A lower bound on the number of permutations in S_n that satisfy the equality $I_n + EX_n = D_n$

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In this note, we prove a lower bound formula for the sequence A301897 in the OEIS. Let S_n be the symmetric group of order n. For permutation $b = (b_1, ..., b_n) \in S_n$, let $I_n(b)$ be the number of inversions in b; $EX_n(b)$ be the smallest number of transpositions needed to transform b into the identity $(1, ..., n)$; and $D_n(b) = \sum_{1 \le i \le n} |b_i - i|$. Cayley (1849) proved that $EX_n(b)$ equals n minus the number of cycles in permutation b. In the references for sequence A062869, the quantity $D_n(b)$ is called the **total distance** of b or the **total displacement** of b.

Diaconis and Graham (1977) proved that, for each integer $n \ge 1$,

$$I_n(b) + EX_n(b) \le D_n(b) \le 2I_n(b), \tag{1}$$

while Hadjicostas and Monico (2015) proved that

$$D_n(b) \le I_n(b) + EX_n(b) + \lfloor n/2 \rfloor (\lfloor n/2 \rfloor - 1).$$

For each integer $n \ge 1$, let $a(n) =$ A301897(n) equal the number of permutations b in S_n that satisfy the equality

$$I_n(b) + EX_n(b) = D_n(b).$$

Let M_n be the set of all permutations b in S_n that satisfy $I_n(b) + EX_n(b) = D_n(b)$. We describe how to construct M_n recursively. Let $M_1 = S_1$. For $n \ge 2$, construct the set M_{n1} from the set M_{n-1} as follows: Take a permutation c in M_{n-1} and an integer i in $\{1, ..., n-1\}$ such that either the number of integers in c to the left of c_i that are greater than c_i is zero or the number of integers in c to the right of c_i that are less than c_i is zero. Replace c_i with n and put c_i at the end to form permutation b in M_{n1}. Construct the set M_{n2} from the set M_{n-1} as follows: take a permutation c in M_{n-1} and attach n at the end to form permutation b in M_{n2}. Finally, define M_n as the union of the sets M_{n1} and M_{n2}. See Hadjicostas and Monico (2013) for more precise details.

Hadjicostas and Monico (2013) proved that, for $n \geq 1$,

$$a(n) \leq U_n := 1 + \sum_{k=2}^{n} \left((k-1)! \sum_{\ell=1}^{k-1} \frac{2}{\ell} - \sum_{\ell=1}^{k-1} (\ell-1)!(k-\ell-1)! \right). \tag{2}$$

This implies that $a(n)/n! = \mathcal{O}(\log n/n)$. In this short note, we prove that

$$a(n) \geq L_n := 1 + \sum_{\ell=2}^{n} (\ell-1) F_{2\ell-3}. \tag{3}$$

Here, F_m is the m-th Fibonacci number defined by $F_0 = 0$, $F_1 = 1$, and $F_m = F_{m-1} + F_{m-2}$ for $m \geq 2$. The generating function of the numbers $(L_n : n \geq 1)$ is given by

$$G(x) = \sum_{n=1}^{\infty} L_n x^n = \frac{x(x^4 - 4x^3 + 9x^2 - 5x + 1)}{(1-x)(1-3x+x^2)^2}.$$

Table 1 contains upper and lower bounds for $a(n) =$ A301897(n), obtained through the formulae (2) and (3), from $n = 1$ to $n = 14$. (Recall that $n = 14$ is the maximum value of n for which we know the exact value of $a(n)$.)

