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A Note on Markoff Forms Determining Quadratic Irrationals with Purely Periodic Continued Fractions

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Abstract

Two equivalent indefinite binary quadratic forms associated with *Markoff* triples are considered. The standard representative forms used by Cassels, corresponding to quadratic irrationals which have not purely periodic continued fractions, and the forms which lead to the purely periodic continued fractions used by Perron. It turns out that these equivalent irrationals differ by -2. The reciprocal of the *Perron* irrationals are also considered. The quadratic forms become in general noninteger.

A) Cassels's version

Indefinite binary quadratic *Markoff* (*Markov*) forms *Mf* are related to Markoff triples $MT = (m, m, m)$ with positive integers $m_1 \leq m_2 \leq m$, satisfying the *Markoff* equation

$$m_1^2 + m_2^2 + m^2 - 3\, m_1\, m_2\, m = 0\,. \tag{1}$$

The sequence A002559 [16] (henceforth A-numbers will be given without this citation) gives the numbers which appear in solutions in strictly increasing order. It is known (see *e.g.*, [1], pp. 38-39, pp. 47-48) that every *Markoff* number $m(n) =$A002559(n), $n \in \mathbb{N}$, appears at least once as largest member of a *Markoff* triple. The uniqueness conjecture ([7], [1],[8]) is that it appears precisely once, that is for given m there exist m_1 and m_2 uniquely. This conjecture has been verified for $m \leq 10^{105}$ in [2]. See *Table 1* for the first 40 ordered triples $MT(n)$. In [22] one finds the *Markoff* tree with more triples. In this note we assume that the conjecture is true. Otherwise one would obtain for certain $m(n)$ different triples, and the numeration of the triples by $MT(n)$ would become wrong)

More properties of *Markoff* triples are: i) The members are pairwise coprime ([4]. Lemma 2, p. 28, [1], Corollary 3.4., p. 48). ii) Except for $n = 1$ and $n = 2$ no repeated numbers appear (*e.g.*, [1], Lemma 3.1, p. 45). iii) If m(n) is odd then $m(n) \equiv 1 \pmod 4$, and otherwise $m(n) \equiv 2 \pmod{32}$ ([7], II, p. 601, and the sharper version [1], Proposition 3,13, p. 55). iv) Each odd prime divisor of $m(n)$, of $3\,m(n) - 2$, and of $3\,m(n) + 2$ is from A002144, the odd primes modulo 4. ([7], III, p. 601).

The *Markoff* forms are defined in [4], p. 31, as follows (note the above remark about the numeration by n) .

$$Mf(n) = f_{m(n)}(x, y) = m\, F_{m(n)}(x, y) = m(n)\, x^2 + (3\, m(n) - 2\, k(n))\, x\, y + (l(n) - 3\, k(n))\, y^2\,, \tag{2}$$

with $l(n) = \dfrac{k(n)^2 + 1}{m(n)}$, and the k-sequence, using now $k_C(n)$ for the form $Mf_C(n)$ (C for *Cassels*), is defined by

$$\begin{aligned} &k_C(n) = \min\{k1(n), k2(n)\}, \text{ with} \\ &m_1(n)\, k1(n) - m_2(n) \equiv 0 \pmod{m(n)}, \text{where } 0 \leq k1(n) < m(n), \\ &m_2(n)\, k2(n) - m_1(n) \equiv 0 \pmod{m(n)}, \text{where } 0 \leq k2(n) < m(n)\,. \end{aligned} \tag{3}$$

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For this k_C-sequence see *Table 1* and A305310. The corresponding $\mathit{Mf}(n)$ coefficients $[a,\, b,\, c]$ are also shown in *Table 1* for $n = 1, 2, ..., 40$. The discriminant of such forms $[a,\, b,\, c]$ is $D = b^2 - 4\,a\,c$. Therefore,

$$D(n) = 9\,m(n)^2 - 4 = b(n)\,(b(n) + 4), \tag{4}$$

with $b(n) := 3\,m(n) - 2$. Note that for even $m(n)$ these forms are not primitive (they have a common divisor > 1). The forms with odd $m(n)$ are primitive and, except for $n = 1$, also reduced (for reduced see [3], p. 21 or [20], p. 112). Sometimes, *e.g.,* in [5], *Table 2*, p. 11, $k(1)$ is taken as 1 (not 0), leading to the primitive reduced form $[1,\, 1\, -1]$. In [1], p. 37, the forms of *Cassels* are used. Also *Dickson* [6], p. 106, uses these forms (with $m \to Q$, $k_c \to \pi$ and $l \to \rho$).

The solution of $f_{m(n)}(x,\, 1) = 0$ determines two equivalent quadratic irrationals for each n. The one with positive square-root is denoted by $\xi(n)$ and is given by

$$\xi(n) = \frac{2\,k(n) - 3\,m(n) + \sqrt{D(n)}}{2\,m(n)}\;. \tag{5}$$

The definition of equivalent real numbers is (see *e.g.,* [19], pp. 54-56, [14], p. 14, or [10], p. 224): $\alpha \sim \beta$ if there exit integers $p,\, q,\, r,\, s$ such that $\alpha = \dfrac{p\,\beta + q}{r\,\beta + s}$ with $(p,\, s - q,\, r) = \pm 1$ (unimodular equivalence). (Sometimes the symbol $\sim$ is used only for the $+1$ case and $\approx$ for the -1 case. In this note mostly the $+1$ case is used). Equivalent irrational numbers are characterized by the fact that their regular continued fractions eventually coincide ([19], Satz 2.24, p. 55, [1], Definition 1.25, p.24). This equivalence of numbers is in relation to the equivalence of forms under unimodular transformations. Two forms $f(\mathbf{A},\, \vec{x})$ and $f(\mathbf{A'},\, \vec{x}')$ with

$$f(\mathbf{A}, \vec{x}) = \vec{x}^{\top}\, \mathbf{A}\, \vec{x}, \quad \text{with} \quad \mathbf{A} = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \quad \text{and} \quad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{6}$$

are said to be properly (or improperly) equivalent, denoted by $f \sim f'$, if $f(\mathbf{A},\, \vec{x}) = f(\mathbf{A'},\, \vec{x}')$, with $\vec{x}' = \mathbf{M}\,\vec{x}$, identically in $\vec{x}$, and $\mathbf{A'} = \mathbf{M}^{-1,\top}\,\mathbf{A}\,\mathbf{M}^{-1}$ with $\mathit{Det}\,\mathbf{M} = +1$ (or -1).

If $f(\mathbf{A}, (\beta,\, 1)^{\top}) = 0$ and $\alpha \sim \beta$, as given above in terms of $\mathbf{M} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ with $\mathit{Det}\,\mathbf{M} = \pm 1$ then $0 = f(\mathbf{A'}, (p\,\beta + q, r\,\beta + s)^{\top}) = f(\mathbf{A'}, (\alpha,\, 1)^{\top})$ has a solution equivalent to β (see [4], p. 19).

