CONNECTED MULTIGRAPHS WITH 4 VERTICES (A290778)

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ABSTRACT. The ordinary generating function of sequence A290778 is derived by adding the contributions of the 6 distinct geometries of the underlying connected simple graph.

1. UNDERLYING GRAPHS

We consider connected multigraphs with 4 vertices and n edges; an obvious minimum number of edges is three, because this is the sparsest way to keep the vertices connected.

The multigraph thus has at least one (non-oriented) edge attached to edge vertex, so all degrees are ≥ 1. Loops are not allowed; the vertices are not labeled.

If all multi-edges are replaced by a single edge, the underlying simple graph has one of 6 shapes shown in the Figure:
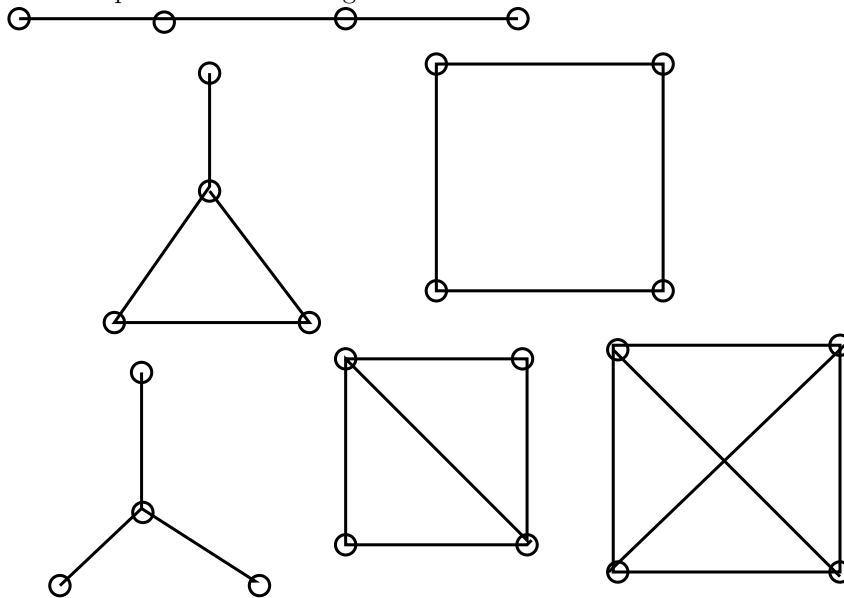
(1) The linear chain with 3 edges.
(2) A triangle with an edge to a lone vertex of degree 1 (4 edges).
(3) The star graph with 3 edges.
(4) The quadrangle (cycle of 4 edges).
(5) A quadrangle with a single diagonal, total of 5 edges.

(6) The complete graph on 4 vertices with 6 edges.

2. Linear Chain

We wish to distribute $n \geq 3$ edges over the 3 edges of the simple graph of the linear chain. This can be done by putting any number of $k \geq 1$ edges in the middle, and distributing the remaining $n - k$ edges over the two edges connected to two endpoints.

Due to the left-right symmetry of the graph, the distribution of the $n - k$ edges can only be done in $\lfloor (n-k)/2 \rfloor$ ways. The total number of graphs of this kind with multi-edges is

$$(1) \qquad a_1(n) = \sum_{k=1}^{n-2} \lfloor \frac{n-k}{2} \rfloor = 0, 0, 0, 1, 2, 4, 6, 9, 12, 16, 20, 25, \ldots (n \geq 0)$$

with generating function

$$(2) \qquad g_1(x) = \sum_{n \geq 0} a_1(n) x^n = \frac{x^3}{(1+x)(1-x)^3}.$$

Apart from the offset this is [3, A002620].

The generating function is the product of $1/(1-x)$ representing the number of ways of placing n vertices at the middle edge, by the factor $x^2/[(1+x)(1-x)^2]$. The latter factor is obtained by considering the symmetry of the cyclic group C_2 that swaps the edges that inhabit the first and last edges of the underlying simple graph without generating a new graph. The cycle index of the group is [1]

$$(3) \qquad Z(C_2) = (t_1^2 + t_2^1)/2,$$

where the associated generating functions are the number of ways of placing n edges without considering the symmetry on any of them:

$$(4) \qquad t_1(x) = t_2(x) = t_3(x) = t_4(x) = \frac{x}{1-x}.$$

Then [3, A004526]

$$(5) \qquad \frac{t_1(x)^2 + t_2(x^2)}{2} = \frac{x^2}{(1+x)(1-x)^2}.$$

3. Triangle

The contribution from the triangular graph is the number of ways of placing $2 \leq k \leq n-2$ edges on the edge to the lone vertex and the edge opposite to it, and then placing the residual $n - k$ edges under the symmetry constraint of the group C_2 that swaps the other two edges:

(6)
$$a_2(n) = \sum_{k=2}^{n-2} (k-1) \lfloor \frac{n-k}{2} \rfloor = 0, 0, 0, 0, 1, 3, 7, 13, 22, 34, 50, 70, 95, 125, 161 (n \geq 0)$$

See [3, A002623]

$$(7) \qquad g_2(x) = \frac{x^4}{(1+x)(1-x)^4}.$$

This generating function is the product of $x^2/(1-x)^2$, see [3, A001477], by $x^2/[(1+x)(1-x)^2]$, where again the latter is (5), the number of ways of distributing n edges symmetrically over two edges of the simple graph.

