LABELED TREES WITH FIXED NODE LABEL SUM

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Abstract. The non-cyclic graphs known as trees may be labeled by assigning positive integer numbers (weights) to their nodes. We count the trees up to 7 nodes that have prescribed sums of labels, or, from the number-theoretic point of view, we count the compositions of positive integers that are constrained by the symmetries of trees.

1. Partitions and Compositions

Whereas partitions of n into some fixed number of parts regard the order of the parts as irrelevant and to not pay attention to the symmetry or order of the parts, the compositions of n take the opposite point of view and consider the group of all permutations of the parts to generate distinct objects. Somewhere in between these extremes one may impose partial restrictions of symmetry on the arrangement of the parts.

The compositions might for example be counted only by one representative of a set of compositions of n into m parts which are inequivalent under cyclic shifts. The associated triangle is [1, A037306] with row sums in [1, A008965].

2. Compositions reduced by Tree's Symmetries

In a similar strategy one may partition integers imposing the symmetry of the graphs known as trees, which essentially attaches the parts as labels to the nodes. We ask how many different labeled trees exist with a given node label sum, i.e., how many different compositions of an integer exist that are different if their associated labeled trees are different, and compute a complete table of these labeled (weighted) trees up to 7 nodes by providing the ordinary generating functions.

The number of compositions of n into m positive parts is a variant of Pascal's triangle [3, §1.2],

$$\bar{C}(n,m) = \binom{n-1}{m-1} \tag{1}$$

with generating function

$$\bar{C}_m(x) = \sum_{n\ge m} \bar{C}(n,m) z^n = \frac{z^m}{(1-z)^m}. \tag{2}$$

The number of partitions of n into m positive parts has the generating function [3, (1.77)]

$$P_m(x) = \frac{x^m}{\prod_{i=1}^{m}(1-x^i)}. \tag{3}$$

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The number of ways of partitioning a number over the labels for a tree with one node has the generating function

$$v_1 = \bar{C}_1(x) \mapsto 1, 1, 1, 1, 1, 1 \ldots (n \ge 1) \tag{4}$$

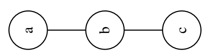
because, whatever the sum is, there is only the single option to put that sum onto the only existing node.

The number of ways of partitioning a number over the labels of the tree with two nodes is

$$v_2 = P_2(x) \mapsto 1, 1, 2, 2, 3, 3, 4, 4 \ldots (n \ge 2) \tag{5}$$

because there are $\lfloor n/2 \rfloor$ ways to push a fraction of the integer on one label (the other gets the remainder). Because the tree is symmetric with respect to swapping the two nodes all variants are exhausted then.

3. Three nodes

There is one tree with 3 nodes.  Its symmetry is that it may be flipped over with an inert middle node, reading the labels from either direction, so the compositions $a+b+c$ and $c+b+a$ are considered the same. The generating function of the middle node indicates that there is one way to insert any positive integer [1, A000012],

$$\bar{C}_1(x) = \sum_{n \ge 1} 1 \times x^n = \frac{x}{1-x} \tag{6}$$

The generating functions of the two terminal elements is the same; because their order does not matter, they represent the number of partitions of a number into two parts [1, A004526]:

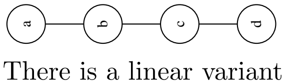
$$P_2(x) = \frac{x^2}{(1+x)(1-x)^2} \mapsto 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7 \ldots (n \ge 2). \tag{7}$$

Convolution with the inert node b gives [1, A002620]

$$v_3(x) = \bar{C}_1(x) P_2(x) = \frac{x^3}{(1+x)(1-x)^3} \mapsto 1, 2, 4, 6, 9, 12, 16, 20, 25, 30 \ldots (n \ge 3) \tag{8}$$

4. Four nodes

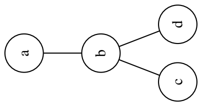
4.1. The linear graph with 4 nodes (n-butane) has the following structure:



