A039770 - Numbers n such that phi(n) is square (File)
Subfamilies and Subsequences of terms

Euler's totient function: phi(n).

In this file, it will be shown that different families of integers belong to the sequence A039770.

Remark: if $a(n) = p_1$^r_1...p_i^r_i..... p_m^r_m is a term of A039770, then, for each i=1,2,...,m, the number p_i^2 * a(n) is also in A039770.

Moreover, if $a(n) = p_1$^r_1...p_i^r_i..... p_m^r_m is a term of a subsequence of A039770, then, for each i=1,2,...,m, the integer p_i^2 * a(n) is also a term of the same subsequence.

There are "primitive" terms called α(n) = p_1^s_1...p_i^s_i..... p_m^s_m, with s_1, s_2,.., s_i,...,s_m = 1 or 2. These "primitive" terms generate an entire subsequence, or sequence, when multiply by p_i^2 .

P. Pollack and C. Pomerance have showed that almost all squares are missing from the range of Euler's phi-function (see link in A039770); the first missing squares {22^2, 34^2, 38^2, 46^2, 58^2, ...} are in A306882. See also A221284 - A221285 with square values taken by phi(k).

Definitions and Examples

I) If a(n) has only one prime factor

In this case, there is only one subsequence b(n), this subsequence contains exactly the prime powers p^{2k+1}, k >=0, where p is prime of the form $m^2 + 1$, so p being in A002496, and m in A005574. These integers b(n) are exactly the terms of the sequence A054755.
If $\mathbf{b(n) = p^{2k+1}}$ with $\mathbf{p = m^2 + 1}$, then $\mathbf{phi(b(n)) = (p^k * m)^2}$ with $\mathbf{m^2 = p - 1}$.

For p = 2, 5, 17, 37, the sequences 2^{2k+1}, 5^{2k+1}, 17^{2k+1} , 37^{2k+1} are respectively in A004171, A013710, A013722 and A262786, with $phi(p^{2k+1}) = (p^k)^2$.

Also, if p is prime of this form $m^2 + 1$, then $phi(p) = p - 1 = m^2$.
These primes in this case are exactly the "primitive" terms **β(n) = p = m² + 1** of this sequence b(n) and **phi(β(n))** = m^2**,** these primitives form the sequence A002496.
Fermat primes in A019434, except 3: {5, 17, 257, 65537} are then a subsequence of A002496 and, if $F_n = 2^{2^n} + 1$ is prime, then $phi(F_n) = 2^{2^n} = [2^{2^{(n-1)}}]^2$.

The new non primitive terms which appear in the general sequence are colored in green.
The "primitive" terms β(n) are exactly the terms of A002496: {2, 5, 17, 37, 197, ...}, and the first few terms b(n) of the sequence A054755 are: {2, 5, 8, 17, 32, 37, 101, 125, 128, ...}.

Examples:
phi(17) = 16 = 4^2; 5^3 = 125 and phi(125) = 10^2.

Remark: the numbers of this subsequence b(n) have both their totient and cototient (=1) which are square, so A054755 is also a subsequence of A063752, square cototients, and of A054754, integers whose the totient and cototient are square (see file in A063752 about square cototients).

II) If a(n) has two distinct prime factors

The new non primitive terms which appear in the general sequence will be colored in green.

These terms form the sequence A324745. The first few terms are {10, 12, 34, 40, 48, 57, 63, 74, 76, 85, 108, 136, 160, 185, 192, 202, 219, ...}. There are two subsequences in this case which are a partition of A324745, these two families are defined from their primitive representation and are described in the following sections.

2.1) Define the first by **$\gamma(n) = p * q$**, $p < q$ with **$\phi(\gamma(n)) = (p-1) * (q-1) = m^2$** is square.
The first "primitive" terms are {10, 34, 57, 74, 85, 185, 202, 219, ...} and form exactly the sequence A247129: semiprimes k such that phi(k) is square.

Couples (p,q) with p < q can be found by solving the Diophantine equation:
$(p-1)*(q-1) = m^2$, with p, q primes. Discussion:

If p = 2, then $q = m^2 + 1$, so, with q prime > 2 in A002496 and m in A005574.

Some solutions: the numbers m must be even, otherwise primes p and q would be both even.

- m = 2, (p-1)*(q-1) = 4, (p,q) = (2,5) with $1*4 = 2^2$,
- m = 4, (p-1)*(q-1) = 16, (p,q) = (2,17) with $1*16 = 4^2$,
- m = 6, (p-1)*(q-1) = 36, (p,q) = (2,37), (3,19) with $1*36 = 2*18 = 6^2$ (2 solutions),
- m = 8, (p-1)*(q-1) = 64, (p,q) = (5,17) with $4*16 = 8^2$,

The first values of m without solutions are: {22, 34, 38, 46, 58, 62, 68, ...} and the number of solutions for each even m is in A306722. Be careful, there is not p*q such that $\phi(p*q) = 68^2$ but there are numbers as 4913 such that $\phi(4913) = 68^2$, it's the same with 114, 128, 136, ...

Some values for (γ(n),p,q,m) = (10,2,5,2), (34,2,17,4), (57,3,19,6), (74,2,37,6), (85,5,17,8), (185,5,37,12), (202,2,101,10), ...