In the *Cassels* case, with $k(n) = k_C(n)$, one obtains the irrationals $\xi_C(n)$ given in *Table 1*. The regular (also called simple) continued fraction representation of $\xi_C(n)$ is eventually periodic but not purely periodic, as is seen in *Table 1*. The (primitive) period is denoted by brackets. *E.g.,* For $n = 5$, $\xi_C(5) = \frac{-63+\sqrt{7565}}{58}$ one has $0, 2_3,\, (1_2,\, 2_4)$ standing for the simple continued faction $[0,\, 2,\, 2\,, 2,\, \text{repeat}(1,\, 1,\, 2,\, 2,\, 2,\, 2)]$. Note that there is an ambiguity in defining the (primitive) period., depending of the choice of the preperiod (see *e.g.,* [19], p. 65). *E.g.,* in [5], Theorem 3, p. 23, for $n \geq 3$, the leading 0 is immediately followed by the period. In the present example this is $0,\, (2_3,\, 1_2,\, 2)$ (also not purely periodic but with a shorter (trivial) preperiod). If these periods are used then the multiplicity indices are not always even. The first and last index is odd (in the example 3 and the not written 1). In our case only even multiplicities appear in the periods. The reason for this choice of the larger preperiod is that then the periods are the ones considered by [18], pp. 5-6.

In the context of Diophantine approximations ([9], [11], [4], [14], [15] [21]) *Perron's* [17] function $M(\xi)$ is defined for any irrational number ξ as supremum of the set of positive numbers c which satisfy

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{c\,q^2} \tag{7}$$

for infinitely many fractions $\frac{p}{q}$ ($p \in \mathbb{Z}$, $q \in \mathbb{N}$, in lowest terms). $M(\xi)$, also called the *Lagrange* number of ξ, is a real unimodular function. *i.e.,* $M(\xi) = M(\xi')$ if $\xi \sim \xi'$. The set $\mathcal{L}$ of all possible values of $M(\xi)$ over all irrational numbers ξ is called the *Lagrange* spectrum (see *e.g.*, [1], p. 8). It takes for $\sqrt{b^2+1} \leq M(\xi) < 3$ countable infinitely many values with the single accumulation point 3. Here b is the largest of the infinitely many partial quotients of the regular continued fraction of the irrational ξ. (see [17] Satz 6. p. 11, and [18], Satz 1, p. 4, and [1], sect. 1.4, pp. 22-28). In this range the *Lagrange* spectrum coincides with the *Markoff* spectrum $\mathcal{M}$ defined in terms of indefinite binary quadratic forms $f(\mathbf{A}, \vec{x})$, for given a, b, and c, with $m(f) := \inf\left\{|\,f\,|\,\middle|\,(x,y) \in \mathbb{Z}^2 \setminus \{0,0\}\right\}$ as $Ma(f) := \frac{\sqrt{Disc(f)}}{m(f)}$, with the discriminant $Disc(f)$ of form f ([12], [13], [5] and [1] , pp. 35-38). Markoff showed, by factoring out a $Disc(f)$ dependent factor that all forms with $Ma(f) < 3$ are equivalent to the forms $f = Mf(n)$ with discriminant $D(n)$, given above in Cassels's version as $Mf_C(n)$, and $\inf\{|\,Mf(n)\,|\} = m(n) =$ A002559(n). Because for values < 3 $\mathcal{L}$ coincides with $\mathcal{M}$ (see [13] and [18], Satz 1, p 4) one has

$$L(n) = Ma(Mf(n)) = \frac{\sqrt{D(n)}}{m(n)} = \frac{\sqrt{9\,m(n)^2 - 4}}{m(n)}, \quad \text{for} \quad n \in 1,\, 2,\, \ldots\,, \tag{8}$$

The values of L(n) are shown in the last column of table *Table 1* for $n = 1, 2, \ldots, 40$.

Instead of *Perron*'s unimodular invariant function $M(\xi)$ whose values < 3 coincide with the Lagrange numbers $L(n)$, in [4], pp. 10, 18 and [21], pp. 80-81, the unimodular invariant function $\lambda(\xi) := \liminf(q\,||q\,\xi||)$ (in [4] called $\nu(\theta)$) for irrational ξ is considered. Here $||x||$ denotes the (positive) difference between ξ and the nearest integer. The spectrum of λ coincides for $> \frac{1}{3}$ with the reciprocal *Lagrange* numbers $\frac{1}{L(n)}$, for $n = 1, 2, \ldots$. See the values given in the last column of *Table 2.*

B) Forms and Perron's irrationals

The purpose of this note is to find the forms which lead to the quadratic irrationals $\xi(n)$ with purely periodic regular continued fractions as used in [18], pp. 5-6. For $n = 1$ the continued fraction of the golden section $\varphi =$ A001622 is repeat(1), denoted in the *Table 2* just by (1). The periods of the continued fractions for $n > 1$ begin with 2 (not 1). It is known (see [11], 10., pp. 31-32) that this so called *Markoff* period $P(n)$ is, for $n \geq 3$, of the structure $(\sigma_0,\, \sigma_0,\, \sigma_1,\, \sigma_1,\, \ldots, \sigma_s,\, \sigma_s)$ with $\sigma_0 = 2$, $\sigma_s = 1$ and $\sigma_i \in \{1,\, 2\}$ for $i \in \{0,\, 1,\, \ldots,\, s\,\}$. The sub-sequence $\sigma_1,\, \sigma_1,\, \ldots \sigma_{s-1},\, \sigma_{s-1}$ is its own reverse sequence. See, *e.g.*, the continued fractions of *Table 2*, for $n \geq 3$. One can also recover the *Markoff* number $m(n)$ from the denominator of the corresponding finite continued fraction $[\sigma_0,\, \sigma_s,\, \sigma_s,\, \sigma_1,\, \sigma_1,\, \ldots, \sigma_{s-1},\, \sigma_{s-1},\, \sigma_0]$ by using the well-known three term recurrence relations for the regular finite continued fraction $[b_0,\, b_1,\, \ldots, b_m] = \frac{A_m}{B_m}$, with $q_k = b_k\, q_{k-1} + q_{k-2}$, $k \in 1, 2, \ldots$, for $q = A$, or B, with inputs $A_{-1} = 1$, $A_0 = b_0$, and $B_{-1} = 0$, $B_0 = 1$, (*e.g.*, [19], p. 24).

The procedure to find the quadratic irrationals $\xi(n)$ and the *Markoff* forms $Mf(n)$ corresponding to the purely periodic continued fractions, shown in *Table 2*, which are obtained from the continued fractions of the *Cassels* version, shown in *Table 1*, is the following.

First the irrationals $\xi(n)$ are determined from the regular purely periodic continued fraction $\xi =$ repeat$(b_0,\, b_1,\, \ldots, b_{L-1})$ (L is the length of the period) by the well known formula ([19], p. 62)

$$\xi = \frac{A_{L-1} - B_{L-2} + \sqrt{(A_{L-1} + B_{L-2})^2 + 4\,(-1)^{L-1}}}{2\,B_{L-1}}, \tag{9}$$

where the three term recurrences for sequences $\{A\}$ and $\{B\}$ given above are used. This produces the $\xi(n)$ given in *Table 2* for $n = 1, 2, \ldots, 40$.