There is a linear variant where the entire string may be flipped over, equivalent to the symmetry of the group of order 2 with cycle index $(x_1^2+x_2)/2$. The individual subgraphs of nodes ab and nodes cd halves have generating functions $\bar{C}_2(x)$, so with $x_i = \bar{C}_2(x^i)$ [1, A005993]

$$v_{4a}(x) = \frac{(\frac{x^2}{(1-x)^2})^2 + \frac{x^4}{(1-x^2)^2}}{2} = \frac{x^4(1+x^2)}{(1-x)^4(1+x)^2} \mapsto 1, 2, 6, 10, 19, 28, 44 \ldots (n \ge 4) \tag{9}$$

4.2. The star graph with 4 nodes has the following structure:



The center node b has the generating function $\bar{C}_1(x)$ and the outer 3 elements are counted irrespective of order, and represent the number of ways of partitioning n into 3 nonnegative parts [1, A069905]

(10)
$$P_3(x) = \frac{x^3}{(1+x)(1+x+x^2)(1-x)^3} \mapsto 1, 1, 2, 3, 4, 5, 7, 8, 10, 12, 14, 16, 19, 21, \dots (n \geq 3)$$

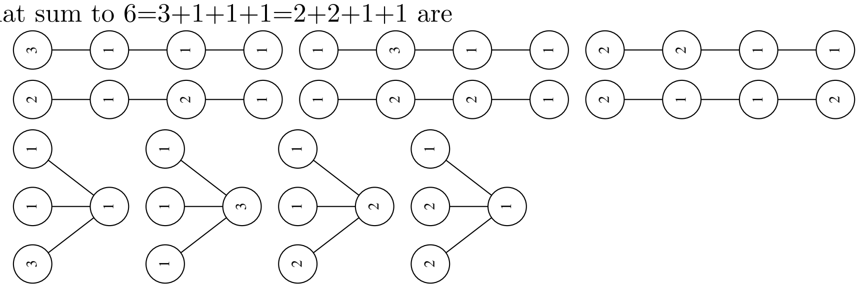
The convolution of these two sequences is [1, A000601]

(11)
$$v_{4b} = \bar{C}_1(x)P_3(x) = \frac{x^4}{(1+x)(1+x+x^2)(1-x)^4} \mapsto 1, 2, 4, 7, 11, 16, 23, 31, 41, 53 \dots (n \geq 4)$$

The total partitions with the symmetries of the two trees with 4 nodes are [1, A301739]

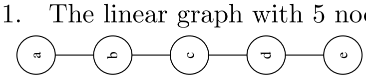
(12)
$$v_4 = v_{4a} + v_{4b} = \frac{x^4(2+2x+2x^2+x^3+x^4)}{(1+x+x^2)(1+x)^2(1-x)^4} \mapsto 2, 4, 10, 17, 30, 44, 67, 91, 126 \dots (n \geq 4)$$

An example for the third nonzero term in that sequence: The 10 trees with labels that sum to 6=3+1+1+1=2+2+1+1 are



5. Five nodes

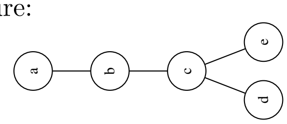
5.1. The linear graph with 5 nodes (n-pentane) has the following structure:



The middle node c is inert by engaging symmetries; the outer elements ab and de have the “palindromic” C_2 symmetry of (9). The convolution of (6) and (9) is [1, A005994]

$$v_{5a} = \bar{C}_1(x)v_{4a} = \frac{x^5(1+x^2)}{(1+x)^2(1-x)^5} \mapsto 1, 3, 9, 19, 38, 66, 110, 170 \dots (n \geq 5) \quad (13)$$

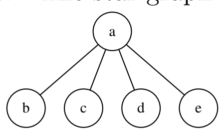
5.2. The bifurcating graph with 5 nodes (2-Methyl-butane) has the following structure:



There is a backbone of 3 nodes abc with full compositorial symmetry as represented by $\bar{C}_3(x)$ [1, A000217], and the two twigs d and e at the end are the number of ways of partitioning into 2 parts (7). The convolution is [1, A001752]

