1,1,0,1,0,1,1,\ldots\}$. In particular, there is a 1, resp. 0, at entry number $n = A(m)$, resp. $n = B(p)$, for every $m \in \boldsymbol{N}$ and $p \in \boldsymbol{N}$. See refs. [21, 5, 17, 18, 12, 6, 7, 20, 19, 2] for details on Wythoff's sequences. For computational purposes (1.2) is more convenient than (1.1) because the irrational φ, the golden ratio, does not enter.

There is a unique representation for every natural number $N \in \boldsymbol{N}$ in terms of Wythoff orbits of 1. This means that every number N can be written uniquely as a composition of Wythoff sequences A and B acting on 1. Because $1 = A(1)$, the composition for $N \geq 2$ can be

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taken to start with $B(1)$. In this way, every number $N \geq 2$ is represented as a Wythoff string ending with $B(1)$, e.g. $(4)_W = AAB(1) \equiv A^2B(1)$, $(5)_W = BB(1) \equiv B^2(1)$. We omit brackets and also use a notation with A replaced by 1 and B replaced by 0 (sometimes 2), e.g. $W(4) = 110$ (or 112), $W(5) = 00$ (or 22). Such Wythoff representations, as we should like to call them, have been the topic of some papers by Hoggatt and Bicknell-Johnson [10, 9]. In the *Appendix* and *Fig.* 1 the reader can find an algorithm for $W(N)$, the Wythoff string corresponding to N.

Another, more familiar, unique representation of a natural number is its partition into nonadjacent Fibonacci numbers. This is usually called the Zeckendorf representation of N (we shall only use its canonical version of the first kind, the one with $1 = F_2$ and without F_1) [22, 4, 8, 3, 20, 2]. E.g. $(4)_Z = F_4 + F_2$, $(5)_Z = F_5$. An obvious string notation is used as well: $Z(4) = 101\cdot$, $Z(5) = 1000\cdot$, where the $\cdot$ is a reminder for the first canonical representation and is positioned next to the F_2 entry.

The conjecture of a direct relationship between these two unique representations is supported by well known identities like

$$A^{k-2}B(1) - 1 = F_k, \quad k = 2,3,\ldots \tag{1.3}$$

$$A^{k-3}B^2(1) - 1 = F_{k-1} + F_{k+1} = L_k, \; k = 3,4,\ldots, \tag{1.4}$$

for the Fibonacci numbers and the Zeckendorf representation of the Lucas numbers. These eqs. can be found, e.g., in ref. [3], as special cases of identity (6.15).

The equivalence of both representations is the content of the following theorem. It will open the passage from one representation to the other without having to know the number N itself.

<u>**Theorem:**</u> (Wythoff-Zeckendorf Bijection)

Part WZ) If, for $N = 2,3,4,\ldots$,

$$(N)_W = A^{i_n}B^{j_n} \ldots A^{i_1}B^{j_1}(1), \; n \in \mathbf{N}, \tag{1.5}$$

with $i_n \in \mathbf{N}_0 \equiv \mathbf{N} \cup \{0\}$, $j_n, i_{n-1}, \ldots, i_1, j_1 \in \mathbf{N}$, then

$$(N-1)_Z = \sum_{k=1}^{s(N)} F_{2k + \sum_{l=1}^{k} r_l(N)} =: F((N)_W, 0), \tag{1.6}$$

where $s(N) = \sum_{k=1}^{n} j_k$ is the total number of B's in the Wythoff string $(N)_W$, and the sequence $\{r_l(N)\}$, for $l = 1, \ldots, s(N)$ is read from the string $(N)_W$, from left to right, as follows

$$
\begin{array}{ll}
 & r_1 = i_n, \\
j_n \geq 2: & r_2 = r_3 = \ldots = r_{j_n} = 0, \\
 & r_{j_n+1} = i_{n-1}, \\
j_{n-1} \geq 2: & r_{j_n+2} = \ldots = r_{j_n+j_{n-1}} = 0, \\
 & \cdot \\
 & \cdot \\
 & \cdot \\
 & r_{j_n+j_{n-1}+\ldots+j_2+1} = i_1, \\
j_1 \geq 2: & r_{j_n+j_{n-1}+\ldots+j_2+2} = r_{s(N)} = 0.
\end{array}
\tag{1.7}
$$

For $N = 1$, $s(1) = 0$ and $(1)_W = A(1)$, $(0)_Z = 0$.

Part ZW) If, for $N \in \{2,3,4,\ldots\}$,

$$
\begin{aligned}
(N)_Z &= \sum_{i=0}^{t(N)} f_i F_{i+2},\ f_i \in \{0,1\},\ f_i f_{i+1} = 0,\ f_{t(N)} = 1, \\
&\equiv f_{t(N)} \cdots f_0 \cdot, \\
&= 10^{k_{m(N)}}1 \ldots 0^{k_3}10^{k_2}10^{k_1} \cdot =: Z(N),\ m(N) \in \mathbf{N},\ k_1 \in \mathbf{N}_0,\ k_2, \ldots, k_{m(N)} \in \mathbf{N},
\end{aligned}
\tag{1.8}
$$

then

$$
(N+1)_W = A^{k_1} B A^{k_2 - 1} B A^{k_3 - 1} \ldots B A^{k_{m(N)} - 1} B(1). \tag{1.9}
$$

Note that $k_1 = 0$ means no zero at the end of the Zeckendorf string of N and no A at the end of the Wythoff composition of $N+1$. For $N = 0$, or $(0)_Z = 0$, one takes $(1)_W = A(1)$. For $\mathrm{N} = 1$, $k_1 = k_{m(1)} = 0$ and $(2)_W = B(1)$.