Then the *Markoff* forms $Mf(n) = Mf(n; x, y) = m(n)\, x^2 + (3\, m(n) - 2\, k(n))\, x\, y + (l(n) - 3\, k(n))\, y^2$ with $l(n) = \frac{k(n)^2 + 1}{m(n)}$ and the Markoff number $m(n) =$ A002559(n) are considered. Now $k(n)$ is determined as the integer solution of the quadratic equation for $k(n)$, given the irrational $\xi(n)$, obtained from $M\mathrm{f}(n;\xi(n),1) = 0$, *viz*

$$k(n)^2 - m(n)\,(2\,\xi(n) + 3)\,k + m(n)^2\,\xi(n)\,(\xi(n) + 3) + 1 = 0. \tag{10}$$

The two solutions are $k(n)_{\pm} = \frac{1}{2}\left(m(n)\,(2\,\xi(n) + 3) \pm \sqrt{D(n)}\right)$. The integer value is obtained for the negative root (cancellation of the roots) called $\tilde{k}(n)$ in *Table 2*. The *Lagrange* numbers $L(n)$ are given in eq. (8) (and *Table 1*).

$$\tilde{k}(n) = \frac{1}{2}\left(m(n)\,(2\,\xi(n) + 3) - \sqrt{D(n)}\right) = \frac{m(n)}{2}\,(2\,\xi(n) + 3 - L(n))\,. \tag{11}$$

This sequence $\{\tilde{k}(n)\}$, for $k = 1, 2, ...,$ is given im A305311.
Hence $\xi(n)$, expressed in terms of $\tilde{k}(n)$, becomes (see also *e.g.*, 5)

$$\xi(n) = \frac{1}{2}\left(L(n) - 3 + 2\,\frac{\tilde{k}(n)}{m(n)}\right)\,. \tag{12}$$

One can compare these $\xi(n)$, given for $n = 1, 2, ..., 11$, in Perron [18], pp. 7-8, with his formula on p. 4 (rewritten with $Q \to m(n), M(\xi) \to L(n)$ and $P \to P(n)$) $\xi = \frac{1}{2}\left(L(n) + 1 + 2\,\frac{P(n)}{m(n)}\right)$. One obtains $P(n) = \tilde{k}(n) - 2\,m(n)$, for $n = 1, 2, \ldots$. It turns out that $P(n) = k_C(n)$. Hence

$$\tilde{k}(n) = k_C(n) + 2\,m(n) \equiv k_C(n) \pmod{m(n)}, \quad \text{for} \;\; n = 1, 2, \ldots\,. \tag{13}$$

This corresponds to the difference of the quadratic irrationals, given by eq. (5) with the corresponding $k(n)$ values (like in eq. (12) for $\tilde{k}(n)$)

$$\xi(n) = \xi_C(n) + 2, \quad \text{for} \;\; n = 1, 2, \ldots\,, \tag{14}$$

which are obviously equivalent $((p,\,q,\,r\,s) = (1,\,2,\,0,\,1))$ as they have to be because the continued fraction of $\xi_C(n)$ ends up in the period of the continued fraction of $\xi(n)$.
The proof of eq. (14) (hence eq. (13)) can be given directly by using the following information about the continued fraction for ξ_C stated in *Theorem* 3, pp. 23-24 of [5]. For $n = 1$ and $n = 2$ this proof is trivial. As mentioned above, for $n \geq 3$ the preperiod can be taken as 0. Then the continued fraction for $\xi_C(n)$ is $[0, \mathrm{repeat}(b_1,\,...,\,b_{2\,l})]$, with the length of the period $2\,l(n)$, and $b_1 = 2 = b_{2\,l}$ and $b_{2\,l-2} = 1 = b_{2\,l-1}$ Therefore, adding 2 results in the purely periodic continued fraction $[\mathrm{repeat}(2,\,2,\,b_2,\,...,\,b_{2\,l-3},\,1,\,1)]$ which is the one for the above constructed $\xi(n)$.
Note that in [1]. pp. 37-38, the *Markoff* forms are the ones of *Cassels* (as mentioned above). However, the quadratic irrationals are given for the purely periodic continued fraction case as γ_m for our $\xi(n)$ with u for $k_C(n)$. Therefore, the forms in [1] do not satisfy $f_m(\gamma_m,\,1) = 0$. See the example for $n = 3$, $m = 5$ given there. The form $f_5(x,\,y) = 5\,x^2 + 11\,x\,y - 5\,y^2$ (like *Cassels*) and therefore $f_5(x,\,1) = 0$ has the solution with positive root $x_+ = \xi_C(3) = \frac{-11 + \sqrt{221}}{10}$, not γ_5 which is $\xi(3) = \frac{9 + \sqrt{221}}{10}$.

From the equivalence of the quadratic irrationals in eq. (14) follows the equivalence of the corresponding forms, as explained above. The rôle of α and β plays $\xi(n)$ and $\xi_C(n)$, respectively. The transformation matrix is $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $\mathbf{M}^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ and thus $\mathbf{A}' = \begin{pmatrix} a & -2\,a + \frac{b}{2} \\ -2\,a + \frac{b}{2} & c + 4\,a - 2\,b \end{pmatrix}$, with $a = m(n)$, $b(n) = 3\,m(n) - 2\,k_C(n)$ and $c = \frac{k_C(n)^2+1}{m(n)} - 3\,k_C(n)$. This produces the new form coefficients for $M\mathrm{f}(n)$

$$M\mathrm{f}(n) = \left[m(n),\, -(m(n) + 2\,k_C(n)),\, \frac{k_C(n)^2 + 1}{m(n)} + k_C(n) - 2\,m_n\right], \tag{15}$$

which is from eq. (2) with $k(n) \to \tilde{k}(n)$

$$Mf(n) \,=\, [m(n),\, 3\,(m(n) \,+\, 2\,\tilde{k}(n)),\, \frac{\tilde{k}(n)^2+1}{m(n)} \,-\, 3\,k(n)]\,. \tag{16}$$

This can be verified also directly using eq. (13).

Using eq. (14) one can convert directly each regular continued fraction for $\xi(n)$ into the one for $\xi_C(n)$. For $n = 1$ one uses for the golden section φ the equation $\varphi - 2 = -1 + \frac{1}{\varphi}$ to obtain the $\xi_C(1)$ representation $[-1, \text{repeat}(1)]$, For $n = 2$ one has for $[\text{repeat}(2)] - 2$ the reciprocal of $\xi(2)$, hence $[0, \text{repeat}(2)]$. If, for $n \geq 3$, the representation of $\xi(n)$ is $[\text{repeat}(2_{2j},\, B(n))]$ with $j \in \mathbb{N}$ then $\xi_C(n) \doteq [0,\, 2_{2j-1}, \text{repeat}(B(n),\, 2_{2j})]$. *E.g.*, in the notation of the *Tables* $\xi(26) \doteq (\{2_4,\, 1_2\}^2,\, 2_2,\, 1_2)$ is mapped to $\xi_C(26) \doteq [0,\, 2_3, \text{repeat}(1_2,\, 2_4,\, 1_2,\, 2_2,\, 1_2,\, 2_4)]$.
Conversely, if a 2 is added the continued fraction representation of $\xi_C(n)$ one obtains the one of $\xi(n)$: if, for $n \geq 3$, $\xi_C(n) \doteq [0, 2_{2j-1}, \text{repeat}(B_C(n),\, 2_{2j})]$ then $\xi(n) \doteq [\text{repeat}(2_{2j},\, B_C(n))]$, for $j \in \mathbb{N}$. For $n = 1$ and 2 the mapping is $[-1, \text{repeat}(1)] \mapsto [\text{repeat}(1)]$ and $[0, \text{repeat}(2)] \mapsto [\text{repeat}(2)]$, respectively.

