It will be noticed that the whole function is unchanged when, for each part p in the partitions, $10-p$ is written for p, any particular function being complementary either to itself or to one of the other functions.

If a partition be

$$p_1^{\pi_1} p_2^{\pi_2} \dots p_t^{\pi_t} \dots,$$

containing a given part p_t, π_t times, where $\pi_t > 0$, and C be its coefficient

$$\Sigma C \frac{(\pi_1 + \pi_2 + \dots + \pi_t - 1 + \dots)!}{\pi_1!\, \pi_2! \dots (\pi_t - 1)! \dots} = \binom{10}{p_t},$$

where the summation is in regard to those partitions which have a part p_t. For instance, taking $p_t = 6$, we are concerned with the third, fourth, ninth, tenth, and eleventh of the functions above appearing. We then have

$$1.60 + 3.5 + 1.5 + 3.20 + 7.10 = 210 = \binom{10}{6}.$$

The theorem is universally applicable, and provides a good check upon the accuracy of the numbers.

10,6644	6	180	87771	3	60
10,6554	9	540	87762	9	540
10,5555	12	60	87753	18	1080
99642	1	60	87744	20	600
99543	2	120	87663	23	1380
99444	2	20	87654	34	4080
98751	1	120	87555	45	900
98742	1	120	86664	42	840
98733	2	120	86655	54	1620
98661	1	60	77772	12	60
98652	3	360	77763	27	540
98643	4	480	77754	45	900
98553	7	420	77664	54	1620
98544	7	420	77655	64	1920
97770	1	20	76665	84	1680
97761	1	60	66666	88	88

The sum of the numbers in the column headed 'No. of Terms' is 2^{15}—a verification.

If a partition in the left-hand column be

$$p_1^{\pi_1} p_2^{\pi_2} \dots p_t^{\pi_t} \dots,$$

containing a given part p_t, π_t times, where $\pi_t > 0$, and C be its coefficient

$$\Sigma C \frac{(\pi_1 + \pi_2 + \dots + \pi_t - 1 + \dots)!}{\pi_1 ! \, \pi_2 ! \dots (\pi_t - 1)!} = 8 \binom{12}{p_t},$$

where the summation is in regard to those partitions which contain the part p_t. Thus for $s = 11$, the sum is

$$24 + 12 + 12 + 12 + 24 + 12 = 96 = 8 \binom{12}{11}.$$

In performing the multiplication of the fifteen factors, which compose $\Pi_{5,2}$, a selection of terms implies the winning of games by certain pairs and the losses by other pairs. If we take the complementary selection so that the pairs who were winners are now losers we see intuitively that the coefficient in $\Pi_{5,2}$ of a particular symmetric function is the same as that of the complementary function obtained by substituting for all parts p the parts $12 - p$.

Ex. gr. the coefficients of the functions (98553), (97743) are the same. The functions are thus associated in pairs; certain functions, such as (76665) which are self-complementary, being outstanding.

It will be gathered that there are so many verifications possible throughout the work that a numerical error in the final result may be regarded as out of the question.

We may notice the particular result, extracted from the Table,

87654 34 4080

which gives the highest number in the 'No. of Terms' column. It shows that the most probable score is

8, 7, 6, 5, 4,

and that this occurs in 4080 out of the 2^{15} events. If this is to be the score with the players in a named order we see that it occurs in 34 out of 2^{15} events. The result

66666 88 88

shows that the five players will end up 'all square' in 88 out of 2^{15} events.

The symmetric functions which arise in these American Tournaments between n players, s players engaging s players in all possible ways, have similar properties whatever values are given to n and s. The actual calculation becomes impracticable very soon with the increase of n. The particular case $n = 6$, $s = 3$ may be set forth because the calculation is not at all laborious. Treated as in the previous cases for $s = 2$, we find the result

$$F_{6,3}$$

$$= (\alpha_1\alpha_2\alpha_3 + \alpha_4\alpha_5\alpha_6)(\alpha_1\alpha_2\alpha_4 + \alpha_3\alpha_5\alpha_6)(\alpha_1\alpha_2\alpha_4 + \alpha_3\alpha_4\alpha_6)(\alpha_1\alpha_2\alpha_6 + \alpha_3\alpha_3\alpha_4)$$
$$\times (\alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_5\alpha_6)(\alpha_1\alpha_3\alpha_5 + \alpha_2\alpha_4\alpha_6)(\alpha_1\alpha_3\alpha_6 + \alpha_2\alpha_4\alpha_5)$$
$$\times (\alpha_1\alpha_4\alpha_5 + \alpha_2\alpha_3\alpha_6)(\alpha_1\alpha_4\alpha_6 + \alpha_2\alpha_3\alpha_5)(\alpha_1\alpha_5\alpha_6 + \alpha_2\alpha_3\alpha_4),$$

as shown in the following Table:

Sym. Function.	Coeff.	No. of Terms.
10,44444	1	6
955533	1	60
866442	1	180
864444	3	90
777333	1	20
775551	1	60
775533	2	180
755553	6	180
666660	1	6
666642	3	90
666444	7	140
555555	12	12
		$1024 = 2^{10}$.

whence we obtain

$$F_{5,2}$$

$$= (3333)\,\alpha_1^{12}$$
$$+ (4432)\,\alpha_1^{11}$$
$$+ \{(5522) + (5441) + (5432) + 3\,(5333) + 2\,(4433)\}\,\alpha_1^{10}$$
$$+ \{(6531) + 2\,(6432) + 2\,(6333) + (5550) + (5541)$$
$$+ 2\,(5532) + 3\,(5442) + 3\,(5433) + 7\,(4443)\}\,\alpha_1^{9}$$
$$+ \{(7522) + (7441) + (7432) + 3\,(7333) + (6640)$$
$$+ 2\,(6622) + 2\,(6541) + 2\,(6532) + 4\,(6442) + 4\,(6433)$$
$$+ 5\,(5542) + 7\,(5533) + 9\,(5443) + 9\,(4444)\}\,\alpha_1^{8}$$
$$+ \{(8432) + (7631) + (7550) + (7541) + 2\,(7532)$$
$$+ 3\,(7442) + 3\,(7433) + 3\,(6632) + 3\,(6551) + 4\,(6542)$$
$$+ 7\,(6533) + 9\,(6443) + 9\,(5552) + 11\,(5543) + 15\,(5444)\}\,\alpha_1^{7}$$
$$+ \{(9333) + (8541) + (8532) + 2\,(8433) + (7722)$$
$$+ (7641) + (7632) + 3\,(7542) + 5\,(7533) + 5\,(7443)$$
$$+ (6660) + 2\,(6651) + 5\,(6642) + 4\,(6633) + 5\,(6552)$$
$$+ 10\,(6543) + 11\,(6444) + 11\,(5553) + 17\,(5544)\}\,\alpha_1^{6}$$
$$+ \&c.$$

It is not necessary for my present purpose to calculate the expression of N with a higher suffix than 6 or to complete the expression of $F_{5,2}$, but it should be remarked that the expression of N_{12-s} can be at once written down from that of N_s by merely substituting for every part p in the latter the part $15-p$. For if

$$N_s = \Sigma\,(a, b, c, d),$$

$$N_{12-s} = N_{15}\,\Sigma\,(-a, -b, -c, -d)$$
$$= (15, 15, 15, 15)\,\Sigma\,(-a, -b, -c, -d)$$
$$= \Sigma\,(15-d, 15-c, 15-b, 15-a).$$

