A DETERMINANT ASSOCIATED TO AN INTEGER LINEAR EQUATION

CHRIS SMYTH

Suppose that $n \geq 2$ and we are given a nonzero n-tuple $\underline{a} = (a_1, \dots, a_n)$ of integers with $\gcd(a_1, \dots, a_n) = g$. We take a basis $\underline{u}_2, \dots, \underline{u}_n$ for the sublattice in $\mathbb{Z}^n$ of the integer solutions to the equation $\sum_{i=1}^n a_i x_i = 0$. We define the determinant $D_{\underline{a}}$ to be the determinant $\det((\underline{a}, \underline{u}_2, \dots, \underline{u}_n)^T)$. It is clear that the modulus of $D_{\underline{a}}$ is independent of the choice of this basis.

Our result is as follows.

Theorem 1. *We have* $|D_{\underline{a}}| = \frac{1}{g} \sum_{i=1}^n a_i^2$.

Proof. First of all, we note that we can assume that $g = 1$, as the result for arbitrary g then follows easily. We can also assume that all the a_i are nonnegative, as again the general case then follows easily. We use strong induction on $m = \min(a_1, \dots, a_n)$. Our induction hypothesis is that there is an $\varepsilon = \pm 1$ such that for $i = 1, \dots, n$ the cofactor A_i of the ith element a_i of the top row of $D_{\underline{a}}$ is equal to εa_i. Then the result follows on expanding $D_{\underline{a}}$ by its top row: $D_{\underline{a}} = \sum_{i=1}^n a_i A_i$.

For the base case $m = 1$, we can assume without loss of generality that $a_1 = 1$. Then $x_2, \dots, x_n \in \mathbb{Z}$ can be chosen arbitrarily, giving $x_1 = -\sum_{i=1}^n a_i x_i$. Thus we can take as a basis of solutions the vectors

$$\underline{u}_j = (-a_j, 0, \dots, 0, 1, 0, \dots, 0) \; (j = 2, \dots, n),$$

where the '1' is in the jth place. Hence we see that $A_1 = 1 = a_1$ while for $j \geq 2$ we have

$$A_j = (-1)^{j-1} \begin{vmatrix} -a_2 & 1 & & & & & & & \\ -a_3 & & 1 & & & & & & \\ \cdot & & & \cdot & & & & & \\ \cdot & & & & \cdot & & & & \\ \cdot & & & & & \cdot & & & \\ -a_{j-1} & & & & & & 1 & & \\ -a_j & & & & & & & & \\ -a_{j+1} & & & & & & & 1 & \\ \cdot & & & & & & & & \cdot \\ \cdot & & & & & & & & & \cdot \\ \cdot & & & & & & & & & & \cdot \\ -a_n & & & & & & & & & & & 1 \end{vmatrix},$$

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where the blank entries are 0. From this we readily obtain $A_j = a_j$. (So $\varepsilon = 1$ for our particular choice of basis.)

Now assume that $m \geq 2$ and that the induction hypothesis holds for all $\underline{a}$ with minimum element strictly less than m. Consider an integer vector $\underline{a}$ with minimum element equal to m. Without loss of generality we can assume that $a_1 = m$. Furthermore, since not all the a_i are divisible by m we can assume, again without loss of generality, that $a_2 = ka_1 + r$, where $1 \leq r < m$. Now consider our equation

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = 0, \tag{1}$$

which we can write as

$$a_1x_0 + rx_2 + a_3x_3 + \cdots + a_nx_n = 0, \tag{2}$$

where $x_0 = x_1 + kx_2$. Then, by the induction hypothesis, we can assume that for the vector $\underline{a}' = (a_1, r, a_3, \ldots, a_n)$ that the ith cofactor A'_i of $D_{\underline{a}'}$ with respect to its top row is, for some $\varepsilon = \pm 1$, equal to $\varepsilon a_i (i \neq 2)$ and εr for $i = 2$. Now, using $x_1 = x_0 - kx_2$, we see that for each basis solution $\underline{u}' = (u_0, u_2, u_3, \ldots, u_n)$ of (2) there is a basis solution $\underline{u} = (u_0 - ku_2, u_2, u_3, \ldots, u_n)$ of (1). Thus the cofactors A_i of (1) are given by $A_1 = A'_1 = \varepsilon a_1$, while for $j = 3, \ldots, n$ we have $A_j = A'_j$ again, obtained by adding k times the jth column of A_j to its first column. Finally, taking the signs of A'_1 and A'_2 into account, we have $A_2 = A'_2 + kA'_1 = \varepsilon(r + ka_1) = \varepsilon a_2$. This proves the inductive step. □

School of Mathematics and Maxwell Institute for Mathematical Sciences, University of Edinburgh, Edinburgh EH9 3FD, Scotland, U.K.

E-mail address: C.Smyth@ed.ac.uk