Identities (1.3) and (1.4) become now special cases of this theorem. Its WZ) *part* invites one to cut off, from the lefthand side of the $W(N)$ string, either the substring $1^{i_n}0$, if $i_n \geq 1$, and record $r_1 = i_n$, or to cut off j_n times 0, if $W(N)$ starts with 0, and record 0 for the first j_n entries of the $\{r_l\}$ sequence. This cutting process is then continued. *E.g.* For $W(N) = 1^3 0^2 1^1 0^3$ (standing for $(N)_W = A^3 B^2 A^1 B^3(1)$), the $\{r_l\}_1^5$ sequence is $\{3,0,1,0,0\}$, and $5 = s(N)$, the number of 0's in $W(N)$ (or B's in $(N)_W$). Therefore,

$$(N-1)_Z = F_{2+3} + F_{4+3+0} + F_{6+3+0+1} + F_{8+3+0+1+0} + F_{10+3+0+1+0+0}$$
$$= F_5 + F_7 + F_{10} + F_{12} + F_{14}.$$

The knowledge of the number N (which happens to be 595 is this example) is not of importance. The converse ZW) *part* of the theorem reads for this example: $Z(N-1) = 101010^2 1010^3 \cdot$, with the $\{k_l\}$ sequence $\{3,1,2,1,1\}$ and $m(N) = 5$, the number of $0-$blocks in $Z(N-1)$. Therefore, $W(N) = 1^3 01^0 01^1 01^0 01^0 0 = 1^3 0^2 10^3$, or $(N)_W = A^3 B^2 A B^3(1)$. Note again that 1^0 means no 1 is present, corresponding to the identity map $A^0 = id$.

We now list some corollaries of this theorem (proved in *section 4*), where our interest is in the WZ) *part.*

$$A(N) - 1 = F(N,1) := \sum_{k=1}^{s(N)} F_{1 + 2k + \sum_{l=1}^{k} r_l(N)}. \tag{1.10}$$

Using (1.6) this implies that a shift of $+1$ in all indices of the Zeckendorf F's of $(N-1)_Z$ produces $A(N)-1$.*

$$B(N) - 2 = F(N,2) := \sum_{k=1}^{s(N)} F_{2 + 2k + \sum_{l=1}^{k} r_l(N)}. \tag{1.11}$$

It will be shown in *section 4* that the generalized Zeckendorf sum

$$F(N,m) := \sum_{k=1}^{s(N)} F_{m + 2k + \sum_{l=1}^{k} r_l(N)} \tag{1.12}$$

satisfies a Fibonacci recursion relation

$$\forall N \in \mathbf{N},\ m = 2,3,\ldots,:\ \ F(N,m+1) = F(N,m) + F(N,m-1), \tag{1.13a}$$

with inputs

$$F(N,0) = N-1,\ F(N,1) = A(N) - 1. \tag{1.13b}$$

It is therefore a generalized Fibonacci sequence and can be written as *

$$F(N,m) = A(N)F_m + NF_{m-1} - F_{m+1}. \tag{1.14}$$

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*That a shift in the indices of the Fibonacci numbers which partition $N-1$ produces the Zeckendorf representation of $A(N)-1$, and a further shift produces $B(N)-2$ is well known. See *e.g.* ref. [20], p. 111.

*This generalized Fibonacci sequence is related to the sequence found in ref.[1], $S^*_{k+1}(N) = F(N,k) + 1$.

Therefore, the alternative representation (1.12) is the Zeckendorf representation of (1.14). Combinations like (1.14) occur in ref [3]. In fact, our theorem will later be proved relying on the following lemma.

Lemma 1: (Carlitz, Scoville, Hoggatt, Jr. [3])

$$\forall N \in \mathbf{N},\ \forall k \in \mathbf{N}_0:\ \ A^kB(N)-1 = A(N)F_{k+2}+(N-1)F_{k+1}. \tag{1.15}$$

This is eq. (6.15), p. 22 of ref. [3] with $a \to A$, $b \to B$, $n \to N$, $j = 1$.

The *Theorem* was originally found after reduction of the following six sets of identities which arose (after some rewriting) in the context of certain Fibonacci chains [14]. They express the self-similarity of the underlying binary substitution sequence $\{h(n)\}$.