We conclude with mentioning yet another version of equivalent quadratic irrationals, *viz* $\xi_H(n) = \frac{1}{\xi(n)}$. Such irrationals are considered in [10], pp. 222-223, and their regular continued fraction representation is $\xi_H(n) \doteq [0, \text{repeat}P(n)]$ if $\xi(n) \doteq \text{repeat}(P(n))$, the purely periodic version. It turns out that the equivalence matrix $\mathbf{M}(n)$ as well as the sequence $k_H(n)$ is not integer for $n \geq 3$, and therefore the Markoff form $Mf_H(n)$ is also not an integer form. If $\xi(n) = \frac{a(n) \,+\, \sqrt{D(n)}}{b(n)}$ then the equivalence matrix is, with

$p(n) := \frac{b(n)\,\sqrt{D(n) \,-\, a(n)^2}}{D(n) \,-\, a(n)^2}$ and $q(n) := \frac{2\,a(n)\,\sqrt{D(n) \,-\, a(n)^2}}{D(n) \,-\, a(n)^2}$,

$$\mathbf{M}(n) \,=\, \begin{pmatrix} p(n) & -q(n) \\ 0 & 1/p(n) \end{pmatrix}. \tag{17}$$

The first eleven values for $[p(n),\, q(n)]$ are: $[1,\, 1]$, $[1,\, 2]$, $[\sqrt{35}/7,\, 9\,\sqrt{35}/35]$, $[\sqrt{247}/19,\, 23\,\sqrt{247}/247]$, $[\sqrt{1189}/41,\, 53\,\sqrt{1189}/1189]$ $[\sqrt{17}/5,\, 6\,\sqrt{17}/17]$, $[\sqrt{11659}/131,\, 157\,\sqrt{11659}/11659]$, $[13\,\sqrt{239}/239,\, 309\,\sqrt{239}/3107]$, $[\sqrt{13774}/142,\, 86\,\sqrt{13774}/6887]$, $[\sqrt{1631}/49,\, 411\,\sqrt{1631}/11417]$, $[\sqrt{265429}/613,\, 791\,\sqrt{265429}/265429]$, ...
The first 20 irrationals $\xi_H(n)$ shown in *Table 3* (where the table should end with a vertical bar).
From eq. (11) follow the corresponding values for $k_H(n)$ also shown in *Table 3.*
Eq2, with $k(n) \to k_H(n)$, yields the coefficients of the *Markoff* forms $Mf_H(n)$: see *Table 3.*

References

[1] Martin Aigner, *Markov's theorem and 100 years of the uniqueness conjecture: a mathematical journey from irrational numbers to perfect matchings*, Springer, 2013.

[2] I. Borosh, More numerical evidence on the uniqueness of the Markoff numbers, https://link.springer.com/article/10.1007/BF01931671, BIT Numerical Mathematics, 15,4 (1975) 351357. (BIT reverse for "Tidskrift för Informationsbehandling").

[3] D. A. Buell, *Binary Quadratic Forms*, Springer, 1989.

[4] J. W. S. Cassels, *An Introduction to Diophantine Approximation*, Cambridge University Press, 1957.

[5] T. W. Cusick and M. S. Flahive, *The Markoff and Lagrange Spectra*, Math. Surveys and Monographs, 30, Am. Math. Soc., Providence, RI, 1989.

[6] L. E. Dickson, *Studies in the Theory of Numbers*, Chelsea Publ. Comp., New York, 1947, Part II, pp. 77ff.

[7] Ferdinand Georg Frobenius, Über die Markoffschen Zahlen, Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, (1913) 458-487. Gesammelte Abhandlungen, Band III, pp. 598-627.

[8] R. K. Guy, *Unsolved Problems in Number Theory*, Second Edition, Springer, 1994, D 12, pp. 166-168.

[9] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Fifth ed., Oxford Science Publications, Clarendon Press, Oxford, 2003, ch. XI, pp. 154-177, and p. 412.

[10] Julian Havil, *The Irrationals*, Princeton University Press, Princeton and Oxford, 2012.

[11] J. F. Koksma, *Diophantische Approximationen*, Ergebmisse der Mathematik und ihrer Grenzgebiete, Vierter Band, Heft 4, Verlag Julius Springer, Berlin, 1936. Also, Chelsea Publ. Co., New York, 1953.

[12] A. Markoff, Sur les formes quadratiques binaires indéfinies., Math. Annalen, 15 (1879) 381-406.

[13] A. Markoff, Sur les formes quadratiques binaires indéfinies. (Sécond mémoire), Math. Annalen, 17 (1880) 379-399.

[14] Ivan Niven, *Irrational Numbers*, Math. Assoc. Am., John Wiley and Sons, 1956, Second printing, 1963.

[15] Ivan Niven, *Diophantine Approximations*, Interscience Publishers,Wiley & Sons, 1936.

[16] The On-Line Encyclopedia of Integer Sequences^{TM}, published electronically at http://oeis.org. 2010.

[17] Oskar Perron, Über die Approximation irrationaler Zahlen durch rationale, pp. 1-17, Sitzungsber. Heidelberger Akademie der Wiss., 1921, 4. Abhandlung, Carl Winters Universitätsbuchhandlung.

[18] Oskar Perron, Über die Approximation irrationaler Zahlen durch rationale, II., op. cit., 8. Abhandlung, pp. 1-12,

[19] Oskar Perron, Die Lehre von den Kettenbrüchen, Band I, dritte Auflage. Teubner Verlagsgesellschaft, Stuttgart, 1954.

[20] A. Scholz and B. Schoeneberg, *Einführung in die Zahlentheorie*, Sammlung Göschen Band 5131, Walter de Gruyter, 1973.

[21] Jörn Steuding, *Diophantine Anlalysis*, Chapman & Hall/CRC, 2005.

[22] Don Zagier, On the Number of Markoff Numbers Below a Given Bound, Math. Comp. 39, Nr. 160, (1982) 709-723.