(14)
$$v_{5b} = \bar{C}_3(x)P_2(x) = \frac{x^5}{(1-x)^5(1+x)} \mapsto 1, 4, 11, 24, 46, 80, 130, 200, 295 \ldots (n \geq 5)$$

5.3. The star graph with 5 nodes has the following structure:



The inert node a is represented by (6) and the 4 outer nodes $bcde$ mean partitioning numbers into 4 parts [1, A026810],

(15)
$$P_4(x) = \frac{x^4}{(1-x)^4(1+x)^2(1+x+x^2)(1+x^2)} \mapsto 1, 1, 2, 3, 5, 6, 9, 11, 15, 18, 23, 27, 34, 39, 47, 54, 64 \ldots (n \geq 4)$$

The convolution of the two sequences is [1, A002621]

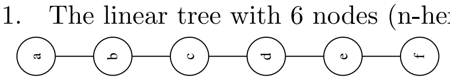
(16)
$$v_{5c} = \bar{C}_1(x)P_4(x) = \frac{x^5}{(1-x)^5(1+x)^2(1+x+x^2)(1+x^2)} \mapsto 1, 2, 4, 7, 12, 18, 27, 38, 53, 71, 94 \ldots (n \geq 5)$$

The sum of the contribution of the three types of trees on 5 nodes is [1, A301740]

(17)
$$v_5 = v_{5a}+v_{5b}+v_{5c} = \frac{x^5(3+3x+6x^2+5x^3+5x^4+2x^5+x^6)}{(1-x)^5(1+x)^2(1+x+x^2)(1+x^2)} \mapsto 3, 9, 24, 50, 96, 164, 267 \ldots (n \geq 5)$$

6. Six nodes

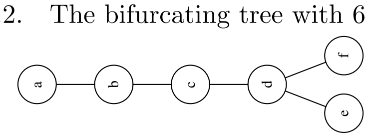
6.1. The linear tree with 6 nodes (n-hexane) has the following structure:



It can be flipped over at the edge from c to d, which represents the C_2 symmetry of two groups of compositions of 3. The individual groups abc and def are counted by $\bar{C}_3(x)$. The cycle index is $(x_1^2 + x_2)/2$, so with $x_i = \bar{C}_3(x^i)$ [1, A005995]

(18)
$$v_{6a} = \frac{x^6(1+3x^2)}{(1-x)^6(1+x)^3} \mapsto 1, 3, 12, 28, 66, 126, 236, 396, 651 \ldots (n \geq 6)$$

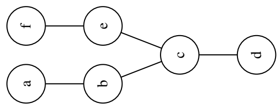
6.2. The bifurcating tree with 6 nodes (2-Methyl-pentane) is:



There is a backbone of the nodes $abcd$ contributing $\bar{C}_4(x)$ [1, A000292], plus the two twigs ef representing the partitions in 2 parts to yield [1, A001753]

(19)
$$v_{6b} = \bar{C}_4(x)P_2(x) = \frac{x^6}{(1-x)^6(1+x)} \mapsto 1, 5, 16, 40, 86, 166, 296, 496, 791 \ldots (n \geq 6)$$

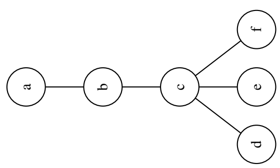
6.3. The symmetric tree with 6 nodes (3-Methyl-pentane)



There is a C_2-symmetry of flipping this over, where the inert nodes are cd contributing $\bar{C}_2(x)$. The cycle index of the group C_2 and the number of compositions of either ab or ef into 2 elements with $x_i = \bar{C}_2(x^i)$ give v_{4a}. The convolution with the factor from the inert nodes cd is

$$v_{6c} = \bar{C}_2(x)v_{4a} = \frac{x^6(1+x^2)}{(1-x)^6(1+x)^2} \mapsto 1, 4, 13, 32, 70, 136, 246, 416 \ldots (n \geq 6) \tag{20}$$

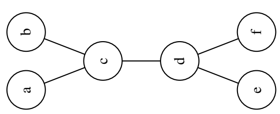
6.4. The tree with 6 nodes and two side chains [(2,2)-Dimethyl-butane] looks as