$$A^k(B(N)+1)-A^{k+2}(N) = F_{k+3}, \tag{1.16a}$$

$$BA^k(B(N)+1)-BA^{k+2}(N) = F_{k+5}, \tag{1.16b}$$

$$A^kB(B(N)+1)-A^{k+4}(N) = F_{k+2}+F_{k+5}, \tag{1.17a}$$

$$BA^kB(B(N)+1)-BA^{k+4}(N) = F_{k+4}+F_{k+7}, \tag{1.17b}$$

$$A^kB(A(N)+1)-A^{k+3}(N) = L_{k+3}, \tag{1.18a}$$

$$BA^kB(A(N)+1)-BA^{k+3}(N) = L_{k+5}. \tag{1.18b}$$

These identities hold for $k \in \mathbf{N}_0$, $N \in \mathbf{N}$ and follow from stronger identities proved later on as corollaries of the *Theorem*:

$$A^k(N)-1 = F(N,k), \tag{1.19a}$$

$$B^k(N) = F_{2k+1}+F(N,2k), \tag{1.19b}$$

$$BA^k(N)-2 = F(N,k+2), \tag{1.19c}$$

$$A^k(B(N)+1) = F_2+F_{k+3}+F(N,k+2), \tag{1.19d}$$

$$BA^k(B(N)+1) = F_3+F_{k+5}+F(N,k+4), \tag{1.19e}$$

$$A^kB(B(N)+1) = F_2+F_{k+2}+F_{k+5}+F(N,k+4), \tag{1.19f}$$

$$BA^kB(B(N)+1) = F_3+F_{k+4}+F_{k+7}+F(N,k+6), \tag{1.19g}$$

$$A^kB(A(N)+1) = F_2+L_{k+3}+F(N,k+3), \tag{1.19h}$$

$$BA^kB(A(N)+1) = F_3+L_{k+5}+F(N,k+5). \tag{1.19i}$$

Note that, in general, the *r.h.s.* is not of the canonical Zeckendorf type. However, these 'pre-Zeckendorf' sums will be used to prove identities (1.16) ff.

All identities (1.19), and many more, follow from repeated application of the *Theorem* using the following lemmas, valid for $N \in \mathbf{N}$, $m \in \mathbf{N}_0$.

Lemma 2a: $$F(A(N), m) = F(N, m+1), \tag{1.20a}$$

Lemma 2b: $$F(B(N), m) = F_{m+2} + F(N, m+2), \tag{1.20b}$$

Lemma 3a: $$F(A(N)+1, m) = F_{m+2} + F(N, m+1), \tag{1.21a}$$

Lemma 3b: $$F(B(N)+1, m) = F_{m+3} + F(N, m+2), \tag{1.21b}$$

In *Fig. 2* the *Theorem* is visualized by depicting a special Fibonacci tree with labeled nodes (corresponding to the Zeckendorf representation) and labeled edges (corresponding to the Wythoff representation). In both cases one reads from bottom to top. The meaning of the edge labels will be explained in *section 2*.

2. ZECKENDORF AND WYTHOFF REPRESENTATIONS

This section serves to fix our notation and recalls some known facts. The unique *Zeckendorf representation* (in its canonical form of the first kind) of a natural number $N \in \mathbf{N}$ [22, 4, 2] has been given in eq. (1.8). The second line of (1.8) is its shorthand notation, a string of 0's and 1's with no adjacent 1's. The third line in (1.8) is yet another form of the same representation. *E.g.* $10^2 10^3 1\cdot$ stands for $10010001\cdot$, which denotes the Zeckendorf representation for $N = 43$, since $(43)_Z = F_9 + F_6 + F_2$. Such representations can be depicted as branches of a Fibonacci tree with nodes labeled by 0's and 1's. See *Fig. 2* for the case of the tree T_7. 0's at the beginning of a path followed from the root at the top of the tree down to the bottom have to be omitted. The leftmost branch, the path with only 0's stands for $(0)_Z = 0$. As example, consider in *Fig. 2* the path 0100100 starting at the bottom. This is shortened to 01001, and stands for the Zeckendorf string $10010\cdot$, representing the number 10.

The fact that the branches of the n level Fibonacci tree T_n generated by the rule $0 \to 01$ and $1 \to 0$, starting with the root 0 at the first level (the top), depicts the Zeckendorf representation of the numbers 0 to $F_{n+1} - 1$ in an ordered manner is well known. Based on the recursive structure of T_n in terms of T_{n-1} and T_{n-2} this can be proved by induction on the level number n using $Z_n(k) = 0Z_{n-1}(k)$ for $k = 0, 1, \ldots, F_n - 1$, and $Z_n(F_n + k) = 01Z_{n-2}(k)$ for $k = 0, 1, \ldots, F_{n-1} - 1$. Here $Z_n(N)$ is the unshortened Zeckendorf string for N which has length n pertaining to T_n. *E.g.* $Z_7(12) = 0010101\cdot$. The inputs are $Z_1(0) = 0\cdot$ and $Z_0(0) = \cdot$.

For the *Wythoff representation* one writes a natural number as composition of the complementary Wythoff sequences $\{A(m)\}_1^\infty$ and $\{B(m)\}_1^\infty$ defined in (1.1). This unique representation with $(1)_W = A(1)$ has been studied in refs. [10, 9]. A given number $N \geq 2$ is written as string of A's and B's ending with $B(1)$, as in eq. (1.5). The same representation is also written as $1^{i_n}0^{j_n}\ldots1^{i_1}0^{j_1}$, where A, resp. B, is replaced by 1, resp. 0. In this case the argument (1) is implicit. The authors of refs. [10, 9] gave a 'Fibonacci composition array' for these Wythoff representations. (Here 'composition' means partition). This array can be depicted as a certain binary tree with different branch lengths and some final leaves of this tree cut.