In agreement with this law it will be noticed that N_6 is unaltered by the substitution referred to. As a direct consequence of this fact the coefficient of α_1^{12-s} in $F_{5,2}$ is obtainable from that of α_1^{s} by the substitution of $9-p$ for p. It will be noticed that the coefficient of α_1^{6} is unaltered by this sub-

stitution. A verification of the expression for N_s is afforded by the circumstance that when it is written out *in extenso* the number of its terms is

$$\binom{12}{s}.$$

Ex. gr. for $s=2$,

$$6 + 12 + 24 + 3 \times 4 + 2 \times 6 = 66 = \binom{12}{2}.$$

Obviously also in $F_{5,2}$ the number of terms in the coefficients of α_1^{s} is

$$\binom{12}{s}.$$

We have now to multiply $F_{5,2}$ by $\Pi_4\,\{= (2220) + (3111)\}$ to obtain the expression of $\Pi_{5,2}$ in descending powers of α_1. If we then insert the part s in each of the partitions which occurs in the coefficient of α_1^{s}, we obtain all that part of $\Pi_{5,2}$ which involves partitions containing the part s. We have thus, obviously, an ample verification of our results. The multiplication of $F_{5,2}$ by $(2220) + (3111)$ is performed with little labour, and the final result is

$$\Pi_{5,2} =$$

Partition of Func.	Coeff.	No. of Terms.	Partition of Func.	Coeff.	No. of Terms.
12,6444	1	20	97752	3	180
12,5553	1	20	97743	7	420
11,7543	1	120	97662	6	360
11,6652	1	60	97653	10	1200
11,6643	1	60	97644	13	780
11,6553	1	60	97554	18	1080
11,6544	2	120	96663	14	280
11,5554	3	60	96654	23	1380
10,8633	1	60	96555	27	540
10,8552	1	60	88860	1	20
10,8543	1	120	88842	3	60
10,8444	3	60	88833	2	20
10,7742	1	60	88761	2	120
10,7661	1	60	88752	5	300
10,7652	2	240	88743	7	420
10,7643	3	360	88662	6	180
10,7553	3	180	88653	13	780
10,7544	5	300	88644	20	600
10,6653	6	360	88554	20	600

We observe also that

$$N_{12}=(\alpha_2\alpha_3\alpha_4\alpha_5)^{15}=(15,\ 15,\ 15,\ 15).$$

We have to find the elementary symmetric functions of the twelve expressions

$$\alpha_2^2\alpha_3^2\alpha_4,\ \ \alpha_2^2\alpha_3\alpha_4^2,\ \ \alpha_2\alpha_3^2\alpha_4^2,\ \ \alpha_2^2\alpha_3^2\alpha_5,\ \ \alpha_2^2\alpha_3\alpha_5^2,\ \ \alpha_2\alpha_3^2\alpha_5^2,$$
$$\alpha_2^2\alpha_4^2\alpha_5,\ \ \alpha_2^2\alpha_4\alpha_5^2,\ \ \alpha_2\alpha_4^2\alpha_5^2,\ \ \alpha_3^2\alpha_4^2\alpha_5,\ \ \alpha_3^2\alpha_4\alpha_5^2,\ \ \alpha_3\alpha_4^2\alpha_5^2,$$

and denoting these by

$$\{1\},\ \ \{1^2\},\ \ \{1^3\},\ \ \&c.,$$

we have

$$1!\,N_1=1!\,\{1\}=(221),$$
$$2!\,N_2=2!\,\{1^2\}=(221)^2-(442),$$
$$3!\,N_3=3!\,\{1^3\}=(221)^3-3\,(221)(442)+2\,(663),$$
$$4!\,N_4=4!\,\{1^4\}=(221)^4-6\,(221)^2(442)+3\,(442)^2$$
$$+\,8\,(221)(663)-6\,(884),$$
$$5!\,N_2=5!\,\{1^5\}=(221)^5-10\,(221)^3(442)+15\,(221)(442)^2$$
$$+20\,(221)^2(663)-20\,(442)(663)-30\,(221)(884)+24\,(10,\ 10,\ 5),$$

etc.

We might proceed to develop the right-hand sides of the relations by means of Hammond's differential operators, and in carrying this out there is more than one mode of procedure.

Having obtained the expressions we can multiply out by $\Pi_{4,2}$, and thus obtain the expression of $\Pi_{5,2}$ in descending powers of α_1. This, as will be seen, leads at once to the desired expression of $\Pi_{5,2}$ as a sum of monomial symmetric functions. Newton's theorem connecting the elementary symmetric functions of a number of quantities with the sums of powers of the same leads to the relations

$$2N_2=(221)\,N_1-(442),$$
$$3N_3=(221)\,N_2-(442)\,N_1+(663),$$
$$4N_4=(221)\,N_3-(442)\,N_2+(663)\,N_1-(884),$$
$$5N_5=(221)\,N_4-(442)\,N_3+(663)\,N_2-(884)\,N_1+(10,\ 10,\ 5),$$

etc.,

relations which are convenient for the successive calculations of N_2, N_3, N_4, N_5, &c.

The convenience arises from the circumstance that the Hammond operators are performed upon products of not more than two monomials. This is easy to carry out with small risk of a numerical error, and a check upon the coefficients is provided by the counting of the terms in the development of the monomials.

Ex. gr. consider the product

$$P=(8444)(221),$$
$$D_{10}D_6D_5D_4P=1,$$
$$D_9D_6^2D_4\qquad=1,$$
$$D_8D_6^2D_5\qquad=1,$$

so that $$P=(10,654)+(9664)+(8665),$$

and counting the terms in the development

$$4\times 12=24+12+12$$

a verification, as it shows that no terms have been omitted.

It will be seen too that when sN_s has been calculated, a certain verification is secured by the fact that each of the coefficients in the resulting linear function of monomials must be divisible by s.

The expressions obtained are

$$N_1=(2210),$$
$$N_2=(4411)+(4330)+(4321)+3\,(4222)+2\,(3322),$$
$$N_3=(6531)+2\,(6432)+2\,(6333)+(5550)+(5541)$$
$$+\,2\,(5532)+3\,(5442)+3\,(5433)+7\,(4443),$$
$$N_4=(8633)+(8552)+(8543)+3\,(8444)+(7751)+2\,(7733)$$
$$+\,2\,(7652)+2\,(7643)+4\,(7553)+4\,(7544)+5\,(6653)$$
$$+\,7\,(6644)+9\,(6554)+9\,(5555),$$
$$N_5=(10,654)+(9853)+(9770)+(9763)+2\,(9754)+3\,(9664)$$
$$+\,3\,(9655)+3\,(8854)+3\,(8773)+4\,(8764)+7\,(8755)$$
$$+\,9\,(8665)+9\,(7774)+11\,(7765)+15\,(7666),$$
$$N_6=(12,666)+(11,874)+(11,865)+2\,(11,766)+(10,1055)$$
$$+\,(10,974)+(10,965)+3\,(10,875)+5\,(10,866)+5\,(10,776)$$
$$+\,(9993)+2\,(9984)+5\,(9975)+4\,(9966)+5\,(9885)$$
$$+\,10\,(9876)+11\,(9777)+11\,(8886)+17\,(8877),$$

etc.,

Part III.

I now consider a tournament in which every pair of players plays every other pair. As the players enter the tournament in a perfectly symmetrical manner, the calculus of symmetric functions is peculiarly fitted to the discussion.

If we take four players, denoted by $\alpha, \beta, \gamma, \delta$, we have three games, α, β v. γ, δ; α, γ v. β, δ; α, δ v. β, γ, and we form the symmetric product

$$(\alpha\beta+\gamma\delta)(\alpha\gamma+\beta\delta)(\alpha\delta+\beta\gamma),$$

and perform the multiplication. To obtain a term of the product we take the first or second term of a factor according as we suppose that the corresponding pair of players wins or loses the game denoted by the factor.

Thus the term formed by the selected product

$$\gamma\delta . \alpha\gamma . \beta\gamma$$

denotes that the pairs

$$\gamma, \delta;\ \alpha, \gamma;\ \beta, \gamma$$

have been successful in the games

$$\alpha, \beta \text{ v. } \gamma, \delta;\ \alpha, \gamma \text{ v. } \beta, \delta;\ \alpha, \delta \text{ v. } \beta, \gamma$$

respectively.

The selection gives the term, of the product,

$$\alpha\beta\gamma^3\delta,$$

shewing that the players $\alpha, \beta, \gamma, \delta$ have been successful in 1, 1, 3, 1 games respectively.

Since the product

$$(\alpha\beta+\gamma\delta)(\alpha\gamma+\beta\delta)(\alpha\delta+\beta\gamma)$$

is symmetrical the product will be, on multiplication, a sum of monomial symmetric functions. It is, in fact,

$$\Sigma\alpha^2\beta^2\gamma^2+\Sigma\alpha^3\beta\gamma\delta,$$

and we observe that of the $2^3=8$ events, four result in the final scores 2, 2, 2, 0 with the players in undefined order and four result in the score 3, 1, 1, 1 with the players in undefined order. These scores are equally probable if it is an even wager on each game. There is only one chance in eight that the final score will be named with the players in a given order of merit. If n players take part in the tournament, four players can be selected in $\binom{n}{4}$ ways, and each of these

selections gives rise to three games; hence the total number of games to be played is $3\binom{n}{4}$, and there are $2^{3\binom{n}{4}}$ events to be analysed.

