FAIRY CHESS LEAPER MINIMUM MOVES ON THE INFINITE CHESSBOARD

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Abstract. The (i,j)-leaper is a generalized figure on a chess board that moves by i squares along one directions and j squares along the perpendicular direction. We ask how many moves it needs at minimum to reach a square (k,l) if starting at square $(0,0)$, and if the chess board (unlike the standard 8×8 board) has no borders that constrain the intermediate positions.

1. Generalized Knight Moves

The (i,j)-leaper on the inifinite chess board leaps in one move from a position (k,l) on a square grid to exactly one of 8 different positions, $(k+i,l+j)$, $(k+i,l-j)$, $(k-i,l+j)$, $(k-i,l-j)$, $(k+j,l+i)$, $(k+j,l-i)$, $(k-j,l+i)$, or $(k-j,l-i)$. The $(1,2)$-leaper is the knight of the standard chess game confined to a 8×8 board.

Definition 1. *$M_{i,j}(k,l)$ is the minimum number of moves the (i,j)-leaper needs to step on square (k,l) starting at square $(0,0)$.*

Because the setup has the 8-fold symmetry of the dihedral group D_8 of order 8 (containing the rotations by multiples of 90° and flips along the horizontal, vertical or the diagonals), one may consider only the target squares in the first octant of the board:

$$M_{i,j}(k,l) = M_{i,j}(k,-l) = M_{i,j}(-k,l) = M_{i,j}(-k,-l) = M_{i,j}(l,k) = M_{i,j}(l,-k) = M_{i,j}(-l,k) = M_{i,j}(-l,-k). \quad (1)$$

In particular there are target squares at maximum Euclidean distance which are multiples of $\sqrt{i^2+j^2}$ that are reached by consecutive moves in the same direction:

$$M_{i,j}(0,0)=0;\quad M_{i,j}(mi,mj)=m \quad (m\ge 0). \quad (2)$$

The leaper's range in a single move is $\sqrt{i^2+j^2}$, so an obvious lower bound on the number of moves is

$$M_{i,j}(k,l)\sqrt{i^2+j^2} \ge \sqrt{k^2+l^2}. \quad (3)$$

Cases where i and j are not coprime are not of interest because the squares that can be reached are topologically the same as the setup where i, j and the scales of the board are all divided by any common integer divisor:

$$\gcd(i,j)=1;\quad 1\le i\le j. \quad (4)$$

The pairs (i,j) we consider are therefore like Table 1, sorted along increasing i^2+j^2.

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i^2+j^2	i	j	i^2+j^2	i	j	i^2+j^2	i	j	i^2+j^2	i	j
2	1	1	41	4	5	85	2	9	130	3	11
5	1	2	50	1	7	85	6	7	130	7	9
10	1	3	53	2	7	89	5	8	137	4	11
13	2	3	58	3	7	97	4	9	145	1	12
17	1	4	61	5	6	101	1	10	145	8	9
25	3	4	65	1	8	106	5	9	146	5	11
26	1	5	65	4	7	109	3	10	149	7	10
29	2	5	73	3	8	113	7	8	157	6	11
34	3	5	74	5	7	122	1	11	169	5	12
37	1	6	82	1	9	125	2	11	170	1	13

TABLE 1. Combinations of coprime positive i and j and their norm i^2+j^2 [5, A037942].

Definition 2. *The area $A_{i,j}(m)$ is the number of squares that can be reached in m or less moves.*

$$A_{i,j}(m) = \#\{(k,l) : M_{i,j}(k,l) \le m\}. \tag{5}$$

In particular for $(i,j) \neq (1,1)$

$$A_{i,j}(0) = 1; \quad A_{i,j}(1) = 9. \tag{6}$$

The symmetry of D_8 means that the target $(0,0)$ contributes 1 to $A_{i,j}$, the targets along the two diagonals and the horizontal or vertical axis, where $k = \pm l$ or $k = 0$ or $l = 0$, contribute with a multiplicity of 4 to $A_{i,j}$, and the other $A_{i,j}(m)$ with a multiplicity of 8. The first difference $A_{i,j}(m+1) - A_{i,j}(m)$ is the number of squares that can be reached in no less than $1+m$ moves. The D_8 symmetry implies these first differences are multiples of 4. In consequence, if $A_{i,j}(m)$ becomes a polynomial $A_{i,j}(m) = \alpha_1 + \alpha_2 m + \alpha_3 m^2$ for sufficiently large m, then $A_{i,j}(m+1) - A_{i,j}(m) = \alpha_1 + \alpha_2 + 2m\alpha_2$ is only a multiple of 4 if α_2 is even and $\alpha_1 + \alpha_2$ is a multiple of 4.

The parity of the target square (k,l) is $k+l \pmod 2$. If $i+j$ is even, the parity of the current square of the leaper stays the same after each move, which essentially—leaving the $(i,j) = (1,1)$ aside—are all entries of Table 1 where i and j are both odd. In consequence the half of the squares of the board where $k+l$ is odd are not reachable when $i+j$ is even [3, 4]. In the following figures 3–18 that display $M_{i,j}(k,l)$ we leave the labels blank along the falling and rising diagonals of the (k,l) grid that cannot be reached from $(0,0)$.

2. Examples

The $(1,1)$-leaper in Figure 1 is some kind of bishop with a stride limited to 1. It reaches the squares on the diagonal of the board in $M_{1,1}(m,m) = m$ moves and the squares on the adjacent row in $M_{1,1}(2m,2m) = 2m$ moves.

$$M_{1,1}(k,l) = \frac{1}{2}(|k+l| + |k-l|). \tag{7}$$

The sequence $A_{1,1}(m) = 1 + 2m(m+1)$ is given in [5, A001844].