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Table legends

Table 1

The first 40 increasingly (for $n = 1, 2$ not strictly) ordered *Markoff* triples $(m_1(n), m_2(n), m(n))$ are shown. The $k_C(n)$ values used by *Cassels* are defined in eq. (3). The Markoff forms of *Cassels* are $Mf_C(n)$, and their coefficients are given. The quadratic irrationals given by the positive solution of $Mf_C(n; x, 1) = 0$ are $\xi_C(n)$, and their approximate values are given. The regular continued fractions are given in a notation like $0, 2, (1_2, 2_2)$ standing for $[0, 2, \text{repeat}(1, 1, 2, 2)]$, The period is enclosed in brackets. Also, a power notation with curly brackets is used, *e.g.*, $\{2_4, 1_2\}^2$ stands for $2_4, 1_2, 2_4, 1_2$. In the last colum the *Lagrange* spectrum numbers $L(n) < 3$ from eq. (8) are given. Their reciprocal values are shown in the last column of *Table 2*.

Table 2

The notation is as in *Table 1* but now for the the *Markoff* forms $Mf(n)$, obtained with $k(n) \to \tilde{k}(n)$ in the general form of eq. (2), which have quadratic irrationals $\xi(n)$ with purely periodic continued fractions (ppconfrac) as positive solutions. The decimal approximations of the reciprocal *Lagrange* numbers in the last column use also the notation, like 3_8 for an eightfold repetition of the digit 3.

Table 3

The notation is as in *Table 2*. Here the noninteger *Markoff* forms $Mf_H(n)$ with their $k_H(n)$ values and quadratic irrationals $\xi_H(n)$ are shown.

Table 1: Markoff triples, Markoff forms (Cassels)

n	$(\mathbf{m_1(n), m_2(n), m(n)})$	$\mathbf{k_C(n)}$	$\mathbf{Mf_C(n)}$	$\boldsymbol{\xi_C}\mathbf{(n)}$	**approx.**	**continued fraction**	$\mathbf{M(\boldsymbol{\xi_C}(n)) = L(n)}$
1	(1, 1, 1)	0	[1, 3, 1]	$\frac{-3+\sqrt{5}}{2}$	−.381966012	$-1, (1)$	$\sqrt{5}$
2	(1, 1, 2)	1	[2, 4, −2]	$-1+\sqrt{2}$	0.414213562	$0, (2)$	$2\sqrt{2}$
3	(1, 2, 5)	2	[5, 11, −5]	$\frac{-11+\sqrt{221}}{10}$	0.386606875	$0, 2, (1_2, 2_2)$	$\frac{\sqrt{221}}{5}$
4	(1, 5, 13)	5	[13, 29, −13]	$\frac{-29+\sqrt{1517}}{26}$	0.382641700	$0, 2, (1_4, 2_2)$	$\frac{\sqrt{1517}}{13}$
5	(2, 5, 29)	12	[29, 63, −31]	$\frac{-63+\sqrt{7565}}{58}$	0.413396697	$0, 2_3, (1_2, 2_4)$	$\frac{\sqrt{7565}}{29}$
6	(1, 13, 34)	13	[34, 76, −34]	$\frac{-19+5\sqrt{26}}{17}$	0.382064563	$0, 2, (1_6, 2_2)$	$\frac{\sqrt{2600}}{17}$
7	(1, 34, 89)	34	[89, 199, −89]	$\frac{-199+\sqrt{71285}}{178}$	0.381980389	$0, 2, (1_8, 2_2)$	$\frac{\sqrt{71285}}{89}$
8	(2, 29, 169)	70	[169, 367, −181]	$\frac{-367+\sqrt{257045}}{338}$	0.414189512	$0, 2_5, (1_2, 2_6)$	$\frac{\sqrt{257045}}{169}$
9	(5, 13, 194)	75	[194, 432, −196]	$\frac{-108+\sqrt{21170}}{97}$	0.386589081	$0, 2, (1_2, 2_2, 1_4, 2_2)$	$\frac{\sqrt{84680}}{97}$
10	(1, 89, 233)	89	[233, 521, −233]	$\frac{-521+\sqrt{488597}}{466}$	0.381968109	$0, 2, (1_{10}, 2_2)$	$\frac{\sqrt{488597}}{233}$
11	(5, 29, 433)	179	[433, 941, −463]	$\frac{-941+\sqrt{1687397}}{866}$	0.413393141	$0, 2_3, (1_2, 2_2, 1_2, 2_4)$	$\frac{\sqrt{1687397}}{433}$
12	(1, 233, 610)	233	[610, 1364, −610]	$\frac{-341+\sqrt{209306}}{305}$	0.381966317	$0, 2, (1_{12}, 2_2)$	$\frac{\sqrt{1687397}}{433}$
13	(2, 169, 985)	408	[985, 2139, −1055]	$\frac{-2139+\sqrt{8732021}}{1970}$	0.414212854	$0, 2_7, (1_2, 2_8)$	$\frac{\sqrt{8732021}}{985}$
14	(13, 34, 1325)	507	[1325, 2961, −1327]	$\frac{-2961+\sqrt{15800621}}{2650}$	0.382641319	$0, 2, (1_4, 2_2, 1_6, 2_2)$	$\frac{\sqrt{15800621}}{1325}$
15	(1, 610, 1597)	610	[1597, 3571, −1597]	$\frac{-3571+\sqrt{22953677}}{3194}$	0.381966056	$0, 2, (1_{14}, 2_2)$	$\frac{\sqrt{22953677}}{1597}$
16	(5, 194, 2897)	1120	[2897, 6451, −2927]	$\frac{-64511+\sqrt{75533477}}{5794}$	0.386606795	$0, 2, (1_2, 2_2, 1_2, 2_2, 1_4, 2_2)$	$\frac{\sqrt{75533477}}{2897}$
17	(1, 1597, 4181)	1597	[4181, 9349, −4181]	$\frac{-9349+\sqrt{157326845}}{8362}$	0.381966018	$0, 2, (1_{16}, 2_2)$	$\frac{\sqrt{157326845}}{4181}$
18	(2, 985, 5741)	2378	[5741, 12467, −6149]	$\frac{-12467+5\sqrt{11865269}}{11482}$	0.414213542	$0, 2_9, (1_2, 2_{10})$	$\frac{\sqrt{296631725}}{5741}$
19	(5, 433, 6466)	2673	[6466, 14052, −6914]	$\frac{-3513+5\sqrt{940706}}{3233}$	0.413393125	$0, 2_3, (1_2, 2_2, 1_2, 2_2, 1_2, 2_4)$	$\frac{\sqrt{94070600}}{3233}$
20	(13, 194, 7561)	2923	[7561, 16837, −7639]	$\frac{-16837+\sqrt{514518485}}{15122}$	0.386589070	$0, 2, (1_2, 2_2, 1_4, 2_2, 1_4, 2_2)$	$\frac{\sqrt{514518485}}{7561}$
...	continued						