We prefer to work with a Fibonacci tree. For this purpose we have stretched some branches and shortened one in order to obtain equal length for all F_{n+1} branches of the Fibonacci tree T_n. Consider *Fig. 2* as example for the tree T_7. The edges are labeled with $L^2, L, L\bullet$ or R. L, resp. R, stands for *left*, resp. *right*, and will later be replaced by A, resp. B, in the Wythoff composition of a number. The $L\bullet$ edges have been introduced to stretch the branches of the array of refs. [10, 9]. If one follows a branch from bottom to top and if one arrives at a $L\bullet$ edge from the right (*i.e.* if the last edge label was R) one does not record this edge label (*i.e.* the $\bullet$) and follows the branch further to the top. If one arrives at such an edge from the left, an L is recorded. A branch starting at the bottom with a 0 node continues upwards with an edge L.

For example, the branch ending over the number 13 is the W line in *Fig. 2* reads from bottom to top $R\bullet R\bullet RL$, or $RRRL$. This translates to a Wythoff composition $BBBA(1)$. Because $A(1) = 1$, such compositions are brought to the unique Wythoff form by omitting$A$'s acting on 1. Therefore, the correct Wythoff representation corresponding to this branch is $(13)_W = BBB(1)$. Or, written as a string of 1's and 0's, $W(13) = 000$. The node labels along this branch correspond to the Zeckendorf representation $Z_7(12) = 0010101\cdot$ which becomes in the shortened form $Z(12) = 10101\cdot$.

We give another example for the T_7 Fibonacci tree. Over number 14 in the W line of *Fig. 2* the branch $LLLLLR$ starts. This corresponds to $(14)_W = A^5B(1)$, or $W(14) = 1^50$. The node labels along this branch correspond to $Z_7(13) = 0100000\cdot$, *i.e.* $Z(13) = 10^5\cdot$. Similarly, for 3 one has $LRLLLL$, or $AB(1)$, which is 10.

Note that in the tree T_n each unshortened edge sequence of a branch is equivalent to a partition of n with only 1's and 2's without regard of order. In this case one replaces L, resp. R, by 1, resp. 2. This explains why the leftmost branch has no labels $L\bullet$ and its last leaf is labeled with L^2. This branch corresponds to $(1)_W = A(1)$ in the shortened form.

If the partitions of N into 1's and 2's are not of interest it is convenient to replace all edge labels of the leftmost branch of the T_n tree with $L\bullet$ labels. This is, for instance, useful to find for the Zeckendorf tree T_n with given node labels the edge labels with the help of the *ZW) part* of the *Theorem*. For this purpose define the unshortened Wythoff strings $W_n(N)$ of an ascending edge sequence of length $n-1$ corresponding to a number $N \in \{1, 2, \ldots, F_{n+1}\}$. These strings are composed of L, R and $\bullet$ symbols. *E. g.* $W_7(8) = LR\bullet R\bullet L$. For such N the general Zeckendorf string $Z(N)$ given in (1.8) is rewritten as unshortened T_n node string $(Z_n(N))_{reversed} = \cdot 0^{k_1}100^{k_2-1}\ldots 100^{k_{m(N)}-1}100^{v(N)-1}$. Here $v(N) = n - m(N) - \sum_{i=1}^{m(N)} k_1 \geq 1$ dummy zeros have been amended in order to have string length n. This reversed T_n node sequence is read from bottom to top. According to (1.9) it translates to the unshortened edge sequence $W_n(N+1) = L^{k_1}R\bullet L^{k_2-1}\ldots R\bullet L^{k_{m(N)}-1}R\bullet L^{v(N)-2}$ if $v(N) \geq 2$. For $v(N) = 1$ this string ends with the last R. We used L for A, and R for B. This translation proves that a node 1 at any level of T_n is always followed upwards by an edge R. A node 0 at some level has an upwards edge L if it is, in the considered branch, not connected downwards to a 1 node. This applies especially to the 0 nodes at the lowest level n of T_n. If in a considered branch a 0 node is connected downwards to a 1 node it carries an upwards $\bullet$ edge. This proves that the rules to label the edges of the Zeckendorf tree T_n with given node labels are exactly the ones which were stated above in connection with *Fig. 2*.

3. PROOF OF THE THEOREM

The starting point is *Lemma 1*, eq. (1.15). Next we show that the r_l sequence defined in eq. (1.7) from the Wythoff representation of N, called $(N)_W$, given by eq. (1.5) can be defined recursively as follows.

Definition 1: Define the sequence $\{(N_l)_W\}_{l=0}^{s(N)}$ and the corresponding sequence $\{r_l\}_{l=1}^{s(N)}$ recursively by

$$(N_l)_W = A^{r_l+1}B(N_{l+1})_W, \quad l = 0, 1, \ldots, s(N) - 1 \tag{3.1}$$

with $N_0 := N \in \mathbf{N}$, and $s(N)$ is defined, for $N \geq 2$, by $N_{s(N)} = 1$. If $N_0 = N = 1$ then one puts

$s(N = 1) = 0$.

It is clear that for $N \in \{2, 3, 4, \ldots\}$ the sequence $\{r_l\}_1^{s(N)}$, defined in (3.1), can be read from $(N)_W$, written as in (1.5), and it is given by (1.7). The formula $s(N) = \sum_{k=1}^{n} j_k$, with the j_k's defined for N in (1.5), is obtained this way.

