Ambiguous two-coloured tilings of rectangular patterns of black and white squares

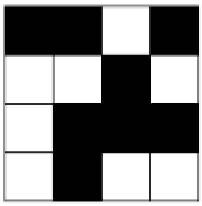
This document considers rectangular patterns, width w and height h, formed of h*w squares, of which exactly half are black and half are white. The 8*8 chessboard is an obvious example.

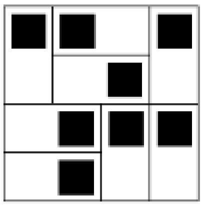
The question is, given any particular pattern, can it be decomposed into (h*w)/2 dominoes, where each domino has one white square and one black square?

The answer for the chessboard is that many such decompositions are possible – see OEIS A004003.

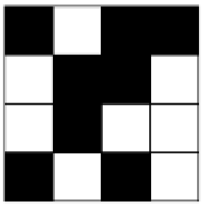
Generally speaking, rectangular patterns as described may be decomposable and undecomposable.

Example of a decomposable pattern, together with one example of its possible decompositions.



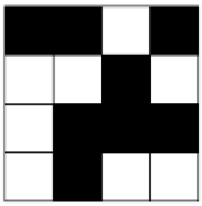


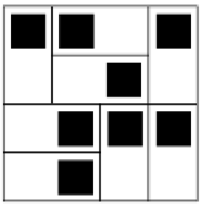
Example of an undecomposable pattern. It is obviously so, as one of one of its white squares has only white neighbours.

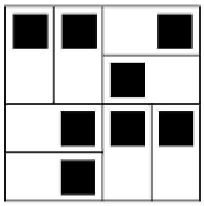


Some patterns may be ambiguous. That is, there may be more than one possible decomposition, as shown below, using the same tiling as above as an example.

Example of an ambiguously decomposable pattern, together with two examples of its possible decompositions.

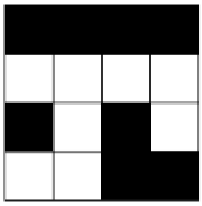


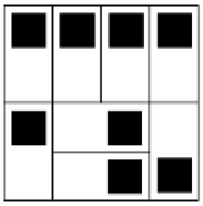




Some patterns, on the other hand, are unambiguous. That is, only one tiling with black and white dominoes is possible.

Example of an unambiguously decomposable pattern, together with the only possible decomposition.





Is there an easy way to determine if a pattern has a single, unambiguous decomposition?

 John Mason – 16 november 2017

Definitions

1. Within a decomposable pattern, a domino is “locally forced” if at least one of two adjacent squares of different colours has only the other as a suitable neighbour.

> That is, in the previous example, the bottom left domino is locally forced because the white, bottom left square, has only one black neighbour.

2. Within a decomposable pattern, a domino is “recursively forced” if each of two adjacent squares of different colours has more than one suitable neighbour, but at least one of the two has only one suitable neighbour once all the squares of other locally or recursively forced dominoes have been taken out of the picture.

> That is, in the previous example, the higher of the two horizontal dominoes is recursively forced, because, while each of its squares has at least two suitable neighbours, taking into account that many of those neighbours belong to locally forced dominoes, this domino itself is reduced to being recursively forced.

Obviously, any pattern composed of only locally and recursively forced dominoes has an unambiguous tiling.

Is the opposite true? That is, is an unambiguous tiling always composed of only locally and recursively forced dominoes? Because if that were true, we would have a relatively quick way to check if any specific rectangular pattern has a single, unambiguous tiling.

Theorem: an unambiguous, two-colour domino tiling of a rectangular pattern, width w and height h, formed of h*w squares, of which exactly half are black and half are white, is always composed of only locally and recursively forced dominoes.

Proof by contradiction. Suppose there exists an unambiguously decomposable rectangular pattern (of size h*w, for even h or w, and exactly half white squares and half black) such that not all its dominoes are locally or recursively forced.

Remove from this pattern all locally or recursively forced dominoes.

We are therefore left with one or more distinct polyominoes, for each of which the following is true: it is an unambiguously decomposable polyomino containing no locally forced dominoes.

Take one such polyomino. Identify at random one of its dominoes, and the white square of that domino. This white square must be adjacent to at least one black square not of the same domino, for otherwise this square would be part of a locally forced domino. Identify then a domino that contains one such black square. Again for this domino, its white square must be adjacent to at least one black square not of the same domino.

Identify in this way a trail of dominoes, with one black square and one white square, such that each white square is adjacent to the black square of a different domino, and so on.

As the number of dominoes in the polyomino is finite, it is clear that the creation of such a trail must at some time form a loop, such that the white square of the last domino is adjacent to the black square of some other, previous domino in the trail. Not necessarily will the loop include the first domino chosen to form the trail, yet the existence of a loop is certain. Note that a loop could be just two dominoes side by side, with opposite black-white orientation.

But a loop is ambiguous. To make an alternative tiling of a loop so constructed, form the dominoes by attaching each white square to the black square of the next domino in the loop and so on.

And this is a contradiction. The polyomino, defined as being unambiguous, contains an ambiguous loop.

Therefore the theorem as stated, that an unambiguous tiling is always composed of only locally and recursively forced dominoes, is proved.