Tribonacci Facts

Jeffrey Shallit
School of Computer Science
University of Waterloo
Waterloo, ON N2L 3G1
Canada
shallit@cs.uwaterloo.ca

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Abstract

We prove some things about the occurrences of letters in the Tribonacci word. (Originally written in November 2016; a crucial typo was corrected July 20 2019.)

1 Introduction

The *Tribonacci numbers* are defined by $T_0 = 0$, $T_1 = 1$, $T_2 = 1$, and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 3$.

From the theory of linear recurrences, we know that if $\alpha_1, \alpha_2, \alpha_3$ are the roots of the cubic equation $X^3 - X^2 - X - 1 = 0$, then there are complex numbers c_1, c_2, c_3 such that

$$T_n = c_1 \alpha_1^n + c_2 \alpha_2^n + c_3 \alpha_3^n.$$

Here c_1, c_2, c_3 are the roots of the cubic

$$44X^3 - 2X - 1 = 0.$$

To fix the ordering, we set

$\alpha_1 \doteq 1.839286755214161132551852564653286600424178746097592246778758$

$\alpha_2 \doteq -0.4196433776070805662759262823266433002120893730487961\,23 + 0.606290729207199369259342197$

$\alpha_3 \doteq -0.419643377607080566275926282326643300212089373048796123 - 0.606290729207199369259342197$

and

$c_1 \doteq 0.336228116994941094225362954014332415157926090020459 28$

$c_2 \doteq -0.168114058497470547112681477007166620 - 0.19832414008114945728227903579631928795 65i$

$c_3 \doteq -0.168114058497470547112681477007166620 + 0.19832414008114945728227903579631928795 65i$

More precisely we have

$$\alpha_1 = \frac{1 + \sqrt[3]{19 - 3\sqrt{33}} + \sqrt[3]{19 + 3\sqrt{33}}}{3}$$
$$c_1 = \frac{\sqrt[3]{3267 - 561\sqrt{33}} + \sqrt[3]{3267 - 561\sqrt{33}}}{66}$$

while

$$|\alpha_2| = |\alpha_3| \doteq 0.73735270576032767520175965050812112334028240692655656 7235613$$

satisfies the equation $X^6 + X^4 + X^2 - 1 = 0$ and has closed form

$$\sqrt{\frac{(17 + 3\sqrt{33})^{1/3} - (-17 + 3\sqrt{33})^{1/3} - 1}{3}}$$

and

$$|c_2| = |c_3| \doteq 0.25999000212203995774095962183820658823125167999078364 7021$$

satisfies the equation $1936X^6 + 88X^4 - 1 = 0$ and has closed form

$$\sqrt{-\frac{1}{66} + \sqrt[3]{\frac{293 + 51\sqrt{33}}{1149984}} + \sqrt[3]{\frac{293 - 51\sqrt{33}}{1149984}}}.$$

2 Tribonacci inequalities

Lemma 1. *For $n \geq 0$ we have $|T_n - c_1\alpha_1^n| \leq 0.520 \cdot 0.738^n$.*

Proof. We have $T_n = c_1\alpha_1^n + c_2\alpha_2^n + c_3\alpha_3^n$, so

$$\begin{aligned}
|T_n - c_1\alpha_1^n| &= |c_2\alpha_2^n + c_3\alpha_3^n| \\
&\leq |c_2||\alpha_2|^n + |c_3||\alpha_3|^n \\
&= 2|c_2||\alpha_2|^n \\
&\leq 2 \cdot 0.260 \cdot 0.738^n \\
&= 0.520 \cdot 0.738^n.
\end{aligned}$$

□

Lemma 2. *For $n \geq 0$ we have*

$$|T_{n+1} - \alpha_1 T_n| \leq 1.342 \cdot .738^n.$$

Proof. From Lemma 1, we have

$$|T_{n+1} - c_1\alpha_1^{n+1}| \le 0.520 \cdot 0.738^{n+1} \tag{1}$$

and $|T_n - c_1\alpha_1^n| \le 0.520 \cdot 0.738^n$. Multiplying this last equation by α_1, we get

$$|\alpha_1 T_n - c_1\alpha_1^{n+1}| \le 0.957 \cdot 0.738^n. \tag{2}$$

Adding (1) to (2) and applying the triangle inequality, we get

$$|T_{n+1} - \alpha_1 T_n| \le 1.341 \cdot 0.738^n.$$

□

Similarly, we can prove

Lemma 3. *$|T_{n+2} - \alpha_1^2 T_n| \le 2.043 \cdot 0.738^n$.*

Lemma 4. *$|T_{n+3} - \alpha_1^3 T_n| \le 3.445 \cdot 0.738^n$.*

The proof is left to the reader.

Lemma 5. *For all $n \ge 0$ we have*
(a) $-.596 < [(n)_T 0]_T - \alpha_1 n < .856$;
(b) $-.883 < [(n)_T 00]_T - \alpha_1^2 n < 1.460$;
(c) $-1.461 < [(n)_T 000]_T - \alpha_1^3 n < 2.298$.