Proof of the WZ) part

Lemma 4: For $n \in \{2, 3, 4, \ldots\}$ and the sequence $\{r_l\}_1^{s(N)}$ defined by (3.1),

$$F((N)_W, 0) := \sum_{k=1}^{s(N)} F_{2k + \sum_{l=1}^{k} r_l(N)} \tag{3.2}$$

is the Zeckendorf representation of some number.

This is clearly true because the sequence $\{r_1\}$ defined in (1.7) is non-negative. Therefore no neighboring Fibonacci numbers appear. Also, only F_k with $k \geq 2$ enter this sum.

Lemma 5:

$$F_m F_{n+1} + F_{m-1} F_n = F_{n+m}, \quad m, n \in \mathbf{N}. \tag{3.3}$$

This is identity (I_{26}) of ref. [8].

Definition 2: With $N_l \equiv (N_l)_W$ given recursively by (3.1) and with $R_l := \sum_{k=1}^{l} (2 + r_k)$ define for $l \in \{1, 2, \ldots, s(N)\}$

$$Q_l \equiv Q(N_l, R_l) := A(N_l) F_{R_l} + (N_l - 1) F_{R_l - 1}. \tag{3.4}$$

Due to *Lemma 1*, $Q_l = A^{R_l - 2} B(N_l) - 1$.

Proposition 1: For $l \in \{1, 2, \ldots, s(N) - 1\}$

$$Q_l = F_{R_l} + Q_{l+1}. \tag{3.5}$$

The proof uses *Lemmas 1* and *5*. With definition (3.1) one has $Q_l = AA^{r_{l+1}+1} B(N_{l+1}) F_{R_l} + (A^{r_{l+1}+1} B(N_{l+1}) - 1) F_{R_l - 1}$, which becomes, after using *Lemma 1* with $k = r_{l+1} + 1$ and $k = r_{l+1}$,

$$\begin{aligned} &= F_{R_l} + (A(N_{l+1}) F_{3 + r_{l+1}} + (N_{l+1} - 1) F_{2 + r_{l+1}}) F_{R_l} + \\ &\quad + (A(N_{l+1}) F_{2 + r_{l+1}} + (N_{l+1} - 1) F_{1 + r_{l+1}}) F_{R_l - 1} \\ &= F_{R_l} + (F_{R_l} F_{3 + r_{l+1}} + F_{R_l - 1} F_{2 + r_{l+1}}) A(N_{l+1}) + \\ &\quad + (F_{R_l} F_{2 + r_{l+1}} + F_{R_l - 1} F_{1 + r_{l+1}})(N_{l+1} - 1). \end{aligned} \tag{3.6}$$

With *Lemma 5* for $m = R_l$, $n = 2 + r_{l+1}$ and $m = R_l$, $n = 1 + r_{l+1}$, this is

$$= F_{R_l} + A(N_{l+1})F_{2+r_{l+1}+R_l} + (N_{l+1} - 1)F_{1+r_{l+1}+R_l}$$
$$= F_{R_l} + Q_{l+1}. \tag{3.7}$$

Proposition 2:

$$\forall N \in \{2,3,4,\ldots\} : (N)_W - 1 = F((N)_W, 0), \tag{3.8}$$

with $F((N)_W, 0)$ defined as in (3.2).

The proof uses *Lemma 1* and *Proposition 1.* We start with $l = 0$ in (3.1), i.e.

$$(N)_W - 1 = (N_0)_W - 1 = A^{r_1}B(N_1)_W - 1 = Q_1$$

by *Lemma 1* and *Definition 2.* Now *Proposition 1* can be applied iteratively for $l = 1, 2, \ldots, s(N) - 1$ until one reaches $N_{s(N)} = 1$. Because $Q_{s(N)} \equiv Q(1, R_{s(N)}) = A(1)F_{R_{s(N)}}$ $= F_{R_{s(N)}}$ one finds, using also the R_k definition

$$(N)_W - 1 = \sum_{k=1}^{s(N)} F_{R_k} = \sum_{k=1}^{s(N)} F_{2k + \sum_{l=1}^{k} r_l} = F((N)_W, 0).$$

By *Lemma 4*, using $(N)_W = N$, we now have $F((N)_W, 0) = (N-1)_Z$.

This concludes the proof of the *WZ) part* of the *Theorem.*

Proof of the ZW) part

Lemma 6: The Zeckendorf representation of N, symbolized by $Z(N) = 10^{k_{m(N)}}1\ldots0^{k_3}10^{k_2}10^{k_1}$, with $N \in \{2,3,4,\ldots\}$, $m(N) \in \boldsymbol{N}$, $k_1 \in \boldsymbol{N}_0$, $k_2, \ldots, k_{m(N)} \in \boldsymbol{N}$, can be written as

$$(N)_Z = \sum_{k=1}^{m(N)} F_{k+1+\sum_{l=1}^{k} k_l}. \tag{3.9}$$

For the proof one counts the positions of the 1's in the Zeckendorf string $Z(N)$. For $N = 1$ one has $(1)_Z = F_2$, *i.e.* $k_1 = k_{m(1)} = 0$.