Before proceeding to general principles I give the complete discussion for the case of five players.

Here there are 15 games and $2^{15}=32768$ events to be examined.

We form the symmetric product

$$(\alpha_1\alpha_2+\alpha_3\alpha_4)(\alpha_1\alpha_2+\alpha_3\alpha_5)(\alpha_1\alpha_2+\alpha_4\alpha_5)(\alpha_1\alpha_3+\alpha_2\alpha_4)(\alpha_1\alpha_3+\alpha_2\alpha_5)(\alpha_1\alpha_3+\alpha_4\alpha_5)$$
$$(\alpha_1\alpha_4+\alpha_2\alpha_3)(\alpha_1\alpha_4+\alpha_2\alpha_5)(\alpha_1\alpha_4+\alpha_3\alpha_5)(\alpha_1\alpha_5+\alpha_2\alpha_3)(\alpha_1\alpha_5+\alpha_2\alpha_4)(\alpha_1\alpha_5+\alpha_3\alpha_4)$$
$$(\alpha_2\alpha_3+\alpha_4\alpha_5)(\alpha_2\alpha_4+\alpha_3\alpha_5)(\alpha_2\alpha_5+\alpha_3\alpha_4),$$

a factor corresponding to each game.

We must expand this into a sum of monomial symmetric functions.

Denoting the product, for the case of n quantities, by $\Pi_{n,2}$ we have

$$\Pi_{5,2}=(\alpha_1\alpha_2+\alpha_3\alpha_4)(\alpha_1\alpha_2+\alpha_3\alpha_5)(\alpha_1\alpha_2+\alpha_4\alpha_5)(\alpha_1\alpha_3+\alpha_2\alpha_4)(\alpha_1\alpha_3+\alpha_2\alpha_5)$$
$$\times(\alpha_1\alpha_3+\alpha_4\alpha_5)(\alpha_1\alpha_4+\alpha_2\alpha_3)(\alpha_1\alpha_4+\alpha_2\alpha_5)(\alpha_1\alpha_4+\alpha_3\alpha_5)(\alpha_1\alpha_5+\alpha_2\alpha_3)$$
$$\times(\alpha_1\alpha_5+\alpha_2\alpha_4)(\alpha_1\alpha_5+\alpha_3\alpha_4)\times\Pi_{4,2},$$

and we have seen that, in the partition notation,

$$\Pi_{4,2}=(2220)+(3111).$$

The product which multiplies $\Pi_{4,2}$ is of degree 12 in α_1, and we will denote it by $F_{5,2}$, so that

$$\Pi_{5,2}=F_{5,2}\Pi_{4,2}.$$

In $\Pi_{5,2}$ the coefficient of α_1^{12} is $(\alpha_2\alpha_3\alpha_4\alpha_5)^3\equiv(3333)$,

and that of α_1^{0} is $(\alpha_2\alpha_3\alpha_4\alpha_5)^6\equiv(6666)$.

We write

$$F_{5,2}=(3333)\left\{\alpha_1^{12}+\frac{N_1}{(1111)}\alpha_1^{11}+\frac{N_2}{(2222)}\alpha_1^{10}+\frac{N_3}{(3333)}\alpha_1^{9}+\ldots+\frac{N_{12}}{(12,\,12,\,12,\,12)}\right\},$$

where it is clear that $N_1=(2210)$ and N_s is equal to the sum of the products, s at a time and unrepeated, of the terms of $\Sigma\alpha_2^2\alpha_3^2\alpha_4$ in respect of the four quantities $\alpha_2, \alpha_3, \alpha_4, \alpha_5$.

$6^35^23^21^2$	2960	7560	22377600
$6^35^232^21$	5048	15120	76325760
$6^35^22^4$	8352	1260	10523520
$6^354^231^2$	4696	15120	71003520
$6^354^22^21$	7968	15120	120476160
6^3543^221	12608	30240	381265920
6^35432^3	20520	10080	206841600
6^353^41	19616	2520	49432320
$6^353^32^2$	31584	5040	159183360
$6^34^41^2$	7328	1260	9233280
6^34^3321	19324	10080	194785920
$6^34^32^3$	31320	1680	52617600
$6^34^23^31$	29800	5040	150192000
$6^34^23^22^2$	47792	7560	361297520
6^343^42	72224	2520	182004480
6^33^6	108288	84	9096192
$6^25^421^2$	2752	3780	10402560
$6^25^3431^2$	7280	15120	110073600
$6^25^342^21$	12336	15120	186520320
$6^25^33^221$	19432	15120	293811840
$6^25^332^3$	31584	5040	159183360
$6^25^24^31^2$	11264	7560	85155840
$6^25^24^2321$	29528	45360	1339390080
$6^25^24^22^3$	47792	7560	361297520
$6^25^243^31$	45368	15120	685964160
$6^25^243^22^2$	72656	22680	1647838080
$6^25^23^42$	109408	3780	413562240
6^254^421	44416	7560	335784960
$6^254^33^21$	67728	15120	1024047360
$6^254^332^2$	108032	15120	1633443840
$6^254^23^32$	161704	15120	2444964480
6^2543^5	240320	1512	363363840
6^24^531	100368	1512	151756416
$6^24^52^2$	159456	756	120548736
$6^24^43^22$	237408	3780	897402240
$6^24^33^4$	351168	1260	442471680
65^531^2	11168	1512	16886016
65^52^21	18896	1512	28570752
$65^44^21^2$	17152	3780	64834560
65^44321	44688	15120	675682560
65^442^3	72224	2520	182004480
65^43^31	68416	2520	172408320
$65^43^22^2$	109408	3780	413562240
65^34^321	66804	10080	673384320
$65^34^23^21$	101528	15120	1535103360

$65^34^232^2$	161704	15120	2444964480
65^343^32	241312	10080	2432424960
65^33^5	357600	504	180230400
65^24^431	149664	7560	1131459840
$65^24^42^2$	237408	3780	897402240
$65^24^33^22$	352472	15120	5329376640
$65^24^23^4$	519968	3780	1965479040
654^61	219360	504	110557440
654^532	512184	3024	1548844416
654^43^3	752464	2520	1896209280
64^72	740880	72	53343360
64^63^2	1084320	252	273248640
5^641^2	25920	252	6531840
5^6321	67104	504	33820416
5^62^3	108288	84	9096192
5^54^221	99760	1512	150837120
5^543^21	151120	1512	228493440
5^5432^2	240320	1512	363363840
5^53^32	357600	504	180230400
5^44^331	221728	2520	558754560
$5^44^32^2$	351168	1260	442471680
$5^44^23^23$	519968	3780	1965479040
5^443^4	765120	630	482025600
5^34^51	323600	504	163094400
5^34^432	752464	2520	1896209280
$5^34^33^3$	1102824	1680	1852744320
5^24^62	1084320	252	273248640
$5^24^53^2$	1583360	756	1197020160
54^73	2265200	72	163094400
4^9	3230080	1	3230080

Sum of numbers in last column is

$$2^{\frac{1}{2}.7.6}(2^{15}-9.2^8+9.8) \text{ or } 64038633472;$$

the most probable score is

6, 5, 5, 4, 4, 4, 3, 3, 2,

which emerges in 5329376640 of the events.

