LOG-CONCAVE PERMUTATIONS

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Abstract. For any $n \geq 10$ there are exactly 12 log-concave permutations of $\{1, \ldots, n\}$.

1. Introduction

A sequence $(x_1, \ldots, x_n)$ of non-negative real numbers is *log-concave* if

$$x_{i-1}x_{i+1} \leq x_i^2$$

for $2 \leq i \leq n-1$. This paper is concerned with log-concave permutations of the set $\{1, \ldots, n\}$. In particular, we prove the following theorem.

Theorem 1. *For any $n \geq 10$ there are exactly 12 log-concave permutations of $\{1, \ldots, n\}$.*

These 12 permutations are

- $\pi_1 = (1, 2, \ldots, n)$,
- $\pi_2 = (2, 3, \ldots, n, 1)$,
- $\pi_3 = (1, 3, 4, \ldots, n, 2)$,
- $\pi_4 = (1, 2, 4, 5, \ldots, n, 3)$,
- $\pi_5 = (1, 3, 5, \ldots, 6, 4, 2)$,
- $\pi_6 = (1, 2, 4, 6, \ldots, 7, 5, 3)$,

and their reverses $\pi_1^r, \ldots, \pi_6^r$. We call these the *basic* (log-concave) permutations. For $n < 10$ these permutations are still log-concave, but for $n \leq 5$ some of them coincide; e.g., for $n = 5$, $\pi_4 = \pi_6$ and $\pi_4^r = \pi_6^r$. The only non-basic log-concave permutations are

- $(1, 3, 6, 5, 4, 2)$,
- $(1, 2, 4, 7, 6, 5, 3), (1, 3, 5, 6, 7, 4, 2), (1, 3, 7, 6, 5, 4, 2)$,
- $(1, 2, 4, 6, 7, 8, 5, 3), (1, 2, 4, 8, 7, 6, 5, 3), (1, 3, 5, 6, 7, 8, 4, 2)$, $(1, 3, 5, 8, 7, 6, 4, 2), (1, 3, 8, 7, 6, 5, 4, 2)$,
- $(1, 2, 4, 6, 9, 8, 7, 5, 3), (1, 3, 5, 7, 8, 9, 6, 4, 2), (1, 3, 9, 8, 7, 6, 5, 4, 2)$,

and their reverses. (For $n \geq 10$, this follows from Theorem 1; for $n < 10$, it follows from a refinement of its proof.) By finding all coincidences of basic permutations this gives that the numbers of log-concave permutations for $n = 0, 1, 2, \ldots$ are

$$1, 1, 2, 4, 8, 10, 14, 18, 22, 18, 12, 12, 12, \ldots$$

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2. Proof of Theorem 1

It is easily checked that the basic permutations are log-concave (and distinct), so it remains to prove that there are no non-basic log-concave permutations for $n \geq 10$. Assume that $n \geq 10$ and that $\pi = (x_1, \ldots, x_n)$ is a non-basic log-concave permutation. Note that any log-concave sequence is also unimodal, i.e., there exists an integer k such that $1 \leq k \leq n$ and

$$x_1 \leq x_2 \leq \cdots \leq x_k \geq x_{k+1} \geq \cdots \geq x_n.$$

(For permutations, all inequalities are of course strict, and $x_k = n$.) We say that a sequence $(y_1, \ldots, y_i)$ *appears* (in π) if either $y_1 = x_j, y_2 = x_{j+1}, \ldots, y_i = x_{j+i-1}$ or $y_1 = x_j, y_2 = x_{j-1}, \ldots, y_i = x_{j-i+1}$ for some j. Note that log-convexity implies that if $(x, x+1)$ appears in π and $x > 1$, then $(x, x+1, x+2, \ldots, n)$ must appear in π.

Let x be the smallest integer such that $x > 1$ and $(x, x+1)$ appears in π. By unimodality, $(n-1, n)$ appears in π, so such an x exists and $x \leq n-1$. Since $x > 1$, $(x, x+1, \ldots, n)$ must appear in π.

If $x = 2$, then $(2, 3, \ldots, n)$ appears in π, so $\pi \in \{\pi_1, \pi_1^r, \pi_2, \pi_2^r\}$. If $x = 3$, then $(3, 4, \ldots, n)$ appears, and by the minimality of x, $(3, 4, \ldots, n, 2)$ must appear. Since $n > 4$, $(n, 2, 1)$ can not appear, so $\pi = \pi_3$ or $\pi = \pi_3^r$. If $x = 4$, the same reasoning shows that $(2, 4, 5, \ldots, n, 3)$ appears, and since $n > 9$, $(n, 3, 1)$ can not appear, so $\pi = \pi_4$ or $\pi = \pi_4^r$. (For $n \leq 9$, also the permutation $(2, 4, 5, \ldots, n, 3, 1)$ is log-concave.)

Thus, we can assume that $x \geq 5$. In this case,

$$(x-2, x, x+1, \ldots, n, x-1, x-3)$$

appears. By log-concavity,

$$n \leq \frac{(x-1)^2}{x-3} = x + 1 + \frac{4}{x-3}.$$

If $5 \leq x \leq 8$, the right hand side is less than 10, so $x \geq 9$. Hence,

$$x \geq n - 1 - \frac{4}{x-3} > n - 2,$$

so we must in fact have $x = n-1$. (For small n we also get the cases $x = n-2$ for $7 \leq n \leq 9$, and $x = n-3$ for $n = 8$.) This implies that either $(2, 4, \ldots, n-1, n, n-2, \ldots, 5, 3)$ (if n is odd) or $(3, 5, \ldots, n-1, n, n-2, \ldots, 4, 2)$ (if n is even) appears. Thus $\pi \in \{\pi_5, \pi_5^r, \pi_6, \pi_6^r\}$. □