Table 1 (continued): Markoff triples, Markoff forms (Cassels)

n	$(m_1(n), m_2(n), m(n))$	$k_C(n)$	$Mf_C(n)$	$\xi_C(n)$	approx.	continued fraction	$M(\xi_C(n)) = L($
21	(34, 89, 9077)	3468	[9077, 20295, −9079]	$\frac{-20295+\sqrt{741527357}}{18154}$	0.382064555	$0, 2, (1_6, 2_2, 1_8, 2_2)$	$\frac{\sqrt{741527357}}{9077}$
22	(1, 4181, 10946)	4181	[10946, 24476, −10946]	$\frac{-6119+\sqrt{67395890}}{5473}$	0.381966012	$0, 2, (1_{18}, 2_2)$	$\frac{\sqrt{67395890}}{5473}$
23	(29, 169, 14701)	6080	[14701, 31925, −15745]	$\frac{-31925+\sqrt{1945074605}}{29402}$	0.414189509	$0, 2_5, (1_2, 2_4, 1_2, 2_6)$	$\frac{\sqrt{1945074605}}{14701}$
24	(1, 10946, 28657)	10946	[28657, 64079, −28657]	$\frac{-64079+\sqrt{7391012837}}{57314}$	0.381966012	$0, 2, (1_{20}, 2_2)$	$\frac{\sqrt{7391012837}}{28657}$
25	(2, 5741, 33461)	13860	[33461, 72663, −35839]	$\frac{-72663+\sqrt{10076746685}}{66922}$	0.414213562	$0, 2_{11}, (1_2, 2_{12})$	$\frac{\sqrt{10076746685}}{33461}$
26	(29, 433, 37666)	15571	[37666, 81856, −40276]	$\frac{-20464+5\sqrt{31921370}}{18833}$	0.413396697	$0, 2_3, (1_2, 2_4, 1_2, 2_2, 1_2, 2_4)$	$\frac{\sqrt{3192137000}}{18833}$
27	(5, 2897, 43261)	16725	[43261, 96333, −43709]	$\frac{-96333+\sqrt{16843627085}}{86522}$	0.386606875	$0, 2, (\{1_2, 2_2\}^3, 1_4, 2_2)$	$\frac{\sqrt{16843627085}}{43261}$
28	(13, 1325, 51641)	19760	[51641, 115403, −51719]	$\frac{-115403+5\sqrt{960045437}}{103282}$	0.382641699	$0, 2, (\{1_4, 2_2\}^2, 1_6, 2_2)$	$\frac{\sqrt{24001135925}}{51641}$
29	(89, 233, 62210)	23763	[62210, 139104, −62212]	$\frac{-34776+\sqrt{2176922306}}{31105}$	0.381980389	$0, 2, (1_8, 2_2, 1_{10}, 2_2)$	$\frac{\sqrt{8707689224}}{31105}$
30	(1, 28657, 75025)	28657	[75025, 167761, −75025]	$\frac{-167761+\sqrt{50658755621}}{150050}$	0.381966011	$0, 2, (1_{22}, 2_2)$	$\frac{\sqrt{50658755621}}{75025}$
31	(5, 6466, 96557)	39916	[96557, 209839, −103247]	$\frac{-209839+\sqrt{83909288237}}{193114}$	0.413393125	$0, 2_3, (\{1_2, 2_2\}^3, 1_2, 2_4)$	$\frac{\sqrt{83909288237}}{96557}$
32	(34, 1325, 135137)	51709	[135137, 301993, −135341]	$\frac{-301993+\sqrt{164358078917}}{270274}$	0.382641320	$0, 2, (1_4, \{2_2, 1_6\}^2, 2_2)$	$\frac{\sqrt{164358078917}}{135137}$
33	(2, 33461, 195025)	80782	[195025, 423511, −208885]	$\frac{-423511+\sqrt{342312755621}}{390050}$	0.414213562	$0, 2_{13}, (1_2, 2_{14})$	$\frac{\sqrt{342312755621}}{195025}$
34	(1, 75025, 196418)	75025	[196418, 439204, −196418]	$\frac{-109801+\sqrt{21701267282}}{98209}$	0.381966011	$0, 2, (1_{24}, 2_2)$	$\frac{\sqrt{86805069128}}{98209}$
35	(13, 77561, 294685)	113922	[294685, 656211, −297725]	$\frac{-656211+\sqrt{781553243021}}{589370}$	0.386589070	$0, 2, (1_2, \{2_2, 1_4\}^3, 2_2)$	$\frac{\sqrt{781553243021}}{294685}$
36	(233, 610, 426389)	162867	[426389, 953433, −426391]	$\frac{-953433+\sqrt{1636268213885}}{852778}$	0.381968109	$0, 2, (1_{10}, 2_2, 1_{12}, 2_2)$	$\frac{\sqrt{1636268213885}}{428389}$
37	(169, 985, 499393)	206855	[499393, 1084469, −534883]	$\frac{-1084469+\sqrt{2244540316037}}{998786}$	0.414212855	$0, 2_7, (1_2, 2_6, 1_2, 2_8)$	$\frac{\sqrt{2244540316037}}{499393}$
38	(1, 196418, 514229)	196418	[514229, 1149851, −514229]	$\frac{-1149851+\sqrt{2379883179965}}{1028458}$	0.381966011	$0, 2, (1_{26}, 2_2)$	$\frac{\sqrt{2379883179965}}{514229}$
39	(5, 43261, 646018)	249755	[646018, 1438544, −652708]	$\frac{-359636+\sqrt{234753331682}}{323009}$	0.386606875	$0, 2, (\{1_2, 2_2\}^4, 1_4, 2_2)$	$\frac{\sqrt{939013326728}}{323009}$
40	(34, 9077, 925765)	353702	[925765, 2069891, −925969]	$\frac{-2069891+\sqrt{7713367517021}}{1851530}$	0.382064563	$0, 2, (\{1_6, 2_2\}^2, 1_8, 2_2)$	$\frac{\sqrt{7713367517021}}{925765}$
...							

Table 2: Markoff triples, Markoff forms (ppconfrac)