Proof. (a) Write n in its canonical Tribonacci representation, say $n = T_{e_1} + T_{e_2} + \cdots + T_{e_s}$ for $e_1 > e_2 > \cdots > e_s$. Then $[(n)_T 0]_T = T_{e_1+1} + T_{e_2+1} + \cdots + T_{e_i+1}$, so

$$[(n)_T 0]_T - \alpha_1 n = \sum_{1 \le i \le s} (T_{e_i+1} - \alpha_1 T_{e_i}).$$

Break up this sum into two pieces, one where $2 \le e_i \le 20$, and one where $e_i > 20$. The latter sum is bounded in absolute value by $\sum_{j \ge 21} 1.341 \cdot 0.738^j \le .009$. The former sum can be bounded by actually computing it for all $n < T_{21} = 121415$. The minimum is achieved at $n = 65915$ and is, rounded down, equal to -0.587. The maximum is achieved at $n = 78748$ and is, rounded up, equal to 0.847. Hence $-0.596 < [(n)_T 0]_T - \alpha_1 n < 0.856$.

In a similar way we can prove (b) and (c). For (b) the maximum of the appropriate sum is (rounded up) 1.446 and is achieved at $n = 78667$. The minimum is (rounded down) -0.869 and is achieved at $n = 65996$. Since $\sum_{j \ge 21} 2.043 \cdot 0.738^j \le .014$, the bound follows.

For (c) the maximum of the appropriate sum is (rounded up) 2.275 and is achieved at $n = 78667$. The minimum is (rounded down) -1.438 and is achieved at $n = 65996$. Since $\sum_{j \ge 21} 3.445 \cdot 0.738^j \le .023$, the bound follows.

□

3 Tribonacci words

We deal with words over the alphabet $\{0, 1, 2\}$. We let ϵ denote the empty word. By $|w|$ we mean the length of the word w and by $|w|_e$ for $e \in \{0, 1, 2\}$ we mean the number of occurrences of the symbol e in w. For words x, y by xy we mean the concatenation of x with y. By w^n we mean the word $\overbrace{ww\cdots w}^{n}$.

Define a sequence of words $(t_i)_{i\geq 0}$ as follows:

$$\begin{aligned} t_0 &= \epsilon \\ t_1 &= 2 \\ t_2 &= 0 \\ t_3 &= 01 \\ t_n &= t_{n-1}t_{n-2}t_{n-3} \text{ for } n \geq 4. \end{aligned}$$

Define the morphism φ as follows:

$$\begin{aligned} \varphi(0) &= 01 \\ \varphi(1) &= 02 \\ \varphi(2) &= 0 \end{aligned}$$

It is now easy to prove the following results by induction:

Lemma 6. *(a) For $n \geq 0$ we have $|t_n| = T_n$;*
(b) For $n \geq 2$ and $e \in \{0, 1, 2\}$ we have $|t_n|_e = T_{n-e-1}$;
(c) For $n \geq 0$ we have $\varphi^n(0) = t_{n+2}$;
(d) For $n \geq 1$ we have $\varphi(t_n) = t_{n+1}$;
(e) For $n \geq 2$ t_n is a prefix of $\mathbf{T} = \varphi^\omega(0)$.

We let $(n)_T$ denote the Tribonacci representation of n and $[w]_T$ be the integer whose Tribonacci representation is given by w.

Lemma 7. *The n'th occurrence of 0 in $\mathbf{T}$ is at position $[(n)_T0]_T$; the n'th occurrence of 1 in $\mathbf{T}$ is at position $[(n)_T01]_T$; the n'th occurrence of 2 in $\mathbf{T}$ is at position $[(n)_T011]_T$. In other words, the n'th occurrence of e in $\mathbf{T}$ is at position $[(n)_T01^e]_T$, for $e \in \{0, 1, 2\}$.*

Remark 8. Here we index $\mathbf{T}$ starting at position 0 and the "first" occurrence is actually the 0'th occurrence. So in this section, we are using 0-origin indexing for both concepts.

Proof. By induction on n. Let $e \in \{0, 1, 2\}$, and let $f_e(n)$ be the position of the n'th occurrence of e in $\mathbf{T}$.

Base case: the base case is $n \leq 4$, and is left to the reader.

For the induction step, assume the claim is true for all $n \leq T_k$ for some $k \geq 4$. We prove it for $T_k < n \leq T_{k+1}$.

There are two cases:

(i) $T_k < n \leq T_k + T_{k-1}$;
(ii) $T_k + T_{k-1} < n \leq T_{k+1}$.

Using Lemma 6 (a) and (e), write $|\mathbf{T}[0..T_{k+e+2} - 1]|$ as $t_{k+e+2} = t_{k+e+1}t_{k+e}t_{k+e-1}$. By Lemma 6 (b) we have $|t_{k+e+1}|_e = T_k$ and $|t_{k+e}|_e = T_{k-1}$.