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NJAS

$7^2 4^5 1^2$	1696	756	1282176
$7^2 4^4 321$	4512	7560	34110720
$7^2 4^4 2^3$	7328	1260	9233280
$7^2 4^3 3^3 1$	7008	5040	35320320
$7^2 4^3 3^2 2^2$	11264	7560	85155840
$7^2 4^2 3^4 2$	17152	3780	64834560
$7^2 4 3^6$	25920	252	6531840
$76^3 531^3$	28	10080	282240
$76^3 52^2 1^2$	56	15120	846720
$76^3 4^2 1^3$	56	5040	282240
$76^3 4321^2$	224	30240	6773760
$76^3 42^3 1$	392	10080	3951360
$76^3 3^3 1^2$	392	5040	1975680
$76^3 3^2 2^2 1$	672	15120	10160640
$76^3 32^4$	1120	2520	2822400
$76^2 5^2 41^3$	96	15120	1451520
$76^2 5^2 321^2$	384	45360	17418240
$76^2 5^2 2^3 1$	672	15120	10160640
$76^2 54^2 21^2$	656	45360	29756160
$76^2 543^2 1^2$	1120	45360	50803200
$76^2 5432^2 1$	1912	90720	173456640
$76^2 542^4$	3168	7560	23950080
$76^2 53^3 21$	3084	30240	93260160
$76^2 53^2 2^3$	5048	15120	76325760
$76^2 4^3 31^2$	1808	15120	27336960
$76^2 4^3 2^2 1$	3072	15120	46448640
$76^2 4^2 3^2 21$	4888	45360	221719680
$76^2 4^2 32^3$	7968	15120	120476160
$76^2 43^4 1$	7648	7560	57818880
$76^2 43^3 2^2$	12336	15120	186520320
$76^2 3^5 2$	18896	1512	28570752
$765^4 1^3$	160	2520	403200
$765^3 421^2$	1064	30240	32175360
$765^3 3^2 1^2$	1808	15120	27336960
$765^3 32^2 1$	3084	30240	93260160
$765^3 2^4$	5104	2520	12862080
$765^2 4^2 31^2$	2880	45360	130636800
$765^2 4^2 2^2 1$	4888	45360	221719680
$765^2 43^2 21$	7744	90720	702535680
$765^2 432^3$	12608	30240	381265920
$765^2 3^4 1$	12064	7560	91203840
$765^2 3^3 2^2$	19432	15120	293811840
$7654^4 1^2$	4512	7560	34110720
$7654^3 321$	11918	60480	720800640
$7654^3 2^3$	19324	10080	194785920

$7654^2 3^3 1$	18404	30240	556536960
$7654^2 3^2 2^2$	29528	45360	1339390080
$76543^4 2$	44688	15120	675682560
7653^6	67104	504	33820416
$764^5 21$	18136	3024	54843264
$764^4 3^2 1$	27792	7560	210107520
$764^4 32^2$	44416	7560	335784960
$764^3 3^3 2$	66804	10080	673384320
$764^2 3^5$	99760	1512	150837120
$75^5 21^2$	1696	1512	2564352
$75^4 431^2$	4512	7560	34110720
$75^4 42^2 1$	7648	7560	57818880
$75^4 3^2 21$	12064	7560	91203840
$75^4 32^3$	19616	2520	49432320
$75^3 4^3 1^2$	7008	5040	35320320
$75^3 4^2 321$	18404	30240	556536960
$75^3 4^2 2^3$	29800	5040	150192000
$75^3 3^4 2$	68416	2520	172408320
$75^3 43^3 1$	28316	10080	285425280
$75^3 43^2 2^2$	45368	15120	685964160
$75^2 4^4 21$	27792	7560	210107520
$75^2 4^3 3^2 1$	42440	15120	641692800
$75^2 4^3 32^2$	67728	15120	1024047360
$75^2 4^2 3^3 2$	101528	15120	1535103360
$75^2 43^5$	151120	1512	228493440
$754^5 31$	63144	3024	190947456
$754^5 2^2$	100368	1512	151756416
$754^4 3^2 2$	149664	7560	1131459840
$754^3 3^4$	221728	2520	558754560
$74^7 1$	93360	72	6721920
$74^6 32$	219360	504	110557440
$74^5 3^3$	323600	504	163094400
$6^5 31^3$	48	504	24192
$6^5 2^2 1^2$	96	756	72576
$6^4 541^3$	160	2520	403200
$6^4 5321^2$	640	7560	4838400
$6^4 52^3 1$	1120	2520	2822400
$6^4 4^2 21^2$	1088	3780	4112640
$6^4 43^2 1^2$	1856	3780	7015680
$6^4 432^2 1$	3168	7560	23950080
$6^4 42^4$	5248	630	3306240
$6^4 3^3 21$	5104	2520	12862080
$6^4 3^2 2^3$	8352	1260	10523520
$6^3 5^3 1^3$	264	1680	443520
$6^3 5^2 421^2$	1744	15120	26369280

As a verification of the linear function

$$L_3(n),$$

we recall that when the symmetric function in $L(n)$ are written out *in extenso* the number of terms is

$$2^{\frac{1}{2}n(n-1)}.$$

To find the number of terms in $L_3(n)$ we observe that

$$L_3(n) = L(n) - K_{n-1}L(n-1) - IL(n-1) + K_{n-1}IL(n-2).$$

The number of terms in a symmetric function included in $L(n-1)$ becomes multiplied by n when it is capitated by K_{n-1} and the unit increase due to I has a similar effect. Also the unit increase and capitation performed upon a function included in $L(n-2)$ has the effect of multiplying the number of terms by $n(n-1)$. Hence the number of terms in $L_3(n)$ is

$$2^{\frac{1}{2}n(n-1)} - 2n.2^{\frac{1}{2}(n-1)(n-2)} + n(n-1).2^{\frac{1}{2}(n-2)(n-3)}$$

or

$$2^{\frac{1}{2}(n-2)(n-3)}\{2^{2n-3} - n.2^{n-1} + n(n-1)\}.$$

For $n=9$, we find

number of terms in $L_3(9) = 64, 038, 633, 472.$

I append the expression of $L_3(9)$.

The first column of numbers gives the coefficients of the symmetric function whose partitions are in the left-hand column. The second column of numbers gives the number of terms in the functions. The right-hand column is the product of the numbers in the first and second columns of numbers. I have verified that the sum of the numbers in the right-hand column is

64, 038, 633, 472,

and have thus the satisfaction of knowing that there is not a single numerical error in the calculation.

Π_9. The Portion $L_3(9)$.

Sym. Function or Score.	Coeff. of Sym. Func.	No. of terms in Sym. Func.	Events in Tournament.
$7^3 5431^3$	4	10080	40320
$7^3 542^2 1^2$	8	15120	120960
$7^3 53^2 21^2$	16	15120	241920
$7^3 532^3 1$	28	10080	282240
$7^3 52^5$	48	504	24192

$7^3 4^3 1^3$	8	1680	13440
$7^3 4^2 321^2$	32	15120	483840
$7^3 4^2 2^3 1$	56	5040	282240
$7^3 43^3 1^2$	56	5040	282240
$7^3 43^2 2^2 1$	96	15120	1451520
$7^3 432^4$	160	2520	403200
$7^3 3^4 21$	160	2520	403200
$7^3 3^3 2^3$	264	1680	443520
$7^2 6^2 431^3$	8	15120	120960
$7^2 6^2 42^2 1^2$	16	22680	362880
$7^2 6^2 3^2 21^2$	32	22680	725760
$7^2 6^2 32^3 1$	56	15120	846720
$7^2 6^2 2^5$	96	756	72576
$7^2 65^2 31^3$	16	15120	241920
$7^2 65^2 2^2 1^2$	32	22680	725760
$7^2 654^2 1^3$	32	15120	483840
$7^2 654321^2$	128	90720	11612160
$7^2 6542^3 1$	224	30240	6773760
$7^2 653^3 1^2$	224	15120	3386880
$7^2 653^2 2^2 1$	384	45360	17418240
$7^2 6532^4$	640	7560	4838400
$7^2 64^3 21^2$	224	15120	3386880
$7^2 64^2 3^2 1^2$	384	22680	8709120
$7^2 64^2 32^2 1$	656	45360	29756160
$7^2 64^2 2^4$	1088	3780.	4112640
$7^2 643^3 21$	1064	30240	32175360
$7^2 643^2 2^3$	1744	15120	26369280
$7^2 63^5 1$	1696	1512	2564352
$7^2 63^4 2^2$	2752	3780	10402560
$7^2 5^3 41^3$	56	5040	282240
$7^2 5^3 321^2$	224	15120	3386880
$7^2 5^3 2^3 1$	392	5040	1975680
$7^2 5^2 4^2 21^2$	384	22680	8709120
$7^2 5^2 43^2 1^2$	656	22680	14878080
$7^2 5^2 432^2 1$	1120	45360	50803200
$7^2 5^2 42^4$	1856	3780	7015680
$7^2 5^2 3^3 21$	1808	15120	27336960
$7^2 5^2 3^2 2^3$	2960	7560	22377600
$7^2 54^3 31^2$	1064	15120	16087680
$7^2 54^3 2^2 1$	1808	15120	27336960
$7^2 54^2 3^2 21$	2880	45360	130636800
$7^2 54^2 32^3$	4696	15120	71003520
$7^2 543^4 1$	4512	7560	34110720
$7^2 543^3 2^2$	7280	15120	110073600
$7^2 53^5 2$	11168	1512	16886016

so that
$$L_1(n) = K_{n-1}L(n-1) - K_{n-1}IL(n-2),$$
$$L_4(n) = IL(n-1) - K_{n-1}IL(n-2),$$
and
$$L_3(n) = L(n) - K_{n-1}L(n-1) - IL(n-1) + K_{n-1}IL(n-2).$$
If we know the expressions for
$$L(2),\ L(3),\ \ldots,\ L(n-1),$$
we only require to calculate $L_3(n)$ in order to obtain the expression of $L(n)$. It thus suffices to calculate the expressions $L_3(n)$ in order to obtain the expressions $L(n)$. Moreover, $L_3(n)$ is a self-complementary function, so that it is only necessary to calculate the coefficients of symmetric monomial functions associated with self-complementary partitions and one half of the remainder. Applying this to the calculation of $L(9)$ we find it to be a linear function of 490 monomial functions; $L_3(9)$ is a linear function of 215 such functions, 33 of which are associated with self-complementary partitions, and therefore the calculation is only necessary of
$$33 + \tfrac{1}{2}(215 - 33) \text{ or } 124$$
coefficients.