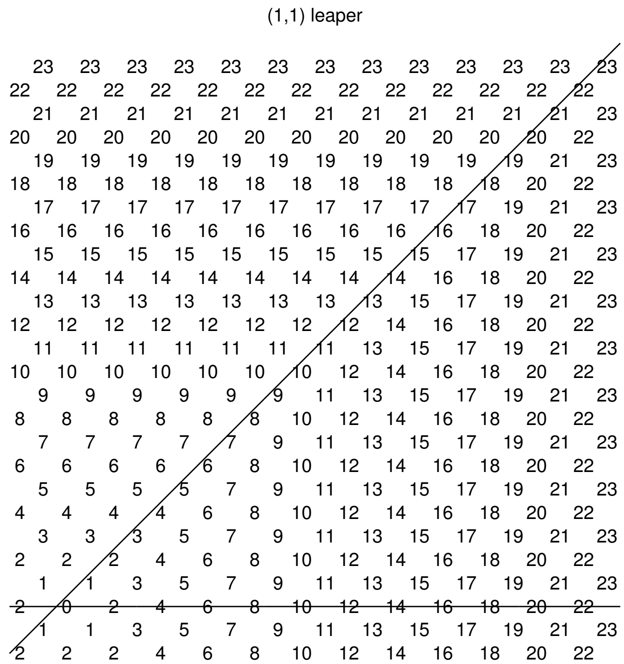
(1,1) leaper
23 23 23 23 23 23 23 23 23 23 23 23 23
22 22 22 22 22 22 22 22 22 22 22 22 22
21 21 21 21 21 21 21 21 21 21 21 21 23
20 20 20 20 20 20 20 20 20 20 20 20 22
19 19 19 19 19 19 19 19 19 19 19 21 23
18 18 18 18 18 18 18 18 18 18 18 20 22
17 17 17 17 17 17 17 17 17 17 19 21 23
16 16 16 16 16 16 16 16 16 16 18 20 22
15 15 15 15 15 15 15 15 15 17 19 21 23
14 14 14 14 14 14 14 14 14 16 18 20 22
13 13 13 13 13 13 13 13 15 17 19 21 23
12 12 12 12 12 12 12 12 14 16 18 20 22
11 11 11 11 11 11 11 13 15 17 19 21 23
10 10 10 10 10 10 10 12 14 16 18 20 22
9 9 9 9 9 9 11 13 15 17 19 21 23
8 8 8 8 8 8 10 12 14 16 18 20 22
7 7 7 7 7 9 11 13 15 17 19 21 23
6 6 6 6 6 8 10 12 14 16 18 20 22
5 5 5 5 7 9 11 13 15 17 19 21 23
4 4 4 4 6 8 10 12 14 16 18 20 22
3 3 3 5 7 9 11 13 15 17 19 21 23
2 2 2 4 6 8 10 12 14 16 18 20 22
1 1 3 5 7 9 11 13 15 17 19 21 23
2 0 2 4 6 8 10 12 14 16 18 20 22
1 1 3 5 7 9 11 13 15 17 19 21 23
2 2 2 4 6 8 10 12 14 16 18 20 22

FIGURE 1.

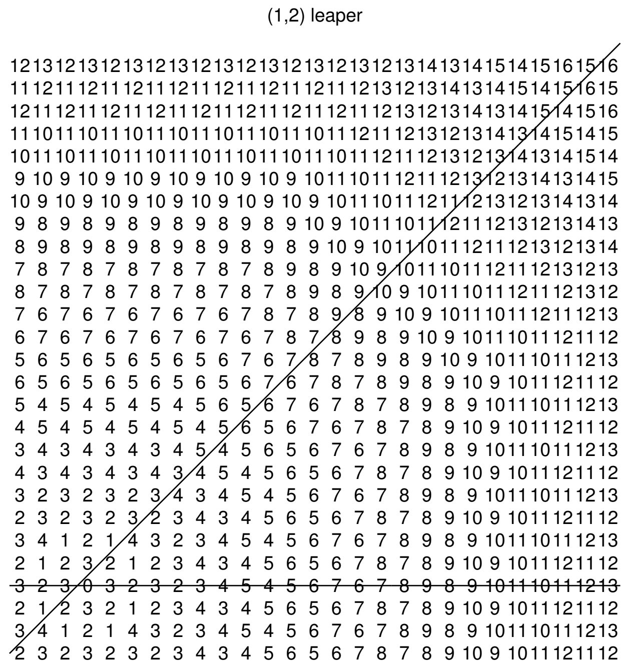

FIGURE 2. The number of moves $M_{1,2}(k,l)$ as defined in [5, A065775] and reduced to the first octant in [5, A183043].

The number of moves for the $(1,2)$-leaper of Figure 2 are registered as follows: $M_{1,2}(n,0)$ in [5, A018837], $M_{1,2}(n,n)$ in [5, A018838] and $A_{1,2}(m)$ in [5, A018836].

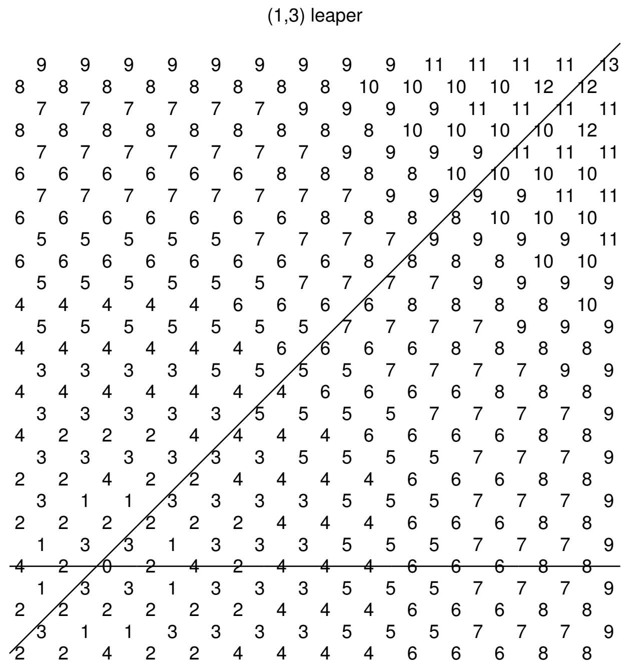

FIGURE 3.

The array of $M_{1,3}(k,l)$ for the $(1,3)$-leaper in Figure 3 is a variant of Figure 2 tilted by 45° and diluted to indicate the squares that cannot be reached. $A_{1,3}(m) = A_{1,2}(m)$. This sort of mapping between a (i,j)-leaper and a (i',j')-leaper may happen for all cases where the norms are related as $i'^2 + j'^2 = 2(i^2 + j^2)$ [2], induced by

$$j = \frac{j' + i'}{2}; \quad i = \frac{j' - i'}{2}. \tag{8}$$

Here in this case the $(1,2)$-leaper has norm 5 and the $(1,3)$-leaper has norm 10, see Table 1.