Let us consider (i). In this case $f_e(n) = T_{k+e+1} + f_e(n - T_k)$. Now $1 \leq n - T_k \leq T_{k-1}$, so by induction we have $f_e(n-T_k) = [(n-T_k)_T 01^e]_T$. Then $f_e(n) = T_{k+e+1} + [(n-T_k)_T 01^e]_T = [(n)_T 01^e]$, as desired.

Now let us consider (ii). In this case $f_e(n) = T_{k+e+1} + T_{k+e} + f_e(n - T_k - T_{k-1})$. Now $1 < n - T_k - T_{k-1} \leq T_{k+1} - T_k - T_{k-1} = T_{k-2}$, so by induction we have $f_e(n - T_k - T_{k-1}) = [(n - T_k - T_{k-1})_T 01^e]_T$. Then $f_e(n) = T_{k+e+1} + T_{k+e} + [(n - T_k - T_{k-1})_T 01^e]_T = [(n)_T 01^e]$, as desired. □

4 Main results

In this section we change our indexing to starting at 1.

Define $A(n)$ to be the sequence A003144 in the OEIS, i.e., the position (starting with position 1) of the n'th occurrence of the letter 0 in the Tribonacci word $\mathbf{T}$ (where the first occurrence is $n = 1$).

Similarly, define $B(n)$ to be sequence A003145 in the OEIS, which is the position of the n'th occurrence of 1, and $C(n)$ to be sequence A003146, which is the position of the n'th occurrence of 2.

So, from Lemma 7, we have

$$A(n) = 1 + [(n-1)_T 0]_T$$
$$B(n) = 1 + [(n-1)_T 01]_T$$
$$C(n) = 1 + [(n-1)_T 011]_T$$

We can now state the main result.

Theorem 9. *For all $n \geq 1$ we have*

$$A(n) - 1 \leq \lfloor \alpha_1 n \rfloor \leq A(n) + 1$$
$$B(n) - 1 \leq \lfloor \alpha_1^2 n \rfloor \leq B(n) + 2$$
$$C(n) - 1 \leq \lfloor \alpha_1^3 n \rfloor \leq C(n) + 3.$$

Proof. From Lemma 5 we get $-.596 < [(n)_T 0]_T - \alpha_1 n < .856$. Since $A(n) = 1 + [(n-1)_T 0]_T$, we get $-.596 < A(n) - 1 - \alpha_1(n-1) < .856$. Hence $.404 < A(n) - \alpha_1 n + \alpha_1 < 1.856$, and, subtracting α_1, we get $-1.436 < A(n) - \alpha_1 n < .017$. Negating, we get $-.017 < \alpha_1 n - A(n) < 1.436$. Adding $A(n)$, we get $A(n) - .017 < \alpha_1 n < A(n) + 1.436$. Taking floors gives us the desired result. This proves the first inequality.

From Lemma 5 we have $-.883 < [(n)_T 00]_T - \alpha_1^2 n < 1.460$ for all $n \geq 0$. Since $B(n) = 1 + [(n-1)_T 01]_T$, we get $-.883 < B(n) - 2 - \alpha_1^2(n-1) < 1.460$ for all $n \geq 1$. Hence $1.117 < B(n) - \alpha_1^2 n + \alpha_1^2 < 3.460$, and, subtracting α_1^2, we get $-2.266 < B(n) - \alpha_1^2 n < .078$. Negating, we get $-.078 < \alpha_1^2 n - B(n) < 2.266$. Adding $B(n)$, we get $B(n) - .078 < \alpha_1^2 n < B(n) + 2.266$. Taking floors gives us the desired result. This proves the second inequality.

From Lemma 5 we get $-1.461 < [(n)_T 000]_T - \alpha_1^3 n < 2.298$ for all $n \geq 0$. Since $C(n) = 1 + [(n-1)_T 0]_T$, we get $-1.461 < C(n) - 4 - \alpha_1^3(n-1) < 2.298$ for all $n \geq 1$. Hence $2.539 < C(n) - \alpha_1^3 n + \alpha_1^3 < 6.298$, and, subtracting α_1^3, we get $-3.684 < C(n) - \alpha_1^3 n < .076$. Negating, we get $-.076 < \alpha_1^3 n - C(n) < 3.684$. Adding $C(n)$, we get $C(n) - .076 < \alpha_1^3 n < C(n) + 3.684$. Taking floors gives us the desired result. This proves the last inequality. □

Remark 10. The closeness of the lower bounds suggests that cases where $A(n) - 1 = \lfloor \alpha_1 n \rfloor$ should be rather rare, and similarly for $B(n) - 1$ and $C(n) - 1$. Indeed, the smallest n for which $A(n) - 1 = \lfloor \alpha_1 n \rfloor$ is $n = 12737$. Similarly, for $B(n) - 1$ it is 329 and and for $C(n) - 1$ it is 2047.