n	$(m_1(n), m_2(n), m(n))$	$\tilde{k}(n)$	Mf(n)	$\xi(n)$	approx.	continued fraction	$\lambda(n) = \frac{1}{L(n)}$
1	(1, 1, 1)	2	[1, −1, −1]	$\frac{1+\sqrt{5}}{2}$	1.618033988	(1)	0.4472135955...
2	(1, 1, 2)	5	[2, −4, −2]	$1+\sqrt{2}$	2.414213562	(2)	0.3535533905...
3	(1, 2, 5)	12	[5, −9, −7]	$\frac{9+\sqrt{221}}{10}$	2.386606875	$(2_2, 1_2)$	0.3363363969...
4	(1, 5, 13)	31	[13, −23, −19]	$\frac{23+\sqrt{1517}}{26}$	2.382641700	$(2_2, 1_4)$	0.3337725078...
5	(2, 5, 29)	70	[29, −53, −41]	$\frac{53+\sqrt{7565}}{58}$	2.413396697	$(2_4, 1_2)$	0.3334214468...
6	(1, 13, 34)	81	[34, −60, −50]	$\frac{15+5\sqrt{26}}{17}$	2.382064563	$(2_2, 1_6)$	0.3333974297...
7	(1, 34, 89)	212	[89, −157, −131]	$\frac{157+\sqrt{71285}}{178}$	2.381980389	$(2_2, 1_8)$	0.3333426853...
8	(2, 29, 169)	408	[169, −309, −239]	$\frac{309+\sqrt{257045}}{338}$	2.414189512	$(2_6, 1_2)$	0.3333359269...
9	(5, 13, 194)	463	[194, −344, −284]	$\frac{86+\sqrt{21170}}{97}$	2.386589081	$(2_2, 1_2, 2_2\, 1_4)$	0.3333353015...
10	(1, 89, 233)	555	[233, −411, −343]	$\frac{411+\sqrt{488597}}{466}$	2.381968109	$(2_2, 1_{10})$	0.3333346977...
11	(5, 29, 433)	1045	[433, −791, −613]	$\frac{791+\sqrt{1687397}}{866}$	2.413393141	$(2_4, 1_2, 2_2, 1_2)$	0.3333337284...
12	(1, 233, 610)	1453	[610, −1076, −898]	$\frac{269+\sqrt{209306}}{305}$	2.381966317	$(2_2, 1_{12})$	0.3333335324...
13	(2, 169, 985)	2378	[985, −1801, −1393]	$\frac{1801+\sqrt{8732021}}{1970}$	2.414212854	$(2_8, 1_2)$	0.3333334096...
14	(13, 34, 1325)	3157	[1325, −2339, −1949]	$\frac{2339+\sqrt{15800621}}{2650}$	2.382641319	$(2_2, 1_4, 2_2, 1_6)$	0.3333333755...
15	(1, 610, 1597)	3804	[1597, −2817, −2351]	$\frac{2817+\sqrt{22953677}}{3194}$	2.381966056	$(2_2, 1_{14})$	0.3333333623...
16	(5, 194, 2897)	6914	[2897, −5137, −4241]	$\frac{5137+\sqrt{75533477}}{5794}$	2.386606795	$(\{2_2, 1_2\}^2, 2_2, 1_4)$	.3333333421...
17	(1, 1597, 4181)	9959	[4181, −7375, −6155]	$\frac{7375+\sqrt{157326845}}{8362}$	2.3819660188	$(2_2, 1_{16})$	0.3333333375...
18	(2, 985, 5741)	13860	[5741, −10497, −8119]	$\frac{10497+5\sqrt{11865269}}{11482}$	2.414213542	$(2_{10}, 1_2)$	0.3333333355...
19	(5, 433, 6466)	15605	[6466, −11812, −9154]	$\frac{2953+5\sqrt{940706}}{3233}$	2.413393125	$(2_4, 1_2, \{2_2, 1_2\}^2)$	0.3333333351...
20	(13, 194, 7561)	18045	[7561, −13407, −11069]	$\frac{13407+\sqrt{514518485}}{15122}$	2.386589070	$(2_2, 1_2, \{2_2, 1_4\}^2)$	0.3333333346...
...	continued						

Table 2 (continued): Markoff triples, Markoff forms (ppconfrac)

n	$(m_1(n), m_2(n), m(n))$	$\tilde{k}(n)$	Mf(n)	$\xi(n)$	approx.	continued fraction	$\lambda(n) = \frac{1}{L(n)}$
21	(34, 89, 9077)	21622	[9077, −16013, −13361]	$\frac{16013+\sqrt{741527357}}{18154}$	2.382064555	$(2_2, 1_6, 2_2, 1_8)$	$0.3_8423237...$
22	(1, 4181, 10946)	26073	[10946, −19308, −16114]	$\frac{4827+\sqrt{67395890}}{5473}$	2.381966012	$(2_2, 1_{18})$	$0.3_8395157...$
23	(29, 169, 14701)	35491	[14701, −26879 − 20791	$\frac{26879+\sqrt{1945074605}}{29402}$	2.414189509	$(2_6, 1_2, 2_4, 1_2)$	$0.3_8367607...$
24	(1, 10946, 28657)	68260	[28657, −50549, −42187]	$\frac{50549+\sqrt{7391012837}}{57314}$	2.381966012	$(2_2, 1_{20})$	$0.3_8342353...$
25	(2, 5741, 33461)	80782	[33461, −61181, −47321]	$\frac{61181+\sqrt{10076746685}}{66922}$	2.414213562	$(2_{12}, 1_2)$	$0.3_8339949...$
26	(29, 433, 37666)	90903	[37666, −68808, −53324]	$\frac{17202+5\sqrt{31921370}}{18833}$	2.413396697	$(\{2_4, 1_2\}^2, 2_2, 1_2)$	$0.3_8338554...$
27	(5, 2897, 43261)	103247	[43261, −76711, −63331]	$\frac{76711+\sqrt{16843627085}}{86522}$	2.386606875	$(\{2_2, 1_2\}^3, 2_2, 1_4)$	$0.3_8337291...$
28	(13, 1325, 51641)	123042	[51641, −91161, −75961]	$\frac{91161+5\sqrt{960045437}}{103282}$	2.382641699	$(\{2_2, 1_4\}^2, 2_2, 1_6)$	$0.3_8336110...$
29	(89, 233, 62210)	148183	[62210, −109736, −91580]	$\frac{27434+\sqrt{2176922306}}{31105}$	2.381980389	$(2_2, 1_8, 2_2, 1_{10})$	$0.3_8335247...$
30	(1, 28657, 75025)	178707	[75025, −132339, −110447]	$\frac{132339+\sqrt{50658755621}}{150050}$	2.381966011	$(2_2, 1_{22})$	$0.3_8334649...$
31	(5, 6466, 96557)	233030	[96557, −176389, −136697]	$\frac{176389+\sqrt{83909288237}}{193114}$	2.413393125	$(2_4, 1_2, \{2_2, 1_2\}^3)$	$0.3_8334127...$
32	(34, 1325, 135137)	321983	[135137, −238555, −198779]	$\frac{238555+\sqrt{164358078917}}{270274}$	2.382641320	$(2_2, 1_4, \{2_2, 1_6\}^2)$	$0.3_8333738...$
33	(2, 33461, 195025)	470832	[195025, −356589, −275807]	$\frac{356589+\sqrt{342312755621}}{390050}$	2.414213562	$(2_{14}, 1_2)$	$0.3_8333528...$
34	(1, 75025, 196418)	467861	[196418, −346468, −289154]	$\frac{86617+\sqrt{21701267282}}{98209}$	2.381966011	$(2_2, 1_{24})$	$0.3_8333525...$
35	(13, 77561, 294685)	703292	[294685, −522529, −431407]	$\frac{522529+\sqrt{781553243021}}{589370}$	2.3865890702	$(2_2, 1_2, \{2_2, 1_4\}^3)$	$0.3_8333418...$
36	(233, 610, 426389)	1015645	[426389, −752123, −627701]	$\frac{752123+\sqrt{1636268213885}}{852778}$	2.381968109	$(2_2, 1_{10}, 2_2, 1_{12})$	$0.3_8333374...$
37	(169, 985, 499393)	1205641	[499393, −913103, −706249]	$\frac{913103+\sqrt{2244540316037}}{998786}$	2.414212855	$(2_8, 1_2, 2_6, 1_2)$	$0.3_8333363...$
38	(1, 196418, 514229)	1224876	[514229, −907065, −757015]	$\frac{907065+\sqrt{2379883179965}}{1028458}$	2.381966011	$(2_2, 1_{26})$	$0.3_8333361..$
39	(5, 43261, 646018)	1541791	[646018, −1145528, −945724]	$\frac{286382+\sqrt{234753331682}}{323009}$	2.386606875	$(\{2_2, 1_2\}^4, 2_2, 1_4)$	$0.3_8333351...$
40	(34, 9077, 925765)	2205232	[925765, −1633169, −1362691]	$\frac{1633169+\sqrt{7713367517021}}{1851530}$	2.382064563	$(\{2_2, 1_6\}^2, 2_2, 1_8)$	$0.3_8333341...$
...							

