SEQUENCE A307663

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Consider the sequence $\{a_n\}_{n\geq 1}$, registered in the OEIS as A307663. It is defined by

$$a_n = (n-1)!\sum_{i=1}^{n}\sum_{j=1}^{i}\binom{i}{j}\frac{i}{j}.$$

It is conjectured that the egf of the sequence, denoted by $A(x)$, is given by

$$\frac{(4x^2-8x+5)\ln\left(\frac{1-x}{1-2x}\right)}{2(1-x)^2}+\frac{x(5x-6)}{4(1-x)^2}.$$

In this note we prove this conjecture. We shall exploit the fact that the egf $B(x)$ of the sequence

$$b_n = n!\sum_{k=1}^{n}\binom{n}{k}\frac{1}{k},$$

registered in the OEIS as A307663, is given by

$$B(x) = \frac{\ln\left(\frac{1-x}{1-2x}\right)}{1-x}.$$

We have

$$\begin{aligned}
A(x) &= \sum_{n\geq 1}\frac{a_n}{n!}x^n\\
&= \sum_{n\geq 1}\sum_{i=1}^{n}\sum_{j=1}^{i}\binom{i}{j}\frac{i}{jn}x^n\\
&= \sum_{n\geq 1}\sum_{i=1}^{n} i\frac{b_i}{i!}\frac{x^n}{n}\\
&= \int_0^x \sum_{n\geq 1}\sum_{i=1}^{n} i\frac{b_i}{i!}t^{n-1}dt
\end{aligned}$$

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\begin{aligned}
&= \int_0^x \sum_{n\geq 1} \sum_{i=1}^{n} \left(\frac{b_i}{i!} t^i\right)' t^{n-i} dt \\
&= \int_0^x \frac{B'(t)}{1-t} dt \\
&= \frac{(4x^2 - 8x + 5) \ln\left(\frac{1-x}{1-2x}\right)}{2(1-x)^2} + \frac{x(5x-6)}{4(1-x)^2}.
\end{aligned}
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References

[1] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., `https://oeis.org`.