Definition 3: Given the sequence $\{k_l\}_{l=1}^{m(N)}$ which defines $Z(N)$, define the sequence $\{r_l\}_{l=1}^{m(N)}$ by

$$r_1 = k_1 \geq 0,\ r_l = k_l - 1 \geq 0,\ l = 2, 3, \ldots, m(N). \tag{3.10}$$

Lemma 7:

$$N \in \{2,3,4,\ldots\} : (N)_Z = \sum_{k=1}^{m(N)} F_{2k+\sum_{l=1}^{k} r_l}. \tag{3.11}$$

For $N=1$ one uses $r_1 = k_1 = k_{m(1)} = 0$. For the proof one replaces in each term of the sum (3.9) the k_l's by the r_l's defined in (3.10).

Lemma 8: With $R_l := \sum_{k=1}^{l}(r_k + 2)$, for $l = 1, \ldots, m(N)$ one has

$$(N)_Z = \sum_{l=1}^{m(N)} F_{R_l}. \tag{3.12}$$

Proof: Rewrite $(N)_Z$ of (3.11) in the R_l variables.

Definition 4: For $l = 0, 1, \ldots, m(N)-1$ define recursively with $\{r_l\}_1^{m(N)}$ given in (3.10)

$$M_l = A^{r_l+1} B M_{l+1},\ M_{m(N)} = 1. \tag{3.13}$$

Lemma 9:

$$M_l = A^{r_l+1} B \cdots A^{r_{m(N)}} B(1). \tag{3.14}$$

This is the solution to the recursion (3.13).

Definition 5: For $l = 1, \ldots, m(N)$, M_l of (3.14) and R_l of *Lemma 9* define

$$P_l \equiv P(M_l, R_l) := A(M_l) F_{R_l} + (M_l - 1) F_{R_l - 1}. \tag{3.15}$$

Proposition 3:

$$l \in \{1, 2, \ldots, m(N)-1\} : P_l = F_{R_l} + P_{l+1}. \tag{3.16}$$

The proof uses (3.13), *Lemma 1* with $k = r_{l+1}+1$ and $k = r_{l+1}$, and *Lemma 5* with $m = r_{l+1}+3$, $n = R_l - 1$ and $m = r_{l+1}+2$, $n = R_l - 1$.

$$\begin{aligned} P_l &= F_{R_l} + (A(M_l)-1)F_{R_l} + (M_l - 1)F_{R_l - 1} \\ &= F_{R_l} + (A(M_{l+1})F_{r_{l+1}+3} + (M_{l+1}-1)F_{r_{l+1}+2})F_{R_l} + \\ &\qquad + (A(M_{l+1})F_{r_{l+1}+2} + (M_{l+1}-1)F_{r_{l+1}+1})F_{R_l - 1} \\ &= F_{R_l} + A(M_{l+1})F_{R_l + r_{l+1}+2} + (M_{l+1}-1)F_{R_l + r_{l+1}+1} \\ &= F_{R_l} + P_{l+1} \end{aligned}$$

Lemma 10:

$$\forall N \in \{2,3,4,\ldots\} : M_0 - 1 = A^{r_1} B \cdots A^{r_{m(N)}} B(1) - 1 = \sum_{l=1}^{m(N)} F_{R_l}. \tag{3.17}$$

From the definition (3.13), *Lemma 1* with $k = r_1$, $N = M_1$ and *Definition 5* one has $M_0 - 1 = P_1$. The lemma then follows after repeated application of *Proposition 3* for

$l = 1, 2, \ldots, m(N) - 1$, using the fact that, due to (3.13) and (3.15), $P_{m(N)} = 1 \cdot F_{R_{m(N)}} + 0$.

Lemma 11:

$$M_0 = A^{r_1}BA^{r_2}B \cdots A^{r_{m(N)}}B(1) = A^{k_1}BA^{k_2 - 1}B \cdots A^{k_{m(N)} - 1}B(1)$$

$$= (N)_Z + 1 = (N+1)_W. \tag{3.18}$$

First use definition (3.13), then rewrite the r_l variable in terms of the k_l ones using (3.10). After this apply *Lemma 10* and *Lemma 8*. The last equality in (3.18) holds because $(N)_Z + 1 = N + 1$ and it is written in the Wythoff representation. The statements made after (1.9) concerning the cases $N = 0$ and $N = 1$ are obvious.

This concludes the proof of *Part ZW)* of the *Theorem*.

4. PROOF OF LEMMAS 2a, 2b, eq. (1.13), LEMMAS 3a, 3b, IDENTITIES (1.19) AND (1.16) TO (1.18)

As corollaries to the *Theorem* we prove in this section the identities (1.16) to (1.18) which were found independently as a consequence of the self-similarity of the binary Fibonacci substitution sequence $\{h_n\}$ in ref. [14]. To this end we first prove the more general identities 1.19a to i). These identities involve the quantity $F(N, m)$ which is defined in (1.12) from the unique Wythoff representation of N, *i.e.* $(N)_W$ given by (1.5) with the sequence $\{r_l\}_1^{s(N)}$($r(N)$ – sequence for short) obtained from (1.7).