Table 3: Markoff forms and irrationals (Havil)

n	$k_H(n)$	$Mf_H(n)$	$\xi_H(n)$	continued fraction
1	1	$[1,\ 1,\ -1]$	$\frac{-1+\sqrt{5}}{2}$	$0,\ (1)$
2	1	$[2,\ 4,\ -2]$	$-1\ +\ \sqrt{2}$	$0,\ (2)$
3	$\frac{30-\sqrt{221}}{7}$	$[5,\ \frac{45+2\sqrt{221}}{7},\ \frac{-396-9\sqrt{221}}{49}]$	$\frac{-9+\sqrt{221}}{14}$	$0,\ (2_2,\ 1_2)$
4	$\frac{221-3\sqrt{1517}}{19}$	$[13,\ \frac{299+6\sqrt{1517}}{19},\ \frac{-7762+69,\sqrt{1517}}{361}]$	$\frac{-23+\sqrt{1517}}{38}$	$0,\ (2_2,\ 1_4)$
5	$\frac{1015-6\sqrt{7565}}{41}$	$[29,\ \frac{1537+12\sqrt{7565}}{41},\ \frac{-79871+1681\sqrt{7565}}{1681}]$	$\frac{-53+\sqrt{7565}}{82}$	$0,\ (2_4,\ 1_2)$
6	$\frac{153-16\sqrt{26}}{5}$	$[34,\ \frac{204+32\sqrt{26}}{5},\ \frac{-5*282+96\sqrt{26}}{25}]$	$\frac{-3+\sqrt{26}}{5}$	$0,\ (2_2,\ 1_6)$
7	$\frac{10502-21\sqrt{71285}}{131}$	$[89,\ \frac{13973+42\sqrt{71285}}{131},\ \frac{-2534636+3297\sqrt{71285}}{17161}]$	$\frac{-157+\sqrt{71285}}{262}$	$0,\ (2_2,\ 1_8)$
8	$\frac{34476-35\sqrt{257045}}{239}$	$[169,\ \frac{52221+70\sqrt{257045}}{239},\ \frac{-15822654+10815\sqrt{257045}}{57121}]$	$\frac{-309+\sqrt{257045}}{478}$	$0,\ (2_6,\ 1_2)$
9	$\frac{12319-45\sqrt{2}\sqrt{10585}}{71}$	$[194,\ \frac{16684+90\sqrt{2}\sqrt{10585}}{71},\ \frac{-162068+3870\sqrt{2}\sqrt{10585}}{5041}]$	$\frac{-86+\sqrt{2}\sqrt{10585}}{142}$	$0,\ (2_2,\ 1_2,\ 2_2\ 1_4)$
10	$\frac{71997-55\sqrt{488597}}{343}$	$[233,\ \frac{95763+110\sqrt{488597}}{343},\ \frac{-45493962+22605\sqrt{488597}}{117649}]$	$\frac{-411+\sqrt{488597}}{686}$	$0,\ (2_2,\ 1_{10})$
11	$\frac{226892-90\sqrt{1687397}}{613}$	$[433,\ \frac{342503+180\sqrt{1687397}}{613},\ \frac{-266796487+71190\sqrt{1687397}}{375769}]$	$\frac{-791+\sqrt{1687397}}{1226}$	$0,\ (2_4,\ 1_2,\ 2_2,\ 1_2)$
12	$\frac{246745-288\sqrt{209306}}{449}$	$[610,\ \frac{328180+576\sqrt{209306}}{449},\ \frac{-204096706+154944\sqrt{209306}}{201601}]$	$\frac{-269+\sqrt{209306}}{449}$	$0,\ (2_2,\ 1_{12})$
13	$\frac{1171165-204\sqrt{8732021}}{1393}$	$[985,\ \frac{1773985+408\sqrt{8732021}}{1393},\ \frac{-3132855709+367404\sqrt{8732021}}{1940449}]$	$\frac{-1801+\sqrt{8732021}}{2786}$	$0,\ (2_8,\ 1_2)$
14	$\frac{2324050-312\sqrt{15800621}}{1949}$	$[1325,\ \frac{3099175+624\sqrt{15800621}}{1949},\ \frac{-8351506877+729768\sqrt{15800621}}{3798601}]$	$\frac{-2339+\sqrt{15800621}}{3898}$	$0,\ (2_2,\ 1_4,\ 2_2,\ 1_6)$
15	$\frac{3382446-377\sqrt{22953677}}{2351}$	$[1597,\ \frac{4498749+754\sqrt{22953677}}{2351},\ \frac{-14649547788+1062009\sqrt{22953677}}{5527201}]$	$\frac{-2817+\sqrt{22953677}}{4702}$	$0,\ (2_2,\ 1_{14}$
16	$\frac{10988321-672\sqrt{75533477}}{4241}$	$[2897,\ \frac{14881889+1344\sqrt{75533477}}{4241},\ \frac{-86351551313+3452064\sqrt{75533477}}{17986081}]$	$\frac{-5137+\sqrt{75533477}}{8482}$	$0,\ (\{2_2,\ 1_2\}^2,\ 2_2,\ 1_4)$
17	$\frac{4636729-987\sqrt{5}\sqrt{31465369}}{1231}$	$[4181,\ \frac{5*6166975+1974\sqrt{5}\sqrt{31465369}}{6155},\ \frac{-52575135994+5*291165\sqrt{5}\sqrt{31465369}}{7576805}]$	$\frac{-5*1475+\sqrt{5}\sqrt{31465369}}{12310}$	$0,\ (2_2,\ 1_{16})$
18	$\frac{39785130-5945\sqrt{11865269}}{8119}$	$[5741,\ \frac{60263277+11890\sqrt{11865269}}{8119},\ \frac{-620289899664+62404665\sqrt{11865269}}{65918161}]$	$\frac{-10497+5\sqrt{11865269}}{16238}$	$0,\ (2_{10},\ 1_2)$
19	$\frac{25298225-13440\sqrt{940706}}{4577}$	$[6466,\ \frac{38188196+26880\sqrt{940706}}{4577},\ \frac{-222111139906+79376640\sqrt{940706}}{20948929}]$	$\frac{-2953+5\sqrt{940706}}{4577}$	$0,\ (2_4,\ 1_2,\ \{2_2,\ 1_2\}^2)$
20	$\frac{74853900-1754\sqrt{514518485}}{11069}$	$[7561,\ \frac{101370327+(3508\sqrt{514518485}}{11069},\ \frac{-1535265995439+23515878\sqrt{514518485}}{122522761}]$	$\frac{-13407+\sqrt{514518485}}{22138}$	$0,\ (2_2,\ 1_2,\ \{2_2,\ 1_4\}^2)$
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