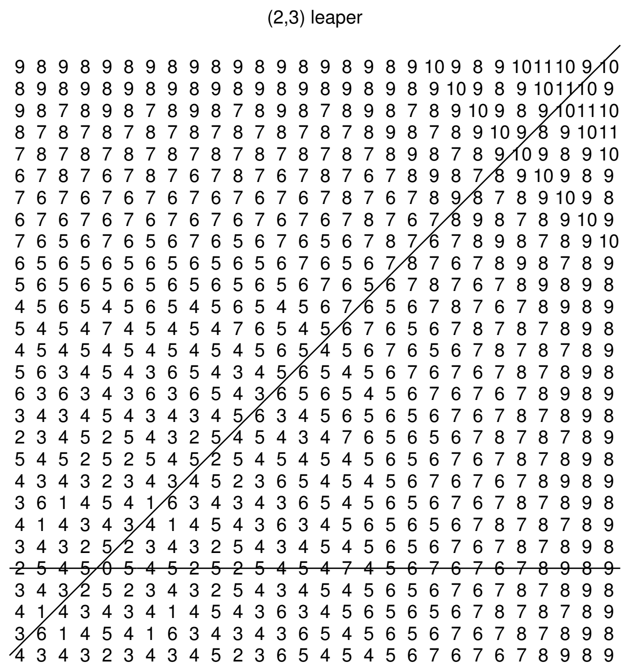

Figure 4.

The number of reachable squares for the $(2,3)$-leaper of Figure 4 are registered in [5, A297740], and the first differences in [5, A018839]. $A_{2,3}(m) = 1, 9, 41, 129, 321, 625, 997, 1413, \ldots$ $(m \geq 0)$, eventually $A_{2,3}(m) = 9 + 30m + 34m^2$ for $m \geq 6$.

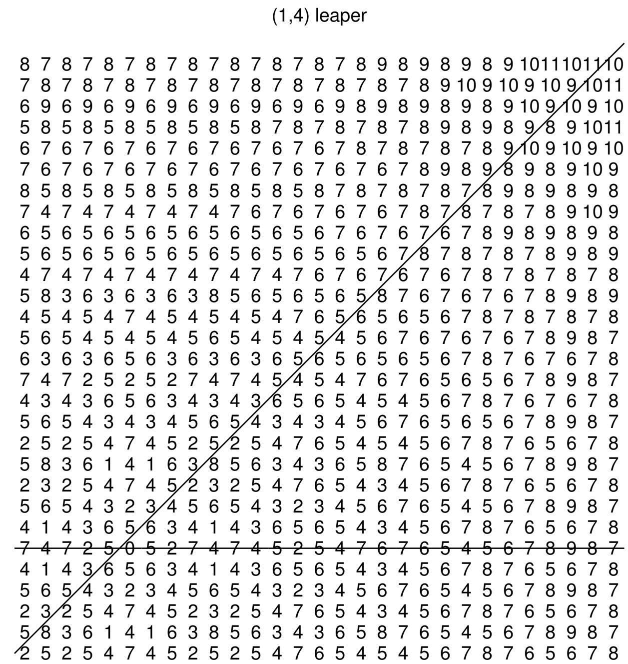

Figure 5.

The number of reachable squares for the $(1,4)$-leaper of Figure 5 is $A_{1,4}(m) = 1, 9, 41, 129, 321, 645, 1093, 1641, \ldots (m \geq 0)$, conjectured $A_{1,4}(m) = 5 - 86m + 46m^2$ for $m \geq 8$.

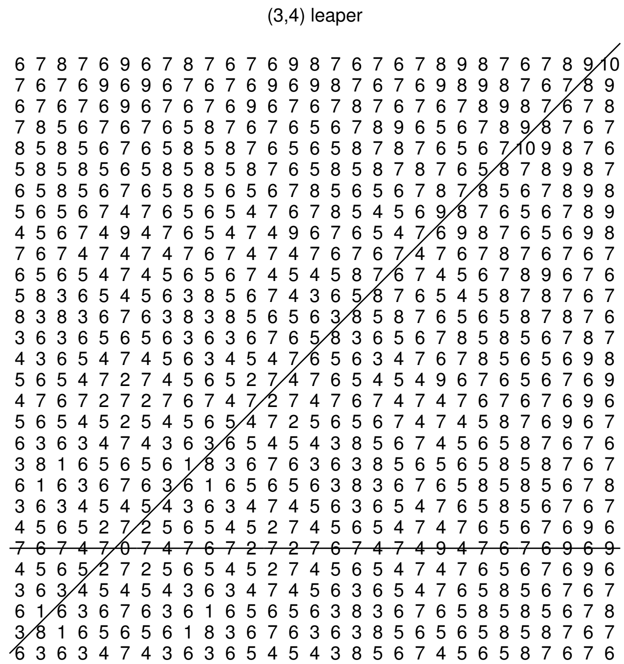

FIGURE 6.

The number of reachable squares for the (3,4)-leaper of Figure 6 is tabulated in [5, A297741]. $A_{3,4}(m) = 1, 9, 41, 129, 321, 681, 1289, 2121, 3081, 4121 \ldots (m \geq 0)$, conjectured $A_{3,4}(m) = -55 + 30m + 62m^2$ for $m \geq 10$.

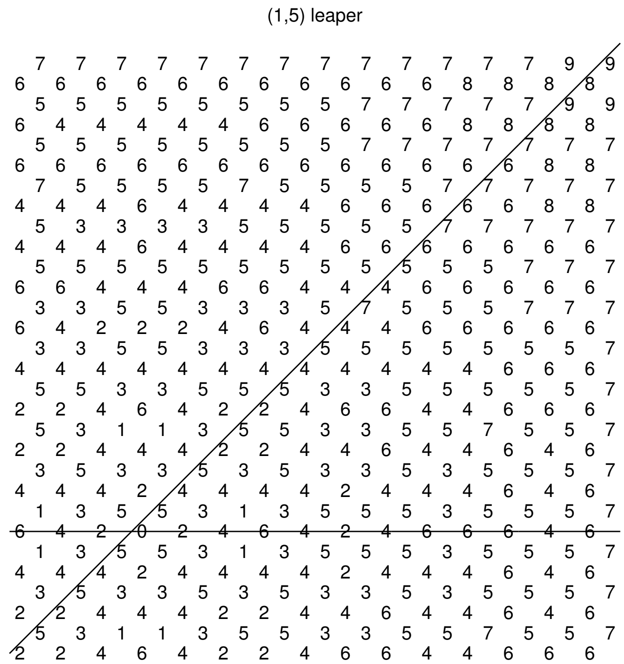

FIGURE 7.