Proof of Lemma 2a and 2b:

These lemmas use *Part WZ)* of the *Theorem* (1.6). (1.20a), *resp.* (1.20b), is proved by comparing the $r(A(N))$ –, *resp.* $r(B(N))$ – sequence, with the $r(N)$ – sequence. If N is replaced by $A(N)$ in (1.5) the $r(A(N))$ – and $r(N)$ – sequences satisfy:

$$r_1(A(N)) = r_1(N) + 1,\ r_k(A(N)) = r_k(N) \text{ for } k = 2, 3, \ldots, s(N).$$

Hence, $s(A(N)) = s(N)$. Therefore, the index of each term of the sum $F(A(N), m)$ can be rewritten, producing the sum for $F(N, m+1)$. This proves *Lemma 2a*.

Similarly, if N is replaced by $B(N)$ in (1.5) one finds

$$r_1(B(N)) = 0,\ r_k(B(N)) = r_{k-1}(N) \text{ for } k = 2, 3, \ldots, s(N) + 1.$$

Hence, $s(B(N)) = s(N) + 1$. The first term in the sum (1.12) for $F(B(N), m)$ is F_{m+2}. The remaining sum is, after an index-shift $k \to k-1$ brought to the form of $F(N, m+2)$. This proves

Lemma 2b.

Next we show that the Zeckendorf sum $F(N,m)$, given by (1.12), satisfies the generalized Fibonacci number recursion (1.13).

Proof of eqs. (1.13 a,b):

(1.13a) is obvious because there N, hence the $r(N)-$sequence, is held fixed. One only needs the ordinary Fibonacci number recursion relation for each term of the sum $F(N,m)$. The inputs (1.13b) follow from the *Theorem*, (1.6), and (1.10) which is a consequence of *Lemma 2a*.

The result (1.14) then follows because $F(N,m) = F(N,1)F_m + F(N,0)F_{m-1}$.*

Proof of Lemmas 3a and 3b:

From (1.14) and the well-known identity $A(A(N)+1) = A^2(N)+2$ one finds $F(A(N)+1,m) = F(A(N),m) + 2F_m + F_{m-1}$. Applying *Lemma 2a* now yields *Lemma 3a*.

Similarly, from the well-known identity $A(B(N)+1) = AB(N)+1$ one finds from (1.14) first $F(B(N)+1,m) = F(B(N),m) + F_m + F_{m-1}$. Applying *Lemma 2b* now yields *Lemma 3b*.

Proof of identities (1.19):

These identities are corollaries to the *WZ) part* (1.6) of the *Theorem*. (1.19a), *resp.* (1.19b), follow after successive application of (1.6) together with (1.20a), *resp.* (1.20b). In the case of (1.19b) one first finds $(B^k(N)-1)_Z = F(N,2k) + \sum_{i=0}^{k-1} F_{2+2i}$. Then only the well-known formula $\sum_{j=1}^{k} F_{2j} = -1 + F_{2k+1}$ has to be used.

Eqs. (1.10), *resp.* (1.11), are the $k=1$ cases of (1.19a), *resp.* (1.19b).

The proof of (1.19c) just needs $F(BA^k(N),0) = F_2 + F(A^k(N),2) = F_2 + F(N,2+k)$ due to (1.20b) and (1.20a).

For the proof of (1.19d) one uses (1.6) and *Lemmas 2a* and *3b*: $F(A^k(B(N)+1),0) = F(B(N)+1,k) = F(N,k+2) + F_{k+3}$.

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*The generalized Fibonacci sequences which start with primitive Wythoff pairs [17] $(A(A(k)), B(A(k)))$, $k = 1,2,3,\ldots$ are therefore $F(B(k),m)$, for $m = 0,1,2,\ldots$. Hence the Wythoff array [15] has entries $W_{m,n} = F(B(m),n-1)$ with $m,n \in \mathbf{N}$. *E.g.* $W_{3,4} = F(7,3) = F_{3+2+0} + F_{3+4+1} = F_5 + F_8 = 26$.

Similarly, (1.19e) needs *Lemmas 2b, 2a, 3b:*
$F(BA^k(B(N)+1),0) = F_2 + F(B(N)+1,k+2) = F_2 + F_{k+5} + F(N,k+4)$.

In the same way the other identities (1.19f to i) follow. The Lucas numbers $L_k = F_{k-1} + F_{k+1}$ also show up.

Note that (1.19b) is the Zeckendorf representation of $B^k(N)$ iff $r_1 \neq 0$, *i.e.* for $A-$numbers N. This is so because $F(N,2k)$ starts with F_{2k+2} if $r_1 = 0$. Of course, $F(N,m)$ is always a Zeckendorf representation. For example,
$(A^k(B(N)+1) - 1 - F_{k+3})_Z = F(N,k+2)$.

Also, $(A^kB(A(N)+1) - 1 - F_{k+4})_Z = F_{k+2} + F(N,k+3)$ from (1.19h).

Proof of identities (1.16), (1.17) and (1.18):

These identities, which can be derived from the self-similarity identities obeyed by Fibonacci chains (based on the self-similarity of the $\{h(n)\}$ sequence), follow immediately from the stronger results (1.19) by subtracting out the relevant Zeckendorf sums $F(N,m)$.

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NOTE ADDED

After completion of this work the author became aware of ref. [13], and he would like to thank Dr. Kimberling for sending him this preprint. The footnote in *section 4* of the present work, together with Lemma 2b, shows that $W(i,j) = F_{j+1} + F(i,j+1)$, which gives the j-th column of the Zeckendorf array $Z(i,j)$ of ref. [13].

