OEIS A300793 Notes

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The sequence in question $(a_n)_{n\in\mathbb{N}}$ is given by

$$a_n := \frac{(-2)^n}{\sqrt{2}} \frac{d^n}{dx^n} \operatorname{arcsinh}\left(\frac{1}{x}\right)\Big|_{x=1} \tag{1}$$

where the first elements turn out to be

n	1	2	3	4	5	6	7	8	9	10	...
a_n	1	3	13	75	561	5355	63405	894915	14511105	263544435	...

In this document, we want to show the following properties of said sequence.

Theorem 1. *The sequence $(a_n)_{n\in\mathbb{N}}$ satisfies $a_n = (-1)^n \sum_{j=0}^{n-1} b(j,n)$ for any $n \in \mathbb{N}$ where $(b(j,n))_{j\in\mathbb{Z},n\in\mathbb{N}}$ is a recursive sequence of integers given by*

$$\begin{gathered} b(0,1) = -1 \qquad b(j,n) = 0 \text{ if } j < 0 \text{ or } j \geq n \\ b(j,n+1) = b(j,n)(2j-n) + b(j-1,n)(2j-3n-1) \text{ for all } n \in \mathbb{N}, j \in \{0,\dots,n\}. \end{gathered} \tag{2}$$

In particular, $(a_n)_{n\in\mathbb{N}}$ is a sequence of integers.

For this we need to structure the n-th derivative of $\operatorname{arcsinh}(\frac{1}{x})$.

Lemma 1. *For all $n \in \mathbb{N}$*

$$\frac{d^n}{dx^n} \operatorname{arcsinh}\left(\frac{1}{x}\right) = \frac{\sum_{j=0}^{n-1} b(j,n) x^{2j}}{x^n (x^2+1)^{n-\frac{1}{2}}} \tag{3}$$

where $(b(j,n))_{j\in\mathbb{Z},n\in\mathbb{N}}$ is the sequence defined in (2).

Proof. Proof by induction. $n = 1$:

$$\frac{d}{dx} \operatorname{arcsinh}\left(\frac{1}{x}\right) = \frac{-\frac{1}{x^2}}{\sqrt{1+\frac{1}{x^2}}} = \frac{-1}{x^2(1+\frac{1}{x^2})^{\frac{1}{2}}} = \frac{b(0,1)}{x(x^2+1)^{\frac{1}{2}}} \qquad \checkmark$$

Now differentiating the right-hand side of (3) yields

$$
\begin{aligned}
\frac{d}{dx}\frac{\sum_{j=0}^{n-1} b(j,n)x^{2j}}{x^n(x^2+1)^{n-\frac{1}{2}}} &= \sum_{j=0}^{n-1} b(j,n)\frac{d}{dx}\frac{x^{2j}}{x^n(x^2+1)^{n-\frac{1}{2}}} \\
&= \sum_{j=0}^{n-1} b(j,n)\frac{x^{n-1}(x^2+1)^{n-\frac{3}{2}}x^{2j}[2j(x^2+1)-n(x^2+1)-(2n-1)x^2]}{x^{2n}(x^2+1)^{2n-1}} \\
&= \sum_{j=0}^{n-1} b(j,n)\frac{x^{2j+2}(2j-3n+1)+x^{2j}(2j-n)}{x^{n+1}(x^2+1)^{n+\frac{1}{2}}} \\
&= \frac{\sum_{j=0}^{n} x^{2j}[b(j-1,n)(2j-3n-1)+b(j,n)(2j-n)]}{x^{n+1}(x^2+1)^{n+\frac{1}{2}}} \\
&\overset{(2)}{=} \frac{\sum_{j=0}^{n} b(j,n+1)x^{2j}}{x^{n+1}(x^2+1)^{n+\frac{1}{2}}}
\end{aligned}
$$

where in the second to last step we made an index change $j \to j-1$ (to recover x^{2j} from x^{2j+2}) and used $b(n,n) = b(-1,n) = 0$. □

Proof of Theorem 1. By Lemma 1,

$$
\begin{aligned}
a_n = \frac{(-2)^n}{\sqrt{2}}\frac{d^n}{dx^n}\operatorname{arcsinh}\left(\frac{1}{x}\right)\Big|_{x=1} &= \frac{(-2)^n}{\sqrt{2}}\left.\frac{\sum_{j=0}^{n-1} b(j,n)x^{2j}}{x^n(x^2+1)^{n-\frac{1}{2}}}\right|_{x=1} \\
&= \frac{(-2)^n}{\sqrt{2}}\frac{\sum_{j=0}^{n-1} b(j,n)}{2^{n-\frac{1}{2}}} = (-1)^n\sum_{j=0}^{n-1} b(j,n)
\end{aligned}
$$

for any $n \in \mathbb{N}$. □