The number of reachable squares for the $(1,5)$-leaper of Figure 7 is $A_{1,5}(m) = 1, 9, 41, 129, 321, 625, 997, 1413, 1885, 2425, \ldots$ $(m \geq 0)$, conjectured $A_{1,5}(m) = 13 - 38m + 34m^2$ for $m \geq 7$.

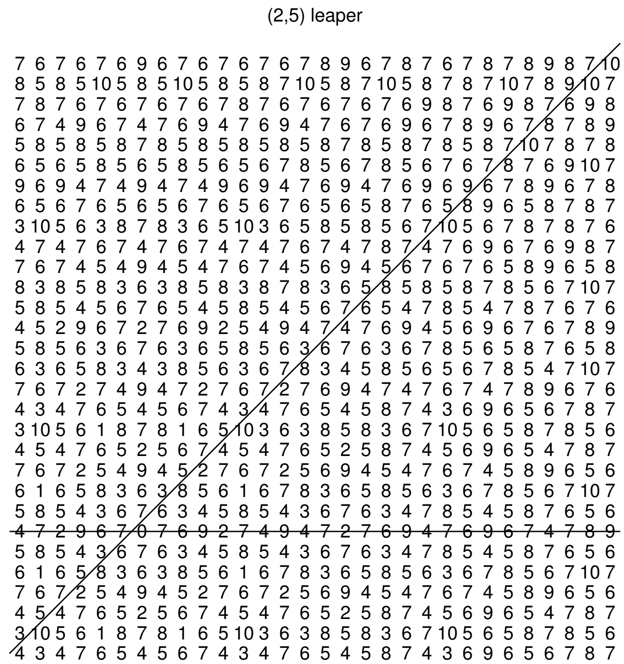

FIGURE 8.

The number of reachable squares for the $(2,5)$-leaper of Figure 8 is $A_{2,5}(m) = 1, 9, 41, 129, 321, 681, 1289, 2149, 3217, 4469, 5853, 7349, 8989, 10793, 12761, 14893, \ldots$ $(m \geq 0)$, conjectured $A_{2,5}(m) = 133 - 246m + 82m^2$ for $m \geq 11$.

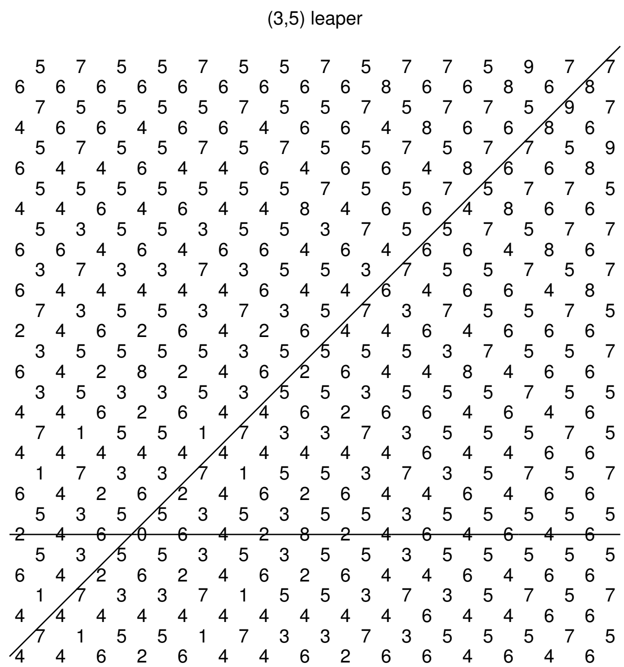

FIGURE 9.

The array of $M_{3,5}(k,l)$ for the $(3,5)$-leaper in Figure 9 is a variant of Figure 5 tilted by 45° and diluted to indicate the squares that cannot be reached. $A_{3,5}(m) = A_{1,4}(m)$.

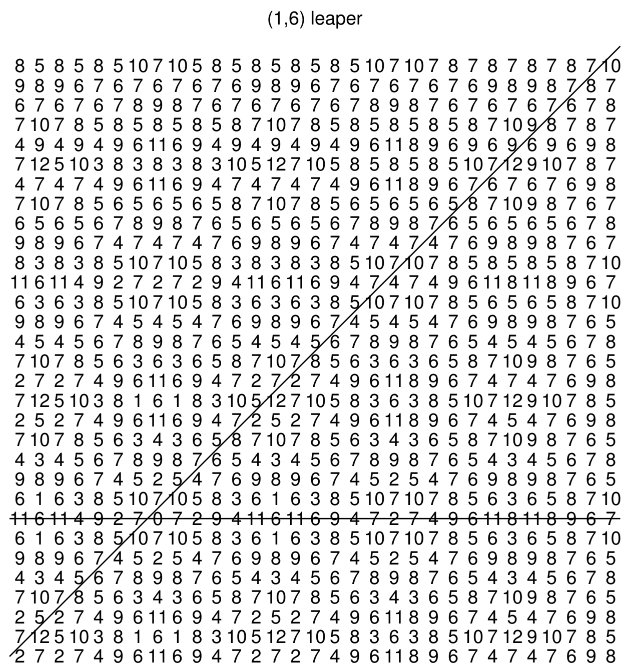

FIGURE 10.

The number of reachable squares for the $(1,6)$-leaper of Figure 10 is $A_{1,6}(m) = 1, 9, 41, 129, 321, 681, 1289, 2189, 3373, 4817, 6481, 8309, 10261, 12365, 14657, \ldots$ $(m \geq 0)$, conjectured $A_{1,6}(m) = -323 - 246m + 94m^2$ for $m \geq 12$.

