ON THE ORDINARY GENERATING FUNCTION OF A307684

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The purpose of this note is to validate the conjecture regarding the ordinary generating function of the integer sequence

$$a_n = \sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} \sum_{i=k}^{\lfloor \frac{n-k}{2} \rfloor} ik(n-i-k),$$

registered as A307684 in the On-Line Encyclopedia of Integer Sequences (OEIS) [2]. Using Faulhaber's formula up to order 5 (e.g., [1, p. 106]), we have

$$\sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} \sum_{i=k}^{\lfloor \frac{n-k}{2} \rfloor} ik(n-i-k)$$

$$= \sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} \sum_{i=k}^{\lfloor \frac{n-k}{2} \rfloor} ikn - \sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} \sum_{i=k}^{\lfloor \frac{n-k}{2} \rfloor} i^2k - \sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} \sum_{i=k}^{\lfloor \frac{n-k}{2} \rfloor} ik^2$$

$$= \sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} \left(\frac{k}{6} \left\lfloor \frac{n-k}{2} \right\rfloor \left(\left\lfloor \frac{n-k}{2} \right\rfloor + 1 \right) \left(3(n-k) - 2 \left\lfloor \frac{n-k}{2} \right\rfloor - 1 \right) + \frac{k^2(k-1)(5k-3n-1)}{6} \right)$$

$$= \begin{cases} \frac{1}{30}s - \frac{3}{4}s^2 + \frac{7}{6}s^3 + \frac{27}{4}s^4 + \frac{54}{5}s^5 & \text{if } n = 6s \\ \frac{1}{5}s + \frac{5}{4}s^2 + 6s^3 + \frac{63}{4}s^4 + \frac{54}{5}s^5 & \text{if } n = 6s+1 \\ \frac{7}{10}s + \frac{25}{4}s^2 + \frac{39}{2}s^3 + \frac{99}{4}s^4 + \frac{54}{5}s^5 & \text{if } n = 6s+2 \\ 1 + \frac{241}{30}s + \frac{103}{4}s^2 + \frac{125}{3}s^3 + \frac{135}{4}s^4 + \frac{54}{5}s^5 & \text{if } n = 6s+3 \\ 2 + \frac{157}{10}s + \frac{185}{4}s^2 + \frac{129}{2}s^3 + \frac{171}{4}s^4 + \frac{54}{5}s^5 & \text{if } n = 6s+4 \\ 7 + \frac{196}{5}s + \frac{349}{4}s^2 + 96s^3 + \frac{207}{4}s^4 + \frac{54}{5}s^5 & \text{if } n = 6s+5 \end{cases}$$

The ogf of the sequence $\left\{\frac{1}{30}s - \frac{3}{4}s^2 + \frac{7}{6}s^3 + \frac{27}{4}s^4 + \frac{54}{5}s^5\right\}_{s \geq 0}$ is given by

$$\frac{1}{30}\frac{x}{(1-x)^2} - \frac{3}{4}\frac{x(x+1)}{(1-x)^3} + \frac{7}{6}\frac{x(x^2+4x+1)}{(1-x)^4} +$$

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$$\frac{27}{4}\frac{x(x+1)(x^2+10x+1)}{(1-x)^5}+\frac{54}{5}\frac{x(x^4+26x^3+66x^2+26x+1)}{(1-x)^6}.$$

We now substitute x with x^6 and add the analogue ogfs of the remaining five cases, each multiplied by x^i, for $i = 1, 2, 3, 4$, and 5, where $n = 6s + i$. The resulting ogf is

$$\frac{x^3(6x^8+14x^7+18x^6+21x^5+23x^4+15x^3+7x^2+3x+1)}{(x-1)^6(x+1)^3(x^2+x+1)^4},$$

exactly what Barker conjectured.

References

[1] J. H. Conway and R. Guy, *The Book of Numbers*, Springer Science & Business Media, 1998.

[2] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., `https://oeis.org`.