There is a backbone of abc with the number of compositions of 3 plus the number of partitions into 3 parts at def [1, A001399],

$$v_{6d} = \bar{C}_3(x)P_3(x) = \frac{x^6}{(1-x)^6(1+x)(1+x+x^2)} \mapsto 1, 4, 11, 25, 50, 91, 155, 250, 386, 575 \ldots (n \geq 6) \tag{21}$$

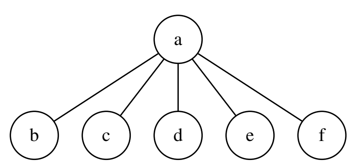
6.5. The tree with 6 nodes and two side chains [(2,3)-Dimethyl-butane] is



The symmetries are a group of order 8, generated by an element of order 2 that swaps $a \leftrightarrow b$ with permutation cycle representation (12), another element that swaps $e \leftrightarrow f$ with cycle representation (56) and an element that flips the graph over with cycle representation (26)(15)(34). The cycle index computed by GAP is $(x_1^6 + 2x_1^4x_2 + x_1^2x_2^2 + 2x_2^3 + 2x_2x_4)/8$ [2]. Insertion of $x_i = \bar{C}_1(x^i)$ yields

$$v_{6e} = \frac{x^6(1-x+2x^2)}{(1-x)^6(1+x)^3(1+x^2)} \mapsto 1, 2, 7, 14, 31, 54, 97, 154 \ldots (n \geq 6) \tag{22}$$

6.6. The star graph with 6 nodes is



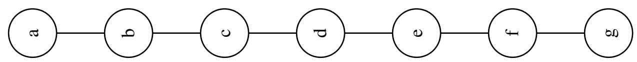
The inert node a and the partitions into 5 parts over $bcdef$ [1, A001401] gives [1, A002622]

$$v_{6f} = \bar{C}_1(x)P_5(x) = \frac{x^6}{(1-x)^6(1+x)^2(1+x+x^2)(1+x^2)(1+x+x^2+x^3+x^4)} \mapsto 1, 2, 4, 7, 12, 19, 29, 42, 60, 83, 113, 150 \ldots (n \geq 6) \tag{23}$$

The total over all 6 trees with 6 nodes is

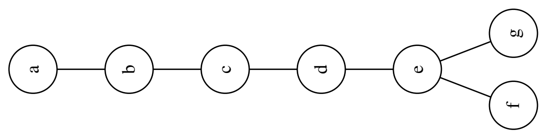
$$(24)\quad v_6 = v_{6a} + v_{6b} + v_{6c} + v_{6d} + v_{6e} + v_{6f}$$
$$= \frac{x^6(6 + 14x + 31x^2 + 49x^3 + 69x^4 + 74x^5 + 72x^6 + 55x^7 + 37x^8 + 17x^9 + 7x^{10} + x^{11})}{(1-x)^6(1+x+x^2+x^3+x^4)(1+x)^3(1+x^2)(1+x+x^2)}$$
$$\mapsto 6, 20, 63, 146, 315, 592 \ldots (n \ge 6)$$

7. Seven nodes

7.1. The linear tree with 7 nodes is 

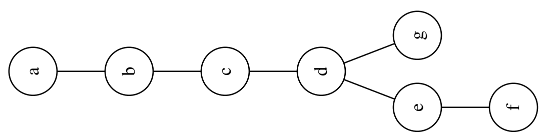
The is an inert middle node d and a left-right symmetry with compositions in 3 parts counted by $\bar{C}_3(x)$ with the aforementioned cycle index already given by (18). The convolution of (6) with (18) yields [1, A018210]

$$(25)\qquad v_{7a} = \frac{x^7(1+3x^2)}{(1-x)^7(1+x)^3} \mapsto 1, 4, 16, 44, 110, 236, 472 \ldots (n \ge 7)$$