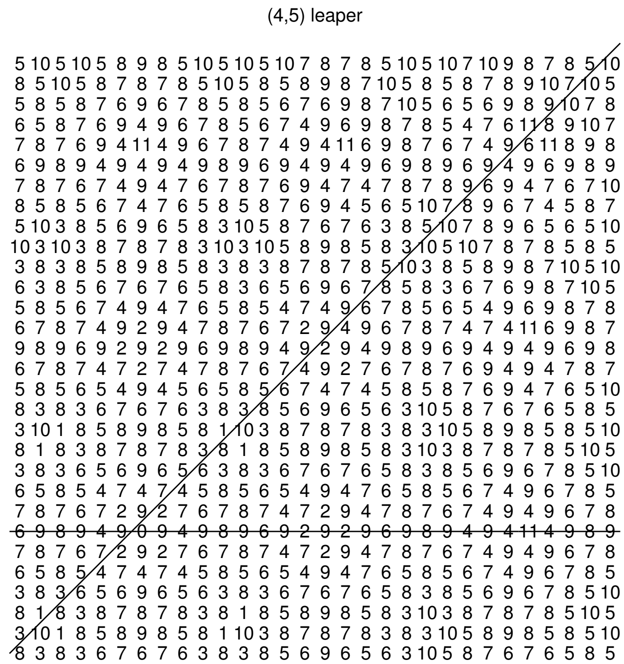

FIGURE 11.

The number of reachable squares for the $(4,5)$-leaper of Figure 11 is $A_{4,5}(m) = 1, 9, 41, 129, 321, 681, 1289, 2241, 3649, 5421, 7393, 9513, 11733, 14065, 16553, 19225, 22089, \ldots (m \geq 0)$, conjectured $A_{4,5}(m) = -215 - 174m + 98m^2$ for $m \geq 15$.

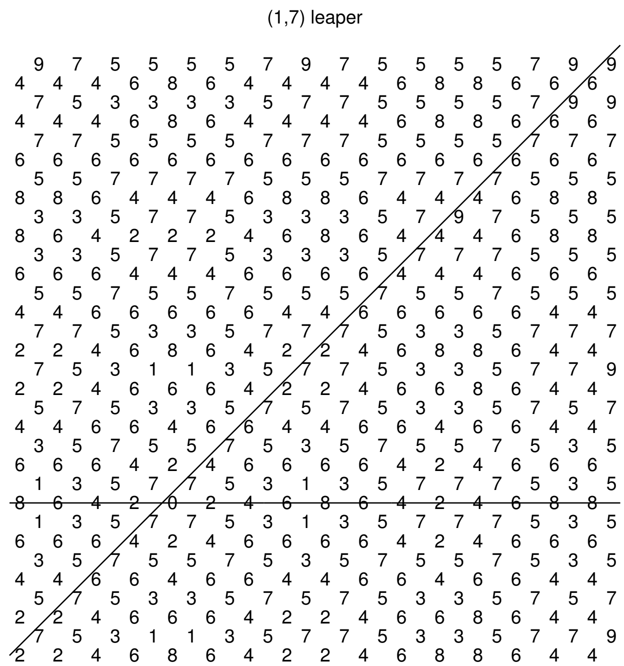

FIGURE 12.

The array of $M_{1,7}(k,l)$ for the $(1,7)$-leaper in Figure 12 is a variant of Figure 6 tilted by $45°$ and diluted to indicate the squares that cannot be reached. $A_{1,7}(m) = A_{3,4}(m)$.

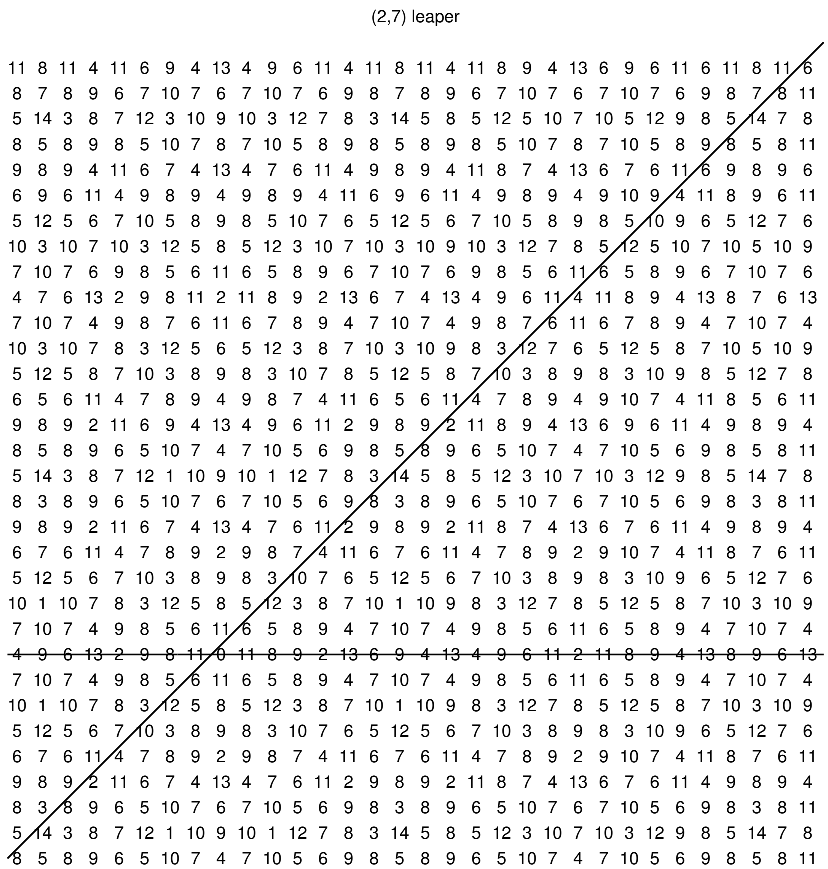

FIGURE 13.

The number of reachable squares for the $(2,7)$-leaper of Figure 13 is $A_{2,7}(m) = 1, 9, 41, 129, 321, 681, 1289, 2241, 3649, 5513, 7773, 10409, 13373, 16601, 20029, 23661, 27557, \ldots$ $(m \geq 0)$, conjectured $A_{2,7}(m) = 261 - 630m + 146m^2$ for $m \geq 15$.

