Generalized Narayana Numbers

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Set

$$N_m(n,k) = \frac{m+1}{n+1}\binom{n+1}{k}\binom{n-m-1}{k-1}$$

for $m \geq 0$ and $0 \leq k \leq n-m$ with the usual binomial-coefficient convention that $\binom{a}{b} = 0$ for $b < 0$ *except* $\binom{-1}{-1} := 1$. For $m = 0$ this gives the usual Narayana numbers $N(n,k)$ A001263, so let us call the $N_m(n,k)$ generalized Narayana numbers and say the N_m triangle is the triangle of m-th order Narayana numbers. Here is a table of values of $N_m(n,k)$ for small m, n, k.

$n \backslash k$	0	1	2	3	4
0	1				
1	0	1			
2	0	1	1		
3	0	1	3	1	
4	0	1	6	6	1

$m = 0$

$n \backslash k$	0	1	2	3	4
1	1				
2	0	2			
3	0	2	3		
4	0	2	8	4	
5	0	2	15	20	5

$m = 1$

$n \backslash k$	0	1	2	3	4
2	1				
3	0	3			
4	0	3	6		
5	0	3	15	10	
6	0	3	27	45	15

$m = 2$

$n \backslash k$	0	1	2	3	4
3	1				
4	0	4			
5	0	4	10		
6	0	4	24	20	
7	0	4	42	84	35

$m = 3$

Values of $N_m(n,k)$ for small m, n, k

Clearly, $N_m(n,k)$ is always nonnegative, and is an integer because

$$N_m(n,k) = \binom{n+1}{k}\binom{n-m-1}{k-1} - \binom{n}{k-1}\binom{n-m}{k}, \tag{1}$$

as a routine calculation shows.

Let $F_m(x,y) = \sum_{n\geq m,\, 0\leq k\leq n-m} N_m(n,k)x^n y^k$ denote the generating function for the m-th order Narayana numbers. It is known [1, Ex, 6.36b] (but Stanley omits the initial 1) that $F_0(x,y) = 1 + xy + x^2(y+y^2) + x^3(y+3y^2+y^3) + \cdots$ is given by

$$F_0(x,y) = \frac{1 + x(1-y) - \sqrt{1-2x(1+y)+x^2(1-y)^2}}{2x}\,. \tag{2}$$

There is a combinatorial interpretation of the m-th order Narayana numbers due to Werner Schulte: $N_m(n,k)$ is the number of nonnegative lattice paths of n upsteps $U = (1,1)$ and $n-m$ downsteps $D = (1,-1)$ with k peaks. (Nonnegative means the path never dips below the horizontal line through its initial point, and a peak is a UD. Clearly, such a path is the same as a Dyck path of semilength n whose last m steps are downsteps with the last m steps removed.) The interpretation can be established using the bijection from Dyck paths to parallelogram polyominoes that takes # peaks in path to width of polyomino [1, solution to Ex. 6.19 ℓ, viewing the sequences of 1s and -1s as Dyck paths], and then using the Lindström-Gessel-Viennot theorem [2, search Lindstrom-Gessel-Viennot lemma] to count the resulting parallelogram polyominoes, viewed as a pair of nonintersecting lattice paths with given endpoints. The theorem counts them as a 2×2 determinant of binomial coefficients that evaluates to (1).

The generating function $F_m(x,y)$ is easily found using the symbolic method [3]—a nonnegative path that starts at the origin and ends at height m decomposes into $m+1$ Dyck paths connected by m upsteps—and is given by

$$F_m(x,y) = x^m F_0(x,y)^{m+1}\,.$$

References

[1] Richard P. Stanley, *Enumerative Combinatorics* Vol. 2, Cambridge University Press, 1999. Exercise 6.19 and related material on Catalan numbers are available online at http://www-math.mit.edu/~rstan/ec/.

[2] Wikipedia.

[3] Philippe Flajolet and Robert Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009.