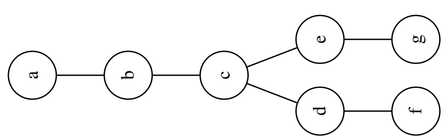
7.2. The structure of 2-Methyl-hexane is
There is a back bone of 5 nodes counted by (2) and the factor of the partitions in 2 parts [1, A001769]:

$$(26)\quad v_{7b} = \bar{C}_5(x)P_2(x) = \frac{x^7}{(1-x)^7(1+x)} \mapsto 1, 6, 22, 62, 148, 314, 610 \ldots (n \ge 7)$$



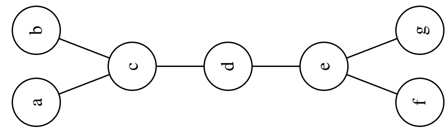
7.3. The structure of 3-Methyl-hexane is
There is no symmetry, so all compositions of (2) contribute [1, A000579]:

$$(27)\qquad v_{7c} = \bar{C}_7(x) = \frac{x^7}{(1-x)^7} \mapsto 1, 7, 28, 84, 210, 461 \ldots (n \ge 7).$$



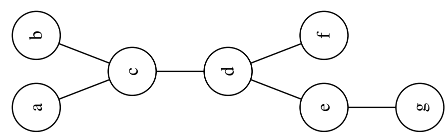
7.4. The structure of 3-Ethyl-pentane is
The inert node c is represented by $\bar{C}_1(x)$ and the symmetry for the group of permutations (16)(24) and (45)(67) that rotate the pairs ab, df and eg around it has cycle index $(x_1^6 + 3x_1^2x_2^2 + 2x_3^2)/6$. With $x_i = \bar{C}_1(x^i)$

$$(28)\quad v_{7d} = \frac{x^7(1+x^2+2x^3+x^4+x^6)}{(1-x)^7(1+x+x^2)^2(1+x)^2} \mapsto 1, 3, 9, 23, 51, 103, 196, 348 \ldots (n \ge 7)$$

7.5. The structure of (2,4)-Dimethyl-pentane is 

The structure has the symmetry that lead to (22) plus an inert node d in the center. The convolution of (22) and (6) is

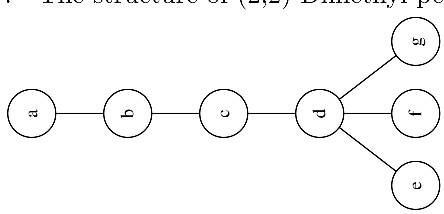
(29)
$$v_{7e} = \bar{C}_1(x) v_{6e} = \frac{x^7(1-x+2x^2)}{(1-x)^7(1+x)^3(1+x^2)} \mapsto 1, 3, 10, 24, 55, 109, 206, 360 \ldots (n \geq 7)$$

7.6. The structure of (2,3)-Dimethyl-pentane is 

There is a C_2 symmetry that swaps positions a and b which gives a factor (7), and the other positions are asymmetric with $\bar{C}_5(x)$. This yields [1, A001769]

(30)
$$v_{7f} = v_{7b} \mapsto 1, 6, 22, 62, 148, 314, 610, 1106 \ldots (n \geq 7)$$

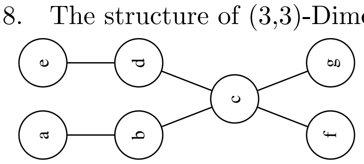
7.7. The structure of (2,2)-Dimethyl-pentane is



The generates the factor $\bar{C}_4(x)$ from the nodes $abcd$ and a factor $P_3(x)$ for the partitions over efg:

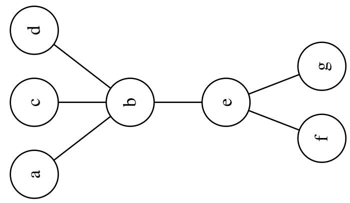
(31)
$$v_{7g} = \bar{C}_4(x) P_3(x) = \frac{x^7}{(1-x)^7(1+x)(1+x+x^2)} \mapsto 1, 5, 16, 41, 91, 182, 337, 587 \ldots (n \geq 7)$$

7.8. The structure of (3,3)-Dimethyl-pentane is



The node c is inert. There is one symmetry operation that swaps nodes f and g with reduced cycle representation (67), and another operation that swaps a with e and b with d with reduced cycle representation (24)(15). GAP computes the cycle index $(x_1^6 + x_1^4 x_2 + x_1^2 x_2^2 + x_2^3)/4$ for this group of order 4.