Table 1: Upper (U_n) and lower (L_n) bounds for $a(n) =$ A301897(n) for $1 \leq n \leq 14$

n	L_n	$a(n)$	U_n
1	1	1	1
2	2	2	2
3	6	6	6
4	21	23	23
5	73	103	107
6	243	511	591
7	777	2719	3807
8	2408	15205	28131
9	7288	88197	235011
10	21661	526018	2192547
11	63471	3206206	22608867
12	183877	19885911	255442467
13	527761	125107063	3138886947
14	1503086	796453594	41684035107

Diaconis and Graham (1977) and Hadjicostas and Monico (2013) examined when

$$I_n(b) + EX_n(b) = D_n(b) = 2I_n(b); \tag{4}$$

i.e., when both inequalities in (1) hold as equalities simultaneously. This happens if and only if $I_n(b) = EX_n(b)$, which in turn happens if and only if permutation b belongs to M_n and has no 3-inversion (i.e., it avoids the pattern 321). A 3-inversion in b is a triplet of integers (i, j, k) in $\{1, ..., n\}$ such that $i < j < k$ but $b_i > b_j > b_k$. The number of permutations in S_n that satisfy both equalities is $F_{2n-1} =$ A001519$(n) =$ A000045$(2n-1)$. We will use this observation to prove lower bound (3) above.

Proof of lower bound (3): From a discussion earlier in the note, we know that

$$a(n) = \#M_n = \#M_{n1} + \#M_{n2} = \#M_{n1} + \#M_{n-1} = \#M_{n1} + a(n-1) \quad \text{for } n \geq 2.$$

We shall prove that

$$\#M_{n1} \geq (n-1)F_{2n-3} \quad \text{for all } n \geq 2. \tag{5}$$

Since $a(1) = 1$, inequality (5) would imply that, for $n \geq 1$,

$$a(n) = 1 + \sum_{\ell=2}^{n}(a(\ell) - a(\ell-1)) \geq 1 + \sum_{\ell=2}^{n} \#M_{\ell 1} = 1 + \sum_{\ell=2}^{n}(\ell-1)F_{2\ell-3} = L_n.$$

To prove inequality (5), define Π_n to be the set of all permutations $b \in S_n$ that satisfy both equalities in (4). From a discussion above, we know that

$$\#\Pi_n = F_{2n-1} \quad \text{for } n \geq 1.$$

Clearly, $\Pi_{n-1} \subseteq M_{n-1}$ for $n \geq 2$. Let $c \in \Pi_{n-1}$. Assume that there is $j \in \{1, \ldots, n-1\}$ such that, in permutation c, the number of integers to the left of c_j that are greater than c_j is not zero and the number of integers to the right of c_j that are less than c_j is not zero. Then c has a 3-inversion, a contradiction.

Hence, for each $i \in \{1, \ldots, n-1\}$, either the number of integers in c to the left of c_i that are greater than c_i is zero or the number of integers in c to the right of c_i that are less than c_i is zero. By replacing c_i with n and putting c_i at the end of c, we form a permutation b in the set M_{n1} (whose construction is described at the beginning of this note).

We claim that this process produces distinct permutations b in M_{n1}. To prove that, assume that the pairs (c, i) and $(\tilde{c}, \tilde{i})$ of permutations $c, \tilde{c} \in \Pi_{n-1}$ and integers $i, \tilde{i} \in \{1, \ldots, n-1\}$ produce permutations b and $\tilde{b}$ in M_{n1}, respectively. Assume $b = \tilde{b}$. Then $b_i = n = \tilde{b}_{\tilde{i}}$, $i = \tilde{i}$, and $c_i = b_n = \tilde{b}_n = \tilde{c}_{\tilde{i}}$. In addition, for $j \neq i = \tilde{i}$,

$$c_j = b_j = \tilde{b}_j = \tilde{c}_j.$$

Thus, $c = \tilde{c}$ and our claim above has been proved.

It follows that

$$\#M_{n1} \geq (n-1)\,(\#\Pi_{n-1}) = (n-1)F_{2n-3}.$$

This proves inequality (5), which implies lower bound (3). □

Remark 1. The methodology used in the proof of lower bound (3) is similar to the proof of upper bound (2) given in Section 5 of Hadjicostas and Monico (2013).

References

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