The conjecture in A284597 follows from Dickson's conjecture

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Theorem 1 *Assume Dickson's conjecture. For every positive integer n there are infinitely many runs of exactly n consecutive natural numbers with nondecreasing tau values, where τ is the number of divisors function A000005.*

Proof The proof is by induction.

For the case $n = 1$, we want to find arbitrarily large x such that $\tau(x-1) > \tau(x) > \tau(x+1)$. Start with y such that $\tau(y) > \tau(y+1)$ (e.g. this will be the case if $y+1$ is prime and y is composite). If y is not an example, $\tau(y-1) \le \tau(y)$. Consider $x = y + ky(y+1)$ for integer $k > 1$. If $1 + ky$ is prime, we have $x + 1 = (y+1)(1+ky)$; $y+1$ and $1+ky$ are coprime, so $\tau(x+1) = \tau(y+1)\tau(1+ky) = 2\tau(y+1)$. Similarly if $1 + k(y+1)$ is prime, we have $x = y(1 + k(y+1))$ so $\tau(x) = 2\tau(y)$. On the other hand, let q be a prime that doesn't divide $y(y+1)$ and N large enough that $N > 2\tau(y)$. There is v such that $y - 1 + vy(y+1) \equiv 0 \mod q^N$. Thus if $k \equiv v \mod q^N$, $q^N \mid x - 1$ and $\tau(x-1) \ge N + 1 > \tau(x) > \tau(x+1)$. There is no congruence condition preventing $1 + ky$ and $1 + k(y+1)$ being prime with $k \equiv v \mod q^N$, since for any finite set of primes $p_j \ne q$, there are k with $k \equiv v \mod q^N$ and $k \equiv 0 \mod p_i$, so that $1 + ky \equiv 1 \mod p_j$ and $1 + k(y+1) \equiv 1 \mod p_j$, while $x - 1 \equiv 0 \mod q^N$ implies $1 + k(y+1) = xy^{-1}$ and $1 + ky = x(y+1)^{-1}$ are nonzero mod q. Thus Dickson's conjecture implies there are infinitely many k for which this is true.

Now for the induction step. Suppose y is the start of a run of exactly n consecutive natural numbers with nondecreasing τ values, i.e.

$$\tau(y-1) > \tau(y) \le \tau(y+1) \le \ldots \le \tau(y+n-1) > \tau(y+n)$$

Let $L = \text{lcm}(y-1, y, \ldots, y+n-1, y+n+1)$, and consider $x = y + kL$ for $k > 1$. If $1+kL/(y+i)$ is prime (where $i \in \{-1, 0, \ldots, n-1, n+1\}$), then $x+i = (y+i)(1+kL/(y+i))$ with $y+i$ and $1+kL/(y+i)$ coprime so $\tau(x+i) = 2\tau(y+i)$. On the other hand, let $q > n$ be a prime that doesn't divide $(y-1)y \ldots (y+n-1)(y+n+1))$, and N large enough that $N+1 > \max(2\tau(y+n-1), 2\tau(y+n+1))$. There is v such that $y+n+vL \equiv 0 \mod q^N$. If $k \equiv v \mod q^N$, $q^N \mid x+n$ and $\tau(x+n) \ge N+1$. Thus if $k \equiv v \mod q^N$ and $1+kL/(y+i)$ is prime for all $i \in \{-1, 1, \ldots, n-1, n+1\}$, then

$$\tau(x-1) > \tau(x) \le \tau(x+1) \le \ldots \tau(x+n-1) < \tau(x+n) > \tau(x+n+1)$$

so x is the start of a run of exactly $n+1$ consecutive natural numbers with nondecreasing τ values. There is no congruence condition preventing this, since for any set of primes $p_j \neq q$, there are k with $k \equiv v \bmod q^N$ and $k \equiv 0 \bmod p_j$, so $1 + kL/(y+i) \equiv 1 \bmod p_j$ for $i \in \{-1, 0, \ldots, n-1, n+1\}$, while $1 + kL/(y+i) = (x+i)(y+i)^{-1}$ are nonzero mod q. Thus Dickson's conjecture implies there are infinitely many k for which this is true.

This completes the proof.