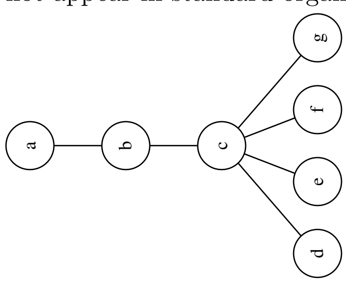
(32)
$$v_{7h} = \frac{x^7(1+x^2)}{(1-x)^7(1+x)^3} \mapsto 1, 4, 14, 36, 84, 172, 330, 588 \ldots (n \geq 7)$$

7.9. The structure of (2,2,3)-Tri-methyl-butane is 

The symmetry of the nodes *abcd* is represented by the star graph with 4 nodes (11), and the symmetry of nodes *efg* has already been presented in (8). The convolution is

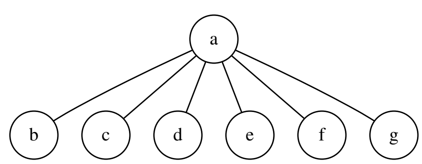
(33)
$$v_{7i} = v_3 v_{4b} = \frac{x^7}{(1-x)^7(1+x)^2(1+x+x^2)} \mapsto 1, 4, 12, 29, 62, 120, 217, 370, 603 \ldots (n \geq 7)$$

7.10. The final two trees with 7 nodes have valences (degrees) larger than 4 and do not appear in standard organic chemistry. First



One factor is the contribution of nodes *abc* as $\bar{C}_3(x)$, the other is the number of ways to partition into 4 parts over the nodes *defg*, $P_4(x)$:

$$(34) \quad v_{7j} = \bar{C}_3(x) P_4(x) = \frac{x^7}{(1-x)^7(1+x)^2(1+x^2)(1+x+x^2)} \mapsto 1, 4, 11, 25, 51, 95, 166, 275, 347 \ldots (n \geq 7)$$

7.11. The star graph with 7 nodes is 

The generating function is the product of the factor by the inert node a and the factor of the partitioning over the 6 nodes *bcdefg*, [1, A288341]

(35)
$$v_{7k} = \bar{C}_1(x) P_6(x) = \frac{x^7}{(1-x)^7(1+x)^3(1+x+x^2)^2(1+x^2)(1+x+x^2+x^3+x^4)(1-x+x^2)} \mapsto 1, 2, 4, 7, 12, 19, 30, 44, 64, 90, 125 \ldots (n \geq 7)$$

$s \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12
1	1											
2	1	1										
3	1	1	1									
4	1	2	2	2								
5	1	2	4	4	3							
6	1	3	6	10	9	6						
7	1	3	9	17	24	20	11					
8	1	4	12	30	50	63	48	23				
9	1	4	16	44	96	146	164		47			
10	1	5	20	67	164	315	437			106		
11	1	5	25	91	267	592	1022				235	
12	1	6	30	126	408	1059	2126					551
13	1	6	36	163	603	1754	4098					
14	1	7	42	213	856	2805	7368					
15	1	7	49	265	1186	4270	12590					

TABLE 1. The number of trees on n nodes with positive integer node labels that sum for each tree to s.