The calculation of the coefficients.

In Part I. it was established that the coefficient of the function, whose partition is
$$(p_1^{\pi_1} p_2^{\pi_2} \ldots p_s^{\pi_s}),$$
in the development of Π_n, is obtained from the operator expression
$$a^{n-p_1-1}(D_{p_1} + a^{-1}D_{p_1-1})^{\pi_1-1}(D_{p_2} + a^{-1}D_{p_2-1})^{\pi_2} \ldots (D_{p_s} + a^{-1}D_{p_s-1})^{\pi_s},$$
where $(1^s) = a^s$ symbolically. We seek therein the portion of the operator product which, on expansion, is free from a, and apply this operator which emerges to Π_{n-1}, but a better way of determining the useful portion of the operator is as follows:—

If we write $\pi_1 + 1$ for π_1 and put
$$\pi_m(\pi_m - 1)(\pi_m - 2) \ldots (\pi_m - k + 1) = \pi_m^k$$
symbolically the operator product may be effectively written
$$a^{n-p-1} . \exp \frac{1}{a}(\pi_p D_p^{-1} D_{p-1} + \pi_{p-1} D^{-1}_{p-1} D_{p-2} + \ldots$$
$$+ \pi_2 D_2^{-1} D_1 + \pi_1 D_1^{-1}) . D_p^{\pi_p} D_{p-1}^{\pi_{p-1}} \ldots D_1^{\pi_1},$$

and we are immediately led to the result
$$D_p^{\pi_p+1} D_{p-1}^{\pi_{p-1}} \ldots D_2^{\pi_2} D_1^{\pi_1} \Pi_n$$
$$= \frac{1}{(n-p-1)!}(\pi_p D_p^{-1} D_{p-1} + \pi_{p-1} D^{-1}_{p-1} D_{p-2}$$
$$+ \ldots \pi_2 D_2^{-1} D_1 + \pi_1 D_1^{-1})^{n-p-1} D_p^{\pi_p} D_{p-1}^{\pi_{p-1}} \ldots D_1^{\pi_1} \Pi_{n-1}.$$
This formula enables us to express the coefficients in Π_n as linear functions of the coefficients in Π_{n-1} and is very convenient to work with. I give some examples from the calculation of Π_9.

To find the coefficient of
$$\Sigma \alpha_1^7 \alpha_2^6 \alpha_3^6 \alpha_4^6 \alpha_5^3 \alpha_6^3 \alpha_7^2 \alpha_8^2 \alpha_9,$$
we have
$$D_7 D_6^3 D_3^2 D_2^2 D_1 \Pi_9$$
$$= (3D_6^{-1}D_5 + 2D_3^{-1}D_2 + 2D_2^{-1}D_1 + D_1^{-1}) D_6^3 D_3^2 D_2^2 D_1 \Pi_8$$
$$= (3D_6^2 D_5 D_3^2 D_2^2 D_1 + 2D_6^3 D_3 D_2^3 D_1$$
$$+ 2D_6^3 D_3^2 D_2 D_1^2 + D_6^3 D_3^2 D_2^2) \Pi_8,$$
and looking to the Table for Π_8 in Part I. and ascertaining the coefficients associated with the partitions
$$(6^2 5 3^2 2^2 1), \quad (6^3 3 2^3 1), \quad (6^3 3^2 2 1^2), \quad (6^3 3^2 2^2),$$
we find $\quad 3 \times 192 + 2 \times 28 + 2 \times 16 + 8 = 672.$

Again, to find the coefficient of
$$\Sigma \alpha_1^6 \alpha_2^6 \alpha_3^6 \alpha_4^5 \alpha_5^5 \alpha_6^2 \alpha_7^2 \alpha_8^2 \alpha_9^2,$$
we have
$$D_6^3 D_5^2 D_2^4 \Pi_9$$
$$= \frac{1}{2!}(2D_6^{-1}D_5 + 2D_5^{-1}D_4 + 4D_2^{-1}D_1)^2 D_6^2 D_5^2 D_2^4 \Pi_8$$
$$= \frac{1}{2!}(2D_6^{-2}D_5^2 + 2D_5^{-2}D_4^2 + 12D_2^{-2}D_1^2 + 8D_6^{-1}D_4$$
$$+ 16D_6^{-1}D_5 D_2^{-1}D_1 + 16D_5^{-1}D_4 D_2^{-1}D_1) D_6^2 D_5^2 D_2^4 \Pi_8$$
$$= (D_5^4 D_2^4 + D_6^2 D_4^2 D_2^4 + 6D_6^2 D_5^2 D_2^2 D_1^2 + 4D_6 D_5^2 D_4 D_2^4$$
$$+ 8D_6 D_5^3 D_2^3 D_1 + 8D_6^2 D_5 D_4 D_2^3 D_1).$$
Whence, operation upon Π_8, and reference to the Π_8 Table in Part I., gives
$$1536 + 544 + 96 + 3712 + 1568 + 896 = 8,352$$
for the coefficient sought.

Set IV. includes those which do not involve $n-1$, but do involve zero.

In fact, briefly:—

Set I. gives $n-1$ yes, zero no.

Set II. gives $n-1$ yes, zero yes.

Set III. gives $n-1$ no, zero no.

Set IV. gives $n-1$ no, zero yes.

The parts $n-1$ zero are complementary in a partition and neither can occur more than once in any partition.

The partitions in Set IV. are complementary to those in Set I. These sets consequently comprise equal numbers of partitions, and the corresponding linear functions involve the same coefficients.

Set II. and Set III. are both self-complementary because every partition

$$\begin{pmatrix} n-1 \text{ yes, zero no} \\ n-1 \text{ no, zero yes} \end{pmatrix}$$

converts into a partition

$$\begin{pmatrix} n-1 \text{ yes, zero no} \\ n-1 \text{ no, zero yes} \end{pmatrix}.$$

Let $N_1(n),\ N_2(n),\ N_3(n),\ N_4(n)$

denote the numbers of partitions in

Set I., Set II., Set III., Set IV.,

respectively, and put $N(n)$ for the whole numbers of partitions appertaining to Π_n. Then

$$N_1(n)+N_2(n)+N_3(n)+N_4(n)=N(n).$$

From the 'Capitation' theorem

$$N_1(n)+N_2(n)=N(n-1).$$

From the 'Unit Increase' theorem, which it should be observed arises directly from the theorem of Part I.,

$$D_k\Pi_n=(1^{n-k-1})\,\Pi_{n-1},$$

by putting $k=0$, we obtain

$$N_2(n)+N_4(n)=N(n-1)$$

equivalent to the relation derived from the 'Capitation' theorem.

Finally, if we take any partition appertaining to Π_{n-2}, viz.,

$$p_1p_2p_3\cdots p_{n-2}.$$

When p_1 may be as large as $n-3$ and p_{n-2} may be zero, we give a unit increase of part and add zero, obtaining

$$p_1+1,\ p_2+1,\ p_3+1,\ \ldots,\ p_{n-2}+1,\ 0,$$

and now capitation with $n-1$ yields

$$n-1,\ p_1+1,\ p_2+1,\ \ldots,\ p_{n-2}+1,\ 0,$$

which is a partition, appertaining to Π_n, of Set II., since

$$n-2+n-1+\binom{n-2}{2}=\binom{n}{2}.$$

Hence $N_2(n)=N(n-2)$,

so that from (3)

$$N_1(n)=N(n-1)-N(n-2),$$

and from (2)

$$N_3(n)=N(n)-2N(n-1)+N(n-2),$$

shewing that $N_1(n)$ and $N_3(n)$ are first and second differences of the numbers $N(n)$.