The general terms are **$c(n) = p^{2s+1} * q^{2t+1}$** with p < q, s,t >= 0, **$\phi(c(n)) = (p^s * q^t * m)^2$**, they form the sequence A324746.
The first few terms are {10, 34, 40, 57, 74, 85, 136, 160, 185, 202, 219, 250, ...}.

Examples:

$85 = 5 * 17$, $(5-1) * (17-1) = 8^2$, so $\phi(85) = 8^2$
$136 = 2^3 * 17^1$, $(2-1)*(17-1) = 4^2$ and $\phi(136) = (2^1 * 17^0 * 4)^2 = 8^2$.

2.2) Define the second by **$\delta(n) = p^2 * q$** with **$\phi(\delta(n) = p * (p-1) * (q-1) = m^2$** is square.
The first "primitive" terms are: {12, 63, 76, 292, 652, 873, ...}.

If p = 2, then $\delta(n) = 2^2 * q$ with $\phi(\delta(n)) = 2 * (q-1) = m^2$, so q belongs to A090698.
If p = 3, then $\delta(n) = 3^2 * q$ with $\phi(\delta(n)) = 6 * (q-1) = m^2$, so q belongs to A090687.
Example: $873 = 3^2 * 97$ and $\phi(873) = 3*2*96 = 24^2$

Some values of (δ(n),p,q,m): (12,2,3,2), (63,3,7,6), (76,2,19,6), (292,2,73,12), (873,3,97,24), ...

The general terms are **d(n) =** $p^{2s} * q^{2t+1}$ with p < q primes, s>=1, t>=0, and such that: **phi(d(n))** = $(p^{s-1} * q^t * m)^2$, they form the sequence A324747.
The first few terms are {12, 48, 63, 76, 108, 192, 292, 304, 432, 567, 652, 873, 972, ...}.

Examples:

$76 = 2^2 * 19$, $2 * 18 = 6^2$ and phi(76) = $6^2 = 36$.

$192 = 2^6 * 3$, $2 * 1 * 2 = 2^2$ and phi(192) = $(2^3 * 3^0)^2 = 64 = 8^2$.

III) If a(n) has three distinct prime factors.

Some brief remarks about these integers which form the sequence A306908. There are three subsequences in this case. The first term with three prime distinct factors is $60 = 2^2 * 3 * 5$, the second one is 114 = 2 * 3 * 19, and the sixteenth term is $468 = 2^2 * 3^2 * 13$.

3.1) Define this first case by **ε(n) = p * q * r** with **phi(ε(n)) = (p-1)*(q-1)*(r-1)** = m^2.
The first primitives are {114, 170, 273, 285, 370, 438, 902, 969, ...}, these primitives are a subsequence of A262406 (Squarefree k such that phi(k) is square).

The general terms are **e(n) =** $p^{2s+1} * q^{2t+1} * r^{2u+1}$ with p<q<r primes, s,t,u >= 0 such that: **phi(e(n)) =** $(p^s * q^t * r^u * m)^2$.
The first few terms are {114, 170, 273, 285, 370, 438, 456, 680, 902, 969, 978, ...}.

Example: 114 = 2*3*19, 1*2*18 = $36 = 6^2$, and phi(114) = 6^2.

3.2) Define the second case by **λ(n) =** $p^2 * q * r$ with **phi(λ(n))** = **p*(p-1)*(q-1)*(r-1) =** m^2.
The first primitives are {60, 126, 204, 315, 364, 380, 444, 825, ...}.

The general terms are **l(n) =** $p^{2s} * q^{2t+1} * r^{2u+1}$ with p,q,r primes, s >= 1, t,u >= 0 such that: **phi(l(n)) =** $(p^{s-1} * q^t * r^u * m)^2$.
The first few terms are:{60, 126, 204, 240, 315, 364, 380, 444, 504, 816, 825, ...}.

Example: $60 = 2^2 * 3 * 5$, $2*1*2*4 = 4^2$, and phi(60) = 4^2.

3.3) Define the third case by **ζ(n) =** $p^2 * q^2 * r$ with **p * q * (p-1) * (q-1) * (r-1) =** m^2 and **phi(ζ(n)** = **p * q * (p-1) * (q-1) * (r-1) =** m^2.
The first primitive terms are {468, 1100, 3924, 4100, 6948, 6975, ...}.

The general terms are **z(n) =** $p^{2s} * q^{2t} * r^{2u+1}$ with p,q,r primes, s,t >=1, u>=0 such that: **phi(z(n)) =** $(p^{s-1} * q^{t-1} * r^u * m)^2$.
The first few terms are {468, 1100, 1872, 3924, 4100, 4212, 4400, 6948, 6975, ...}

Example: $468 = 2^2 * 3^2 * 13$ and phi(468) = 12^2 (Remark: cototient(468) = 3^2).

IV) **If a(n) has four distinct prime factors.**

There is a new sequence {546, 570, 630, 1020, 1365, ...} and other subsequences with four distinct prime factors that can be found with similar conditional requirements as displayed here.

Bernard Schott, March 17 2019