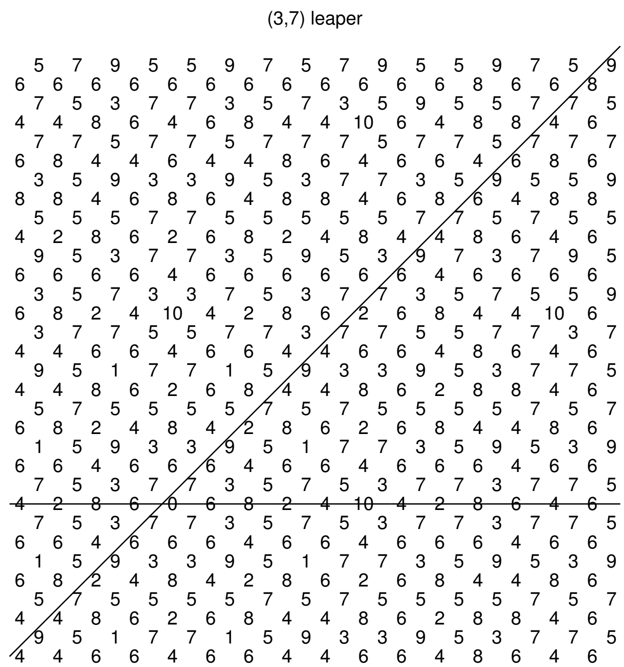

FIGURE 14.

The array of $M_{3,7}(k,l)$ for the $(3,7)$-leaper in Figure 14 is a variant of Figure 8 tilted by 45° and diluted to indicate the squares that cannot be reached. $A_{3,7}(m) = A_{2,5}(m)$.

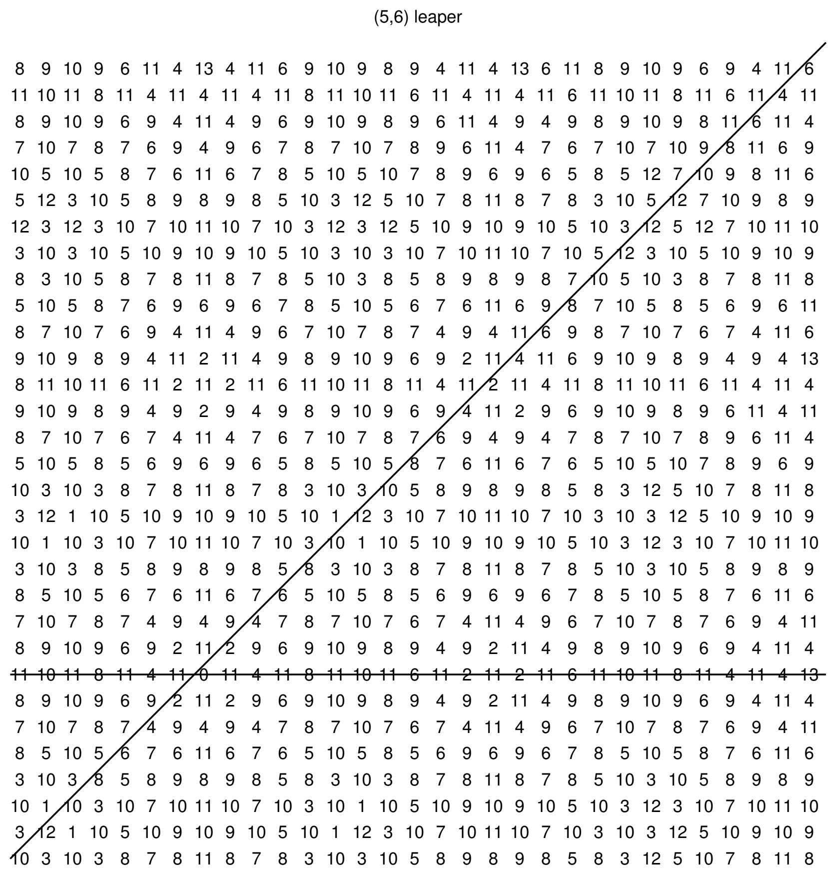

FIGURE 15.

The number of reachable squares for the (5, 6)-leaper of Figure 15 is $A_{5,6}(m) =$ $1, 9, 41, 129, 321, 681, 1289, 2241, 3649, 5641, 8361, 11605, 15125, 18893, 22821, 26881, 31105, 35553,$ $40261, 45241, 50501, \ldots (m \geq 0)$, conjectured $A_{5,6}(m) = -739 - 278m + 142m^2$ for $m \geq 19$.

(1,8) leaper

13	10	13	8	11	6	9	4	9	4	9	4	9	6	11	8	13	10	13	8	11	6	9	4	9	4	9	4
10	3	10	3	10	5	12	7	14	9	14	7	12	5	10	3	10	3	10	3	10	5	12	7	14	9	14	9
15	8	15	6	13	4	11	2	9	2	9	2	11	4	13	6	15	8	15	8	13	6	11	4	9	4	9	4
8	3	8	3	10	5	12	7	14	9	14	7	12	5	10	3	8	3	8	3	10	5	12	7	14	9	14	9
13	10	13	8	11	6	9	4	7	4	7	4	9	6	11	8	13	10	13	8	11	6	9	4	7	4	7	4
6	5	6	5	8	7	10	9	12	11	12	9	10	7	8	5	6	5	6	5	8	7	10	9	12	11	12	9
11	12	11	10	9	8	7	6	5	6	5	6	7	8	9	10	11	12	11	10	9	8	7	6	5	6	5	6
4	7	4	7	6	9	8	11	10	13	10	11	8	9	6	7	4	7	4	7	6	9	8	11	10	13	10	11
9	14	9	12	7	10	5	8	3	8	3	8	5	10	7	12	9	14	9	12	7	10	5	8	3	8	3	8
2	9	2	9	4	11	6	13	8	15	8	13	6	11	4	9	2	9	2	9	4	11	6	13	8	15	10	13
9	16	7	14	5	12	3	10	1	8	1	10	3	12	5	14	7	16	9	14	7	12	5	10	3	8	3	10
2	7	2	9	4	11	6	13	8	15	8	13	6	11	4	9	2	7	2	9	4	11	6	13	8	15	10	13
9	14	9	12	7	10	5	8	3	6	3	8	5	10	7	12	9	14	9	12	7	10	5	8	3	6	3	8
4	5	4	7	6	9	8	11	10	13	10	11	8	9	6	7	4	5	4	7	6	9	8	11	10	13	10	11
11	12	11	10	9	8	7	6	5	4	5	6	7	8	9	10	11	12	11	10	9	8	7	6	5	4	5	6
6	3	6	5	8	7	10	9	12	11	12	9	10	7	8	5	6	3	6	5	8	7	10	9	12	11	12	9
13	10	13	8	11	6	9	4	7	2	7	4	9	6	11	8	13	10	13	8	11	6	9	4	7	2	7	4
8	1	8	3	10	5	12	7	14	9	14	7	12	5	10	3	8	1	8	3	10	5	12	7	14	9	14	9
15	8	15	6	13	4	11	2	9	0	9	2	11	4	13	6	15	8	15	8	13	6	11	4	9	2	9	4
8	1	8	3	10	5	12	7	14	9	14	7	12	5	10	3	8	1	8	3	10	5	12	7	14	9	14	9
13	10	13	8	11	6	9	4	7	2	7	4	9	6	11	8	13	10	13	8	11	6	9	4	7	2	7	4
6	3	6	5	8	7	10	9	12	11	12	9	10	7	8	5	6	3	6	5	8	7	10	9	12	11	12	9
11	12	11	10	9	8	7	6	5	4	5	6	7	8	9	10	11	12	11	10	9	8	7	6	5	4	5	6
4	5	4	7	6	9	8	11	10	13	10	11	8	9	6	7	4	5	4	7	6	9	8	11	10	13	10	11
9	14	9	12	7	10	5	8	3	6	3	8	5	10	7	12	9	14	9	12	7	10	5	8	3	6	3	8
2	7	2	9	4	11	6	13	8	15	8	13	6	11	4	9	2	7	2	9	4	11	6	13	8	15	10	13
9	16	7	14	5	12	3	10	1	8	1	10	3	12	5	14	7	16	9	14	7	12	5	10	3	8	3	10
2	9	2	9	4	11	6	13	8	15	8	13	6	11	4	9	2	9	2	9	4	11	6	13	8	15	10	13