It may be gathered that if we know the numbers

$$N(2),\ N(3),\ \ldots,\ N(n-1),$$

we know $N_1(n),\ N_2(n)$, and $N_4(n)$,

and the value of $N(n)$ only depends upon our finding the value of $N_3(n)$. In fact, $N(n)$ is expressible entirely in terms of the numbers $N_3(n)$. We readily find the relation

$$N(n)-1=(n-2)\,N_3(3)+(n-3)\,N_3(4)+\ldots+N_3(n).$$

Consider next the linear functions of monomials corresponding to the sets of partitions, and write

$$L_1(n)+L_2(n)+L_3(n)+L_4(n)=L(n),$$

where $\Pi_n\equiv L(n)$.

Denote by K_{n-1}, when applied to the partition of a symmetric function, capitation with $n-1$ and by I. a unit increase of part and the addition of the part zero. Then

$$L_1(n)+L_2(n)=K_{n-1}L(n-1),$$

$$L_2(n)+L_4(n)=IL(n-1),$$

$$L_2(n)=K_{n-1}IL(n-2),$$

$5^4 4310$	80	1680	134400
$5^4 42^2 0$	160	840	134400
$5^4 421^2$	320	840	268800
$5^4 3^2 20$	272	840	228480
$5^4 3^2 1^2$	544	420	228480
$5^4 32^2 1$	928	840	779520
$5^4 2^4$	1536	70	107520
$5^3 4^3 10$	132	1120	147840
$5^3 4^2 320$	436	3360	1464960
$5^3 4^2 31^2$	872	1680	1464960
$5^3 4^2 2^2 1$	1480	1680	2486400
$5^3 43^3 0$	712	1120	797440
$5^3 43^2 21$	2348	3360	7889280
$5^3 432^3$	3824	1120	4282880
$5^3 3^4 1$	3664	280	1025920
$5^3 3^3 2^2$	5904	560	3306240
$5^2 4^4 20$	688	840	577920
$5^2 4^4 1^2$	1376	420	577920
$5^2 4^3 3^2 0$	1112	1680	1868160
$5^2 4^3 321$	3640	3360	12230400
$5^2 4^3 2^3$	5904	560	3306240
$5^2 4^2 3^3 1$	5632	1680	9461760
$5^2 4^2 3^2 2^2$	9040	2520	22780800
$5^2 43^4 2$	13712	840	11518080
$5^2 3^6$	20640	28	577920
$54^5 30$	1720	336	577920
$54^5 21$	5584	336	1876224
$54^4 3^2 1$	8576	840	7203840
$54^4 32^2$	13712	840	11518080
$54^3 3^3 2$	20676	1120	23157120
$54^2 3^5$	30960	168	5201280
$4^7 0$	2640	8	21120
$4^6 31$	12960	56	725760
$4^6 2^2$	20640	28	577920
$4^5 3^2 2$	30960	168	5201280
$4^4 3^4$	46144	70	3230080

Sum of numbers in last column $= 2^{28}$.

The most probable score is 5, 4, 4, 4, 3, 3, 3, 2, which arises in 23157120 out of 2^{28} events.

Part II.

The application of Symmetric Functions to this question naturally arises because the players enter the tournament in a symmetrical manner. The fortunes of the players do not depend upon a random fixing of pairs of opponents, the pairing is fixed *à priori* by the nature of the contest. The analysis of the $2^{\frac{1}{2}n(n-1)}$ events depends upon the expression of the symmetric function

$$\Pi_n = (\alpha_1 + \alpha_2)(\alpha_1 + \alpha_3)(\alpha_2 + \alpha_3)\ldots(\alpha_{n-1} + \alpha_n)$$

as a linear function of monomial symmetric functions. The partitions of these monomials may be roughly specified.

Each of them has the content $\frac{1}{2}n(n-1)$ and either n or $n-1$ parts. Admitting zero as part, we may say that there are always n parts, one of which may be zero. The highest part cannot exceed $n-1$ and only one part can be so large.

If a certain partition presents itself so also does the partition obtained by substituting, for every part p, the part $n-p-1$. We call these complementary partitions. Every partition having the above specifications does not present itself.

Several questions come up for consideration:—

(1) How many partitions occur for a given value of n?

(2) What is the most convenient and expeditious way of calculating the coefficients?

(3) How can the results be best verified?

I recall that in Part I. it was shown that the whole linear function appertaining to Π_n becomes by 'Capitation' by n a portion of the linear function appertaining to Π_{n+1}; also that if the same linear function be subjected to a unit increase of part and the part zero added the result is a portion of the linear function appertaining to Π_{n+1}; further the complementary symmetric functions, corresponding to complementary partitions, are affected with the same coefficients.

The theorems are guides to what follows.

It is convenient to divide the partitions appertaining to Π_n into four sets.

Set I. includes all partitions which involve the part $n-1$, but do not involve the part zero.

Set II. includes those which involve both the parts $n-1$ and zero.

Set III. includes those which involve neither $n-1$ nor zero.

$754^2 32^2 1$	164	10080	1653120
$754^2 2^4$	272	840	228480
$7543^4 0$	80	1680	134400
$7543^3 21$	266	6720	1787520
$7543^2 2^3$	436	3360	1464960
$753^5 1$	424	336	142464
$753^4 2^2$	688	840	577920
$74^5 10$	24	336	8064
$74^4 320$	80	1680	134400
$74^4 31^2$	160	840	134400
$74^4 2^2 1$	272	840	228480
$74^3 3^3 0$	132	1120	147840
$74^3 3^2 21$	436	3360	1464960
$74^3 32^3$	712	1120	797440
$74^2 3^4 1$	688	840	577920
$74^2 3^3 2^2$	1112	1680	1868160
$743^5 2$	1720	336	577920
73^7	2640	8	21120
$6^3 43210$	2	6720	13440
$6^3 431^3$	4	1120	4480
$6^3 42^3 0$	4	1120	4480
$6^3 42^2 1^2$	8	1680	13440
$6^3 3^3 10$	4	1120	4480
$6^3 3^2 2^2 0$	8	1680	13440
$6^3 3^2 21^2$	16	1680	26880
$6^3 32^3 1$	28	1120	31360
$6^3 2^5$	48	56	2688
$6^2 5^2 3210$	4	10080	40320
$6^2 5^2 31^3$	8	1680	13440
$6^2 5^2 2^3 0$	8	1680	13440
$6^2 5^2 2^2 1^2$	16	2520	40320
$6^2 54^2 210$	8	10080	80640
$6^2 54^2 1^3$	16	1680	26880
$6^2 543^2 10$	16	10080	161280
$6^2 5432^2 0$	32	10080	322560
$6^2 54321^2$	64	10080	645120
$6^2 542^3 1$	112	3360	376320
$6^2 53^3 20$	56	3360	188160
$6^2 53^3 1^2$	112	1680	188160
$6^2 53^2 2^2 1$	192	5040	967680
$6^2 532^4$	320	840	268800
$6^2 4^3 310$	28	3360	94080
$6^2 4^3 2^2 0$	56	1680	94080
$6^2 4^3 21^2$	112	1680	188160
$6^2 4^2 3^2 20$	96	5040	483840