4. Star Graph

The contribution from the star graph is the number of ways of partitioning n into 3 positive integers [3, A069905],

$$(8) \qquad a_3(n) = 0, 0, 0, 1, 1, 2, 3, 4, 5, 7, 8, 10, 12, 14, 16 (n \ge 0),$$

$$(9) \qquad g_3(x) = \frac{x^3}{(1+x)(1-x)^2(1-x^3)}.$$

Alternatively this expression is obtained if we consider the symmetry group of order 6 of the underlying simple graph, which can be generated by (i) the group C_3 of the triangle combined with (ii) the mirror symmetry along a diagonal.

```
with(group):
g := permgroup(3, {[[1, 2, 3]], [[2, 3]]}) ;
for i in elements(g) do
    print(i) ;
end do;
```

The cycle index obtained with this Maple code is [2, p 57]

$$(10) \qquad Z(tri) = \frac{t_1^3 + 3t_1 t_2 + 2t_3^1}{6}.$$

Insertion of (4) gives (9).

5. Square

The symmetry group of the square is a group of order 8 which essentially is generated by rotation by 90 degrees or flips along the horizontal or vertical axis or diagonals.

```
with(group):
g := permgroup(4, {[[1, 2, 3, 4]], [[1, 3]]}) ;
for i in elements(g) do
    print(i) ;
end do;
```

The cycle index is [4, Fig 3][1]

$$(11) \qquad Z(D_8) = \frac{t_1^4 + 2t_4^1 + 2t_1^2 t_2^1 + 3t_2^2}{8}$$

The enumeration theorem turns this into the generating function

$$(12) \qquad g_4(x) = \frac{x^4(1 - x + x^2)}{(1 + x^2)(1 + x)^2(1 - x)^4}.$$

where [3, A005232]

$$(13) \qquad a_4(n) = 0, 0, 0, 0, 1, 1, 3, 4, 8, 10, 16, 20, 29, 35, 47 (n \ge 0).$$

6. Square with Diagonal

In the square with a diagonal edge, the diagonal stays inert under the symmetry operations, and contributes a factor $x/(1-x)$ to the generating function. The symmetry group of the four other edges allows a flip along any of the two diagonals and generates a group of order 4:

```
with(group):
g := permgroup(4,{[[2, 4], [1, 3]], [[1, 4], [2, 3]]}) ;
for i in elements(g) do
    print(i) ;
end do;
```

The cycle index is

$$(14) \qquad Z(sqdia) = \frac{t_1^4 + 3t_2^2}{4}.$$

Insertion of (4) into the enumeration theorem yields $x^4(1-x+x^2)/[(1+x)^2(1-x)^4]$ [3, A053307], and convolved with the inert factor

$$(15) \qquad g_5(x) = \frac{x^5(1-x+x^2)}{(1+x)^2(1-x)^5};$$

$$(16) \qquad a_5(n) = 0, 0, 0, 0, 0, 1, 2, 6, 11, 22, 36, 60, 90, 135, 190, 266, 357, 476 (n \geq 0)$$

7. Complete Graph

The cycle index of the complete graph K_4 can be taken from the Wikipedia article of the cycle index [1],

$$(17) \qquad Z(K_4) = \frac{t_1^6 + 9t_1^2t_2^2 + 8t_3^2 + 6t_2t_4}{24}.$$

Insertion of (4) into the enumeration theorem yields

$$(18) \qquad g_6(x) = \frac{x^6(1-x+x^2+x^4+x^6-x^7+x^8)}{(1-x)^6(1+x)^2(1+x^2)(1+x+x^2)^2},$$

with [3, A003082]

$$(19) \qquad a_6(n) = 0, 0, 0, 0, 0, 0, 1, 1, 3, 6, 11, 18, 32, 48, 75, 111, 160, 224, 313 (n \geq 0)$$

8. Summary

The full generating function is

(20)
$$g_1(x) + g_2(x) + g_3(x) + g_4(x) + g_5(x) + g_6(x) = \frac{x^3(-x^{10} + x^9 + 2x^7 - x^6 + x^5 - 3x^4 + x^2 + x + 2)}{(x-1)^6(1+x)^2(1+x^2)(1+x+x^2)^2}$$

which expands to [3, A290778]

(21)
$$a(n) = 0, 0, 0, 2, 5, 11, 22, 37, 61, 95, 141, 203, 288, 393, 531, 704, 918, 1180, 1504 (n \geq 0)$$

References

1. F. Freudenstein, *The basic concepts of polya's theory of enumeration, with application to the structual classification of mechanisms*, Jn. Mechanisms **3** (1967), 275–290.
2. G. Pólya, Robert E. Tarjan, and Donald R. Woods, *Notes on introductory combinatorics*, Birkhäuser, 1983.
3. Neil J. A. Sloane, *The On-Line Encyclopedia Of Integer Sequences*, Notices Am. Math. Soc. **50** (2003), no. 8, 912–915, http://oeis.org/. MR 1992789 (2004f:11151)
4. Alan Tucker, *Polya's enumeration formula by example*, Math. Mag. **47** (1974), no. 5, 248–256.

URL: `http://www.mpia.de/~mathar`

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