The total over all 11 trees with 7 nodes is

$$(36)\quad v_7 = v_{7a} + v_{7b} + v_{7c} + v_{7d} + v_{7e} + v_{7f} + v_{7g} + v_{7h} + v_{7i} + v_{7j} + v_{7k}$$

$$= x^7(11 + 26x + 68x^2 + 120x^3 + 196x^4 + 257x^5 + 320x^6 + 332x^7 + 327x^8 + 272x^9 + 211x^{10} + 134x^{11} + 80x^{12} + 33x^{13} + 12x^{14} + 2x^{15})$$

$$\times \frac{1}{(1-x)^7(1+x)^3(1+x+x^2)^2(1+x^2)(1+x+x^2+x^3+x^4)(1-x+x^2)}$$

$$\mapsto 11, 48, 164, 437, 1022, 2126, 4098, 7368, 12590 \ldots (n \geq 7)$$

8. Summary (Vertex Labeled)

The overview of Table 1 fills sequences (8), (12), (17), (24) and (36) into the columns. Its diagonal contains the number of unlabeled trees [1, A000055]. The first sub-diagonal contains the number of unlabeled rooted trees [1, A000081]. The row sums 1,2,3,7,14,... are [1, A036250]. The second column are repeated integers [1, A004526]. The third column is [1, A002620], the forth [1, A301739], and the fifth [1, A301740].

9. One or two edges

A number may as well be partitioned over the edges of trees, leading to edge-labeled trees. The symmetry considerations of the previous sections remain essentially intact; the compositions of n into as many parts as there are edges are roughly obtained by reducing the degrees of freedom by one (as the number of edges in a tree is one less than the number of nodes). Zero or one edge (suffix still counting the nodes in the tree) gets all the weight:

$$(37)\qquad e_1 = e_2 = \bar{C}_1(x) \mapsto 1, 1, 1, 1, 1, \ldots .$$

Two edges in the tree of 3 nodes have the C_2 symmetry:

(38) $$e_3 = P_2(x) \mapsto 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, \ldots.$$

10. Three edges

10.1. The linear tree with 4 nodes contributes

(39) $$e_{4a} = \bar{C}_1(x) P_2(x) = v_3 \mapsto 1, 2, 4, 6, 9, \ldots (n \geq 3)$$

10.2. The star graph with 4 nodes partitions the weight over 3 symmetric edges as in (10):

(40) $$e_{4b} = P_3(x) \mapsto 1, 1, 2, 3, 4, 5, 7, 8, 10, 12 \ldots (n \geq 3)$$

The number of ways of partitioning n over the edges of trees with 4 nodes is

(41) $$e_4 = e_{4a} + e_{4b} = \frac{x^3(2 + x + x^2)}{(1-x)^3(1+x)(1+x+x^2)} \mapsto 2, 3, 6, 9, 13, 17, 23, 28, 35, 42, 50, 58, \ldots (n \geq 3)$$

11. Four Edges

11.1. On five nodes we get from the linear graph

(42) $$e_{5a} = v_{4a} = \frac{x^4(1+x^2)}{(1-x)^4(1+x)^2} \mapsto 1, 2, 6, 10, 19, 28, 44, \ldots (n \geq 4).$$

11.2. In the bifurcating graph the two edges along the backbone contribute $\bar{C}_2(x)$ and the two edges in the twigs contribute $P_2(x)$ [1, A002623]:

(43) $$e_{5b} = \bar{C}_2(x) P_2(x) = \frac{x^4}{(1-x)^4(1+x)} \mapsto 1, 3, 7, 13, 22, 34, 50, 70, \ldots (n \geq 4)$$

11.3. The star graph contributes [1, A026810]

(44) $$e_{5c} = P_4(x) = \frac{x^4}{(1-x)^4(1+x)^2(1+x+x^2)(1+x^2)} \mapsto 1, 1, 2, 3, 5, 6, 9, 11, 15, 18, \ldots (n \geq 4)$$

The number of ways of partitioning n over the edges of trees with 5 nodes is

(45) $$e_5 = e_{5a} + e_{5b} + e_{5c} = \frac{x^4(3 + 3x + 6x^2 + 5x^3 + 5x^4 + 2x^5 + x^6)}{(1-x)^4(1+x)^2(1+x+x^2)(1+x^2)} \mapsto 3, 6, 15, 26, 46, 68, 103, 141, 195, 253, \ldots (n \geq 4)$$