$6^2 4^2 3^2 1^2$	192	2520	483840
$6^2 4^2 32^2 1$	328	5040	1653120
$6^2 4^2 2^4$	544	420	228480
$6^2 43^4 0$	160	840	134400
$6^2 43^3 21$	532	3360	1787520
$6^2 43^2 2^3$	872	1680	1464960
$6^2 3^5 1$	848	168	142464
$6^2 3^4 2^2$	1376	420	577920
$65^3 4210$	14	6720	94080
$65^3 41^3$	28	1120	31360
$65^3 3^2 10$	28	3360	94080
$65^3 32^2 0$	56	3360	188160
$65^3 321^2$	112	3360	376320
$65^3 2^3 1$	196	1120	219520
$65^2 4^2 310$	48	10080	483840
$65^2 4^2 2^2 0$	96	5040	483840
$65^2 4^2 21^2$	192	5040	967680
$65^2 43^2 20$	164	10080	1653120
$65^2 43^2 1^2$	328	5040	1653120
$65^2 432^2 1$	560	10080	5644800
$65^2 42^4$	928	840	779520
$65^2 3^4 0$	272	840	228480
$65^2 3^3 21$	904	3360	3037440
$65^2 3^2 2^3$	1480	1680	2486400
$654^4 10$	80	1680	134400
$654^3 320$	266	6720	1787520
$654^3 31^2$	532	3360	1787520
$654^3 2^2 1$	904	3360	3037440
$654^2 3^3 0$	436	3360	1464960
$654^2 3^2 21$	1440	10080	14515200
$654^2 32^3$	2348	3360	7889280
$6543^4 1$	2256	1680	3790080
$6543^3 2^2$	3640	3360	12230400
$653^5 2$	5584	336	1876224
$64^5 20$	424	336	142464
$64^5 1^2$	848	168	142464
$64^4 3^2 0$	688	840	577920
$64^4 321$	2256	1680	3790080
$64^4 2^3$	3664	280	1025920
$64^3 3^3 1$	3504	1120	3924480
$64^3 3^2 2^2$	5632	1680	9461760
$64^2 3^4 2$	8576	840	7203840
643^6	12960	56	725760
$5^5 210$	24	336	8064
$5^5 1^3$	48	56	2688

643^320	14	840	11760
643^31^2	28	420	11760
643^22^21	48	1260	60480
6432^4	80	210	16800
63^50	24	42	1008
63^421	80	210	16800
63^32^3	132	140	18480
5^33210	2	840	1680
5^331^3	4	140	560
5^32^30	4	140	560
$5^32^21^2$	8	210	1680
5^24^2210	4	1260	5040
$5^24^21^3$	8	210	1680
5^243^210	8	1260	10080
5^2432^20	16	1260	20160
5^24321^2	32	1260	40320
5^242^31	56	420	23520
5^23^320	28	420	11760
$5^23^31^2$	56	210	11760
$5^23^22^21$	96	630	60480
5^232^4	160	105	16800
54^3310	14	840	11760
54^32^20	28	420	11760
54^321^2	56	420	23520
54^23^220	48	1260	60480
$54^23^21^2$	96	630	60480
54^232^21	164	1260	206640
54^22^4	272	105	28560
543^40	80	210	16800
543^321	266	840	223440
543^22^3	436	420	183120
53^51	424	42	17808
53^42^2	688	105	72240
4^510	24	42	1008
4^4320	80	210	16800
4^431^2	160	105	16800
4^42^21	272	105	28560
4^33^30	132	140	18480
4^33^221	436	420	183120
4^332^3	712	140	99680
4^23^41	688	105	72240
$4^23^32^2$	1112	210	233520
43^52	1720	42	72240
3^7	2640	1	2640

Sum of numbers in last column $= 2^{21}$.

The most probable score is 4, 4, 3, 3, 3, 2, 2, which arises in 233520 out of 2^{21} events.

Eight Players. $\Pi_8 = (\alpha_1+\alpha_2)(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)\ldots(\alpha_7+\alpha_8)$.

76543210	1	40320	40320
765431^3	2	6720	13440
76542^30	2	6720	13440
76542^21^2	4	10080	40320
7653^310	2	6720	13440
7653^22^20	4	10080	40320
7653^221^2	8	10080	80640
76532^31	14	6720	94080
7652^5	24	336	8064
764^3210	2	6720	13440
764^31^3	4	1120	4480
764^23^210	4	10080	40320
764^232^20	8	10080	80640
764^2321^2	16	10080	161280
764^22^31	28	3360	94080
7643^320	14	6720	94080
7643^31^2	28	3360	94080
7643^22^21	48	10080	483840
76432^4	80	1680	134400
763^50	24	336	8064
763^421	80	1680	134400
763^32^3	132	1120	147840
75^33210	2	6720	13440
75^331^3	4	1120	4480
75^32^30	4	1120	4480
$75^32^21^2$	8	1680	13440
75^24^2210	4	10080	40320
$75^24^21^3$	8	1680	13440
75^243^210	8	10080	80640
75^2432^20	16	10080	161280
75^24321^2	32	10080	322560
75^242^31	56	3360	188160
75^23^320	28	3360	94080
$75^23^31^2$	56	1680	94080
$75^23^22^21$	96	5040	483840
75^232^4	160	840	134400
754^3310	14	6720	94080
754^32^20	28	3360	94080
754^321^2	56	3360	188160
754^23^220	48	10080	483840
$754^23^21^2$	96	5040	483840

7. The attached tables give the complete results as far as the tournament of eight players inclusive.

The first column gives in partition notation the symmetric functions that arise from the function Π; the second column the coefficients of these functions; the third column the numbers of terms in the functions when they are written out *in extenso*; the fourth column numbers are the products of the numbers in the second and third columns.

In the table for eight players the sixth row is interpretable as follows.

The final score of the tournament is

7, 6, 5, 3, 3, 2, 2, 0

in 40320 out of 2^{28} events.

The same final score arises for the players in a definite order in 4 out of 2^{28} events.

The development of Π_8 is possibly the most elaborate symmetric function that has ever been computed.

Three Players. $\Pi_3 = (\alpha_1+\alpha_2)(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)$.

Sym. Function or Score.	Coeff. of Sym. Func.	No. of terms in Sym. Func.	Events in Tournament.
210	1	6	6
1^3	2	1	2

Four Players. $\Pi_4 = (\alpha_1+\alpha_2)(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)\dots(\alpha_3+\alpha_4)$.

A571(4) = 4

A274098

3210	1	24	24
31^3	2	4	8
2^30	2	4	8
2^21^2	4	6	24

Sum of numbers in last column $= 2^6$.

Five Players. $\Pi_5 = (\alpha_1+\alpha_2)(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)\dots(\alpha_4+\alpha_5)$.

A571(5) = 9

43210	1	120	120
431^3	2	20	40
42^30	2	20	40
42^21^2	4	30	120
3^310	2	20	40
3^22^20	4	30	120
3^221^2	8	30	240
32^31	14	20	280
2^5	24	1	24

Sum of numbers in the last column $= 2^{10}$.

Six Players. $\Pi_6 = (\alpha_1+\alpha_2)(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)\dots(\alpha_5+\alpha_6)$.

543210	1	720	720
5431^3	2	120	240
542^30	2	120	240
542^21^2	4	180	720
53^310	2	120	240
53^22^20	4	180	720
53^221^2	8	180	1440
532^31	14	120	1680
52^5	24	6	144
4^3210	2	120	240
4^31^3	4	20	80
4^23^210	4	180	720
4^232^20	8	180	1440
4^2321^2	16	180	2880
4^22^31	28	60	1680
43^320	14	120	1680
43^31^2	28	60	1680
43^22^21	48	180	8640
432^4	80	30	2400
3^50	24	6	144
3^421	80	30	2400
3^32^3	132	20	2640

Sum of numbers in last column $= 2^{15}$.

The most probable score is 4, 3, 3, 2, 2, 1, which arises in 8640 out of 2^{15} events.

A274098

Seven players. $\Pi_7 = (\alpha_1+\alpha_2)(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)\dots(\alpha_6+\alpha_7)$.

6543210	1	5040	5040
65431^3	2	840	1680
6542^30	2	840	1680
6542^21^2	4	1260	5040
653^310	2	840	1680
653^22^20	4	1260	5040
653^221^2	8	1260	10080
6532^31	14	840	11760
652^5	24	42	1008
64^3210	2	840	1680
64^31^3	4	140	560
64^23^210	4	1260	5040
64^232^20	8	1260	10080
64^2321^2	16	1260	20160
64^22^31	28	420	11760

6. The application to the American Tournament is obtained by writing

$$\Pi_n = \Sigma C \frac{(p_1^{\pi_1} p_2^{\pi_2} \ldots p_s^{\pi_s})}{n!/(\pi_1!\,\pi_2!\ldots\pi_s!)},$$

and then C denotes the number of events out of the $2^{\binom{n}{2}}$ possible events, which give, without specification of the players, the final score

$$p_1, p_1 \ldots \pi_1 \text{ times}, \; p_2, p_2 \ldots \pi_2 \text{ times}, \; \ldots, \; p_s, p_s \ldots \pi_s \text{ times}.$$

Thus we write, when $n=3$,

$$\Pi_3 = 6\frac{(210)}{6} + 2\,(1^3),$$

indicating that the final score will be

2, 1, 0 in 6 events,

1, 1, 1 in 2 events,

2^3

If we write instead $\Pi_3 = (210) + 2\,(1^3)$,

we learn that only 1 event gives the score 2, 1, 0 when the order of the players is specified and that as before 2 events give the score 1, 1, 1 because the order of the players is of no moment.