FIGURE 16.

The number of reachable squares for the $(1, 8)$-leaper of Figure 16 is $A_{1,8}(m) = 1, 9, 41, 129, 321, 681, 1289, 2241, 3649, 5573, 8005, 10921, 14281, 18029, 22093, 26385, 30861, \ldots$ ($m \geq 0$), conjectured $A_{1,8}(m) = -1811 - 486m + 158m^2$ for $m \geq 16$.

(4,7) leaper

10	3	10	7	12	9	12	3	10	3	12	9	12	7	10	3	10	5	12	11	12	5	10	3	12	7	12
11	10	11	4	7	4	9	8	13	8	9	4	7	4	11	10	11	6	7	4	9	6	13	10	9	6	7
6	7	6	7	10	9	8	7	4	7	8	9	10	7	6	7	6	7	10	11	8	7	4	7	8	9	12
9	12	7	10	3	10	5	10	11	10	5	10	3	10	7	12	9	10	3	10	5	10	9	10	7	10	3
2	9	4	11	10	13	6	9	2	9	6	13	10	11	4	9	2	11	8	13	8	9	4	9	4	13	10
9	12	7	8	5	6	5	10	11	10	5	6	5	8	7	12	9	8	5	6	5	10	9	10	7	6	5
8	3	8	7	10	9	8	5	8	5	8	9	10	7	8	3	8	7	10	11	8	5	8	5	8	9	12
11	10	11	4	11	2	11	8	13	8	11	2	11	4	11	10	11	6	11	2	11	6	13	10	11	4	11
8	3	8	7	12	9	10	3	8	3	10	9	12	7	8	3	8	5	12	11	10	5	8	3	10	7	12
9	10	9	6	5	6	7	8	11	8	7	6	5	6	9	10	9	6	5	6	7	8	11	10	7	6	5
4	9	4	9	10	11	6	9	2	9	6	11	10	9	4	9	4	9	8	11	8	9	2	9	6	11	10
9	14	7	10	1	10	5	12	11	12	5	10	1	10	7	14	9	10	3	10	3	12	9	12	7	10	3
4	7	4	9	10	11	6	7	4	7	6	11	10	9	4	7	4	9	8	11	8	7	4	7	6	11	10
9	10	9	6	7	4	7	8	11	8	7	4	7	6	9	10	9	6	7	4	7	8	11	10	7	4	7
10	1	10	7	12	9	10	3	10	3	10	9	12	7	10	1	10	5	12	11	10	5	10	3	10	7	12
11	10	11	4	9	2	9	8	13	8	9	2	9	4	11	10	11	6	9	2	9	6	13	10	9	4	9
6	5	6	7	10	9	8	5	6	5	8	9	10	7	6	5	6	7	10	11	8	5	6	5	8	9	12
9	12	7	8	3	8	5	10	11	10	5	8	3	8	7	12	9	8	3	8	5	10	9	10	7	8	3
2	11	4	11	10	13	6	11	0	11	6	13	10	11	4	11	2	11	8	13	8	11	2	11	4	13	10
9	12	7	8	3	8	5	10	11	10	5	8	3	8	7	12	9	8	3	8	5	10	9	10	7	8	3
6	5	6	7	10	9	8	5	6	5	8	9	10	7	6	5	6	7	10	11	8	5	6	5	8	9	12
11	10	11	4	9	2	9	8	13	8	9	2	9	4	11	10	11	6	9	2	9	6	13	10	9	4	9
10	1	10	7	12	9	10	3	10	3	10	9	12	7	10	1	10	5	12	11	10	5	10	3	10	7	12
9	10	9	6	7	4	7	8	11	8	7	4	7	6	9	10	9	6	7	4	7	8	11	10	7	4	7
4	7	4	9	10	11	6	7	4	7	6	11	10	9	4	7	4	9	8	11	8	7	4	7	6	11	10
9	14	7	10	1	10	5	12	11	12	5	10	1	10	7	14	9	10	3	10	3	12	9	12	7	10	3
4	9	4	9	10	11	6	9	2	9	6	11	10	9	4	9	4	9	8	11	8	9	2	9	6	11	10

FIGURE 17.

The number of reachable squares for the $(4,7)$-leaper of Figure 17 is $A_{4,7}(m) =$ $1, 9, 41, 129, 321, 681, 1289, 2241, 3649, 5641, 8361, 11649, 15341, 19481, 23901, 28509, 33357,$ $38469, 43901, \ldots$ $(m \geq 0)$, conjectured $A_{4,7}(m) = 593 - 798m + 178m^2$ for $m \geq 17$.