The author likes to thank Drs. A.S. Fraenkel and K.B. Stolarsky for pointing out references, especially ref. [2], where an up to date list of references can be found.

In a letter Dr. T.C. Brown proposed to try an alternative proof of the theorem based on the recursive structure of the Wythoff-Zeckendorf tree T_n shown in *Fig. 2* for $n = 7$.

APPENDIX: THE W(N) ALGORITHM

$W(N)$, the Wythoff string corresponding to the Wythoff representation $(N)_W$ of N (see (1.5)), can be computed from the sequences $\{h(n)\}_1^N$ and $\{z(n)\}_1^N$, where $z(n) := \sum_{k=1}^{n} h(k) = A(n+1) - (n+1)$ counts the number of $A-$numbers not exceeding n. The point is that $z(A(k)) = k$, for $k \in \boldsymbol{N}$, and $z(B(k)) = A(k)$, for $k \in \boldsymbol{N}_0$. Therefore, $z^2(B(k)) \equiv z(z(B(k))) = k$, for $k \in \boldsymbol{N}$. (No confusion with $Z(n)$, the Zeckendorf string for n, should arise.) $A(n)$, $B(n)$, and $z(n)$ are called $a(n)$, $b(n)$, and $e(n)$, respectively, in ref [3].

Now start with some $N \in \boldsymbol{N}$. Record as first entry of the string $W(N)$ (the leftmost entry) a 1, *resp.* a 0, if $N = A(\widehat{N})$, *resp.* $N = B(\widehat{N})$. Thus the first entry is $h(N)$. Then continue this procedure for $\widehat{N}$ in place of$N$, *i.e.* compute $z(N)$, *resp.* $z^2(N)$, if $h(N)$ was 1, *resp.* 0. etc.

Continue this process until $z(.)$ or $z^2(.)$ becomes 1.
E.g. $N = 58$, $h(58) = 1$, $z(58) = 36$, $h(36) = 0$, $z^2(36) = 14$, $h(14) = 1$, $z(14) = 9$, $h(9) = 1$, $z(9) = 6$, $h(6) = 1$, $z(6) = 4$, $h(4) = 1$, $z(4) = 3$, $h(3) = 1$, $z(3) = 2$, $h(2) = 0$, $z^2(2) = 1$. Hence $W(58) = 10111110 = 101^5 0$, or $(58)_W = ABA^5B(1)$. The $r(58)-$sequence, defined in (1.7), is $\{1,5\}$, and $s(58) = 2$. Due to the *WZ) part* of the *theorem* $(57)_Z = F_{2+1} + F_{4+6} = F_3 + F_{10}$.

FIG. 1 shows a structural computer program for the computation of $W(N)$.

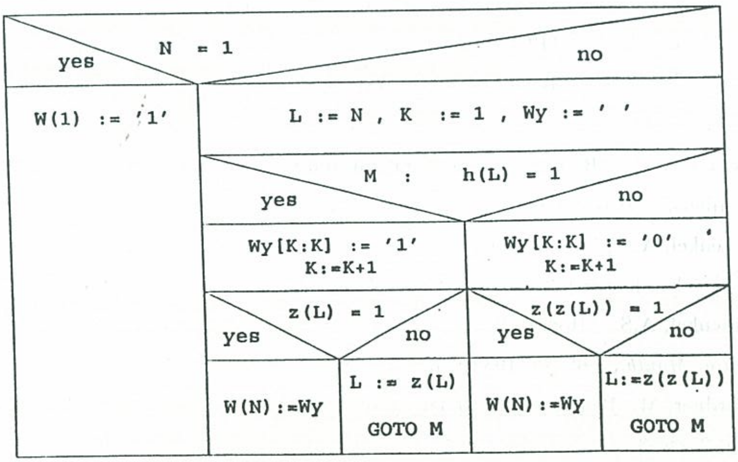

FIG 1: The W(N) Algorithm

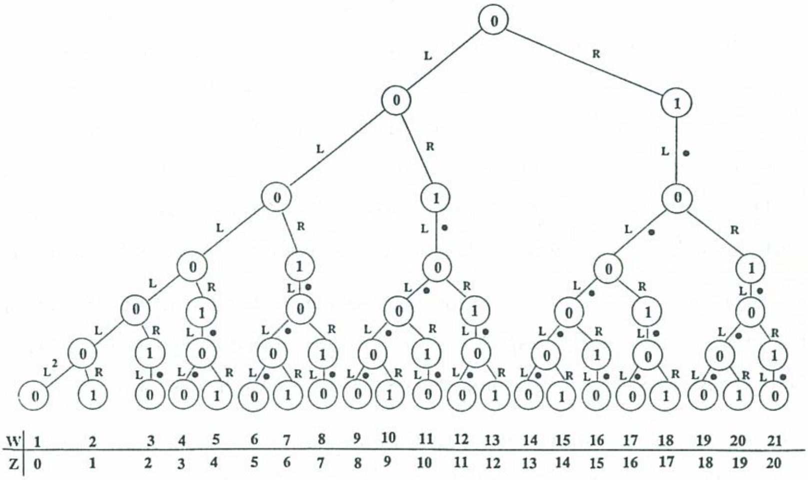

Figure 2: The Wythoff-Zeckendorf Tree $(WZ)_7$

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