12. Five Edges

12.1. The linear graph on 6 nodes contributes

(46) $$e_{6a} = v_{5a} \mapsto 1, 3, 9, 19, 38, 66, 110, 170, \ldots (n \geq 5)$$

12.2. The bifurcating graph on 6 nodes contributes

(47) $$e_{6b} = v_{5b} \mapsto 1, 4, 11, 24, 46, 80, 130, 200, 295, \ldots (n \geq 5)$$

12.3. The graph of 3-Methyl-Pentane contributes

(48) $$e_{6c} = \bar{C}_1(x) v_{4a} = v_{5a} = \frac{x^5(1+x^2)}{(1-x)^5(1+x)^2} \mapsto 1, 3, 9, 19, 38, 66, 110, 170, 255, 365, \ldots (n \geq 5)$$

$s\backslash n$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1										
2	1	1	1									
3	1	1	1	2								
4	1	1	2	3	3							
5	1	1	2	6	6	6						
6	1	1	3	9	15	16	11					
7	1	1	3	13	26	43	37	23				
8	1	1	4	17	46	88	116		47			
9	1	1	4	23	68	169	273			106		
10	1	1	5	28	103	287	585				235	
11	1	1	5	35	141	467	1104					551
12	1	1	6	42	195	711	1972					
13	1	1	6	50	253	1051	3270					
14	1	1	7	58	330	1489	5222					
15	1	1	7	68	412	2063	7958					

TABLE 2. The number of trees on n nodes with positive integer edge labels that sum for each tree to s. The row sums are (discarding the 1 from the single node) 1,2,4,9,21,55,146...

12.4. The tree of (2,2)-Dimethyl-Butane contributes [1, A057524]

(49)
$$e_{6d} = \bar{C}_2(x)P_3(x) = \frac{x^5}{(1-x)^5(1+x)(1+x+x^2)} \mapsto 1, 3, 7, 14, 25, 41, 64, 95, 136, 189, \ldots (n \geq 5)$$

12.5. The cycle index for the permutation group generated by (12) and (34) and (13)(24) (reduced cycle notation) is $(x_1^4 + 2x_1^2x_2 + 3x_2^2 + 2x_4)/8$ where $x_i = \bar{C}_1(x^i)$:

(50)
$$e_{6e} = \bar{C}_1(x)\frac{x_1^4 + 2x_1^2x_2 + 3x_2^2 + 2x_4}{8} = \frac{x^5(1-x+x^2)}{(1-x)^5(1+x)^2(1+x^2)} \mapsto 1, 2, 5, 9, 17, 27, 43, 63, 92, 127 \ldots (n \geq 5).$$

12.6. The star graph on 6 nodes has 5 freely permutable edges [1, A026811],

$$e_{6f} = P_5(x) \mapsto 1, 1, 2, 3, 5, 7, 10, 13, 18, 23, 30, \ldots (n \geq 5). \quad (51)$$

The generating function for the edge-labeled graphs on 6 nodes is in total:

(52)
$$e_6 = e_{6a} + e_{6b} + e_{6c} + e_{6d} + e_{6e} + e_{6f}$$
$$= \frac{x^5(6 + 10x + 21x^2 + 29x^3 + 38x^4 + 36x^5 + 33x^6 + 22x^7 + 14x^8 + 5x^9 + 2x^{10})}{(1-x)^5(1+x)^2(1+x+x^2)(1+x^2)(1+x+x^2+x^3+x^4)}$$
$$\mapsto 6, 16, 43, 88, 169, 287, 467, 711, \ldots (n \geq 5)$$

13. Summary (Edge Labeled)

Table 2 distributes sequences (38), (41), (45), (52) along its columns. The diagonal shows the number of unlabeled trees [1, A000055].

References

1. OEIS Foundation Inc., *The On-Line Encyclopedia Of Integer Sequences*, (2013), http://oeis.org/. MR 1992789 (2004f:11151)
2. The GAP Group, *GAP – groups, algorithms, and programming, version 4.8.10*, 2018.
3. Richard P. Stanley, *Enumerative combinatorics*, 2 ed., vol. 1, Cambridge University Press, 2011. MR 1442260

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