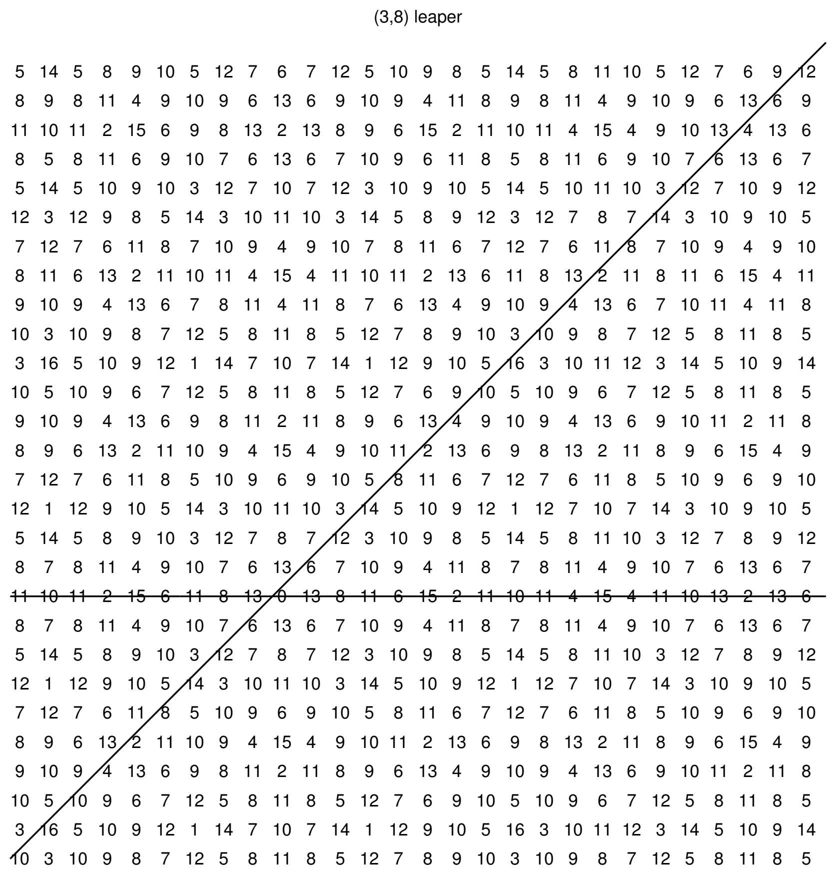

Figure 18.

The number of reachable squares for the $(3,8)$-leaper of Figure 18 is $A_{3,8}(m) =$ $1, 9, 41, 129, 321, 681, 1289, 2241, 3649, 5641, 8361, 11721, 15593, 19989, 24853, 30113, 35661, 41469, 47597, 54101, 61017, 68345, 76085 \ldots (m \geq 0)$.

3. Numerical Aspects

The 8 different moves (ignoring the case $i = j = 1$ with 4 different moves) from any starting square rephrase the computation of $M_{i,j}(k, l)$ to the problem of finding 8 different coefficients of the equation

(9)
$$\begin{pmatrix} k \\ l \end{pmatrix} = s_0 \begin{pmatrix} j \\ i \end{pmatrix} + s_1 \begin{pmatrix} i \\ j \end{pmatrix} + s_2 \begin{pmatrix} -i \\ j \end{pmatrix} + s_3 \begin{pmatrix} -j \\ i \end{pmatrix} + s_4 \begin{pmatrix} -j \\ -i \end{pmatrix} + s_5 \begin{pmatrix} -i \\ -j \end{pmatrix} + s_6 \begin{pmatrix} i \\ -j \end{pmatrix} + s_7 \begin{pmatrix} j \\ -i \end{pmatrix}$$

minimizing $\sum_i s_i$, where the $s_i \geq 0$ count the number of moves of type i that contribute. Because combining moves of the type that cancel each other is a waste to minization of the total number of moves, one of the coefficients in each pair of (s_i, s_{i+4}) remains zero. This allows to rephrase the problem to finding 4 different coefficients of the equation

$$\begin{pmatrix} k \\ l \end{pmatrix} = s_0 \begin{pmatrix} j \\ i \end{pmatrix} + s_1 \begin{pmatrix} i \\ j \end{pmatrix} + s_2 \begin{pmatrix} -i \\ j \end{pmatrix} + s_3 \begin{pmatrix} -j \\ i \end{pmatrix} = \begin{pmatrix} (s_1 - s_2)i + (s_0 - s_3)j \\ (s_1 + s_2)j + (s_0 + s_3)i \end{pmatrix}, \tag{10}$$

minimizing $\sum_i |s_i|$, where s_i are signed integers. These are 2 equations with 4 unknown integers $s_1 \pm s_2$ and $s_0 \pm s_3$.

Remark 1. *Sum and difference of the 2 equations yield*

$$k + l = (s_0 + s_1)(i + j) + (s_2 - s_3)(j - i); \tag{11}$$
$$k - l = -(s_2 + s_3)(i + j) + (s_0 - s_1)(j - i). \tag{12}$$

If the leaper is transformed with (8), this reads

$$k + l = (s_2 - s_3)i' + (s_0 + s_1)j' \tag{13}$$
$$k - l = -(s_2 + s_3)j' + (s_0 - s_1)i'. \tag{14}$$

If we re-mix also the coefficients $s'_0 = s_0$, $s'_1 = -s_3$, $s'_2 = -s_2$, $s'_3 = -s_1$, $k+l = k'$, and $k - l = l'$, we have again Equation (10) but all variables are primed. Since this transformation keeps $\sum_i |s_i| = \sum_i |s'_i|$ constant, this shows that the transformation (8) is indeed mapping minimum solutions of the (i, j)-leaper onto minimum solutions fo the (i', j')-leaper.

The D_8 symmetrie means we may consider $k \pm l$ and $j \pm i$ non-negative. Equation (10) turns into the congruences

$$(s_0 - s_3)j \equiv k \pmod{i}; \tag{15}$$
$$(s_0 + s_3)i \equiv l \pmod{j}; \tag{16}$$
$$(s_1 + s_2)j \equiv l \pmod{i}; \tag{17}$$
$$(s_1 - s_2)i \equiv k \pmod{j}. \tag{18}$$

The fundamental values for $s_0 \pm s_3$ and $s_1 \pm s_2$ are for example obtained by Fermat's Little Theorem [1, Theor. 5.20]. All equivalent solutions are given by the fundamental solutions plus multiples of the moduli. In a plot with s_0 the horizontal coordinate and s_3 the vertical coordinate and in another plot with s_1 the horizontal coordinate and s_2 the vertical coordinate these candidates are on the regular grid where the rising and falling diagonals defined by these sets of solutions intersect.

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