For $n=4$, we write

$$\Pi_4 = 24\frac{(3210)}{24} + 8\frac{(31^3)}{4} + 8\frac{(2^30)}{4} + 24\frac{(2^21^2)}{6},$$

indicating that the final score will be

3, 2, 1, 0 in 24 events,

3, 1, 1, 1 in 8 events,

2, 2, 2, 0 in 8 events,

2, 2, 1, 1 in 24 events,

2^6

while if we write

$$\Pi_4 = (3210) + 2\,(31^3) + 2\,(2^30) + 4\,(2^21^2),$$

we see that if the order of the players be specified the events are in numbers 1, 2, 2, 4 for the scores

(3, 2, 1, 0), (3, 1, 1, 1), (2, 2, 2, 0), (2, 2, 1, 1),

respectively.

In the above case $n=4$ we see that an absolute winner emerges in exactly half of the events. It is an even wager. A tie of two players comes out in 24 events and a tie of three players in 8 events.

The case of five players is better arranged as under

43210	1	120	120
431^3	2	20	40
{ 3^310	2	20	40
{ 42^30	2	20	40
{ 42^21^2	4	30	120
{ 3^22^20	4	30	120
3^221^2	8	30	240
32^31	14	20	280
2^5	24	1	24
			$1024 = 2^{10}$

The left-hand column gives the partitions of the monomial symmetric functions due to Π_5, *or* the partitions which give the final scores in the tournament. The second column gives the coefficients of the symmetric functions in the first column as they arise in Π_5. The third column gives the numbers of terms in the parallel symmetric functions, and the fourth column, being the product of the two preceding columns, gives the numbers of events which appertain to the scores in the first column.

We gather that the most probable score is

3, 2, 2, 2, 1,

and that it emerges from 280 events.

An absolute winner emerges in

120 + 40 + 40 + 120 + 280 or 600 events.

A tie of two winners emerges in 120 + 240 or 360 events.

A tie of three winners emerges in or 40 events.

A tie of five winners emerges in or 24 events.

1024

The remaining terms can be derived from the theorem

$$D_{n-m}\Pi_n = (1^{m-1})\,\Pi_{n-1},$$

which readily leads to a convenient calculus.

By the well-known properties of D_k we have

$$D_k(1^m)\,\Pi_n = \{(1^{m-1})\,D_{k-1} + (1^m)\,D_k\}\,\Pi_{n-1},$$

and it is convenient to denote (1^m) by a^m symbolically, so that

$$D_k a^m \Pi_n = (a^{m-1}D_{k-1} + a^m D_k)\,\Pi_{n-1} = a^{m-1}(aD_k + D_{k-1})\,\Pi_{n-1},$$

leading to

$$D_{k_1}D_{k_2}a^m\Pi_n = a^{m-2}(aD_{k_1} + D_{k_1-1})(aD_{k_2} + D_{k_2-1})\,\Pi_{n-1},$$

and generally to

$$D_{p_1}^{\pi_1}D_{p_2}^{\pi_2}\ldots D_{p_s}^{\pi_s}a^m\Pi_n$$

$$= a^{m-\Sigma\pi}(aD_{p_1}+D_{p_1-1})^{\pi_1}(aD_{p_2}+D_{p_2-1})^{\pi_2}\ldots(aD_{p_s}+D_{p_s-1})^{\pi_s}\,\Pi_{n-1}.$$

Now $\quad D_{p_1}\Pi_n = a^{n-p_1-1}\Pi_{n-1},$

$$D_{p_1}^{\pi_1}\Pi_n = a^{n-p_1-1}(D_{p_1} + a^{-1}D_{p_1-1})^{\pi_1-1}\,\Pi_{n-1},$$

and in general $\quad D_{p_1}^{\pi_1}D_{p_2}^{\pi_2}D_{p_3}^{\pi_3}\ldots D_{p_s}^{\pi_s}\Pi_n$

$$= a^{n-p_1-1}(D_{p_1} + a^{-1}D_{p_1-1})^{\pi_1-1}(D_{p_2} + a^{-1}D_{p_2-1})^{\pi_2}$$

$$\times(D_{p_3} + a^{-1}D_{p_3-1})^{\pi_3}\ldots(D_{p_s} + a^{-1}D_{p_s-1})^{\pi_s}\,\Pi_{n-1},$$

a result which can be put into other forms by varying the order in which the operations

$$D_{p_1}^{\pi_1},\ D_{p_2}^{\pi_2},\ D_{p_3}^{\pi_3},\ \ldots,\ D_{p_s}^{\pi_s}$$

are taken.

If $\quad (p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots p_s^{\pi_s})$

be the partition of a symmetrical function which occurs in Π_n, the result of the operation

$$D_{p_1}^{\pi_1}D_{p_2}^{\pi_2}D_{p_3}^{\pi_3}\ldots D_{p_s}^{\pi_s}\Pi_n$$

must be that part of

$$a^{n-p_1-1}(D_{p_1} + a^{-1}D_{p_1-1})^{\pi_1-1}(D_{p_2} + a^{-1}D_{p_2-1})^{\pi_2}\ldots$$

$$\times(D_{p_s} + a^{-1}D_{p_s-1})^{\pi_s}\,\Pi_{n-1},$$

which on development of the *operator* is free from a. This part is a linear function of products of D operators, which when performed upon Π_{n-1} gives the value of

$$D_{p_1}^{\pi_1}D_{p_2}^{\pi_2}D_{p_3}^{\pi_3}\ldots D_{p_s}^{\pi_s}\Pi_n,$$

or in other words it gives the coefficient of symmetric function

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots p_s^{\pi_s})$$

in the expression of Π_n as a linear function of monomial symmetric functions.

5. **A** few examples will make the method clear to the reader. The expression Π_5 has been given in Art. 3. Suppose that we desire to derive therefrom the coefficient of symmetric function $(3^4 21)$ in the expression of Π_6. We have

$$D_3^4 D_2 D_1 \Pi_6 = a^2(D_3 + a^{-1}D_2)^3(D_2 + a^{-1}D_1)(D_1 + a^{-1})\,\Pi_5$$

$$= (a^3D_3^3 + 3aD_3^2D_2 + 3D_3D_2^2 + \ldots)(D_2 + a^{-1}D_1)(D_1 + a^{-1})\,\Pi_5,$$

where the fourth term in the first bracket is omitted because it has no bearing upon the result.

The part free from a upon the right-hand side is

$$(D_3^3D_1 + 3D_3^2D_2^2 + 3D_3^2D_2D_1^2 + 3D_3D_2^3D_1)\,\Pi_5,$$

and, on referring to the expression of Π_5, this is

$$(2 + 3\times 4 + 3\times 8 + 3\times 14) = 2 + 12 + 24 + 42 = 80;$$

so that a portion of Π_6 is $80\,(3^4 21)$.

To verify this we will find the coefficient of (432^4) by the same method; we know that this is also 80 because (432^4) and $(3^4 21)$ are complementary partitions. We have

$$D_4D_3D_2^4\Pi_6 = a(D_3 + a^{-1}D_2)(D_2 + a^{-1}D_1)^4\Pi_5$$

$$= (aD_3 + D_2)(D_2^4 + 4a^{-1}D_2^3D_1 + \ldots)\,\Pi_5$$

$$= (+4D_3D_2^3D_1 + D_2^5 + \ldots)\,\Pi_5$$

$$= 56 + 24 = 80.$$

The method is very convenient because it is allied to a means of verifying results, which provides a constant check of the work.

of which these two selected terms are types (or members), occur with the same coefficient in the development.

Thus, *e.g.*, in Π_5 we have the corresponding terms

$$\alpha_1\alpha_1\alpha_1\alpha_1\alpha_2\alpha_2\alpha_2\alpha_3\alpha_5\alpha_4 \equiv \alpha_1^4\alpha_2^3\alpha_3\alpha_4\alpha_5,$$

$$\alpha_2\alpha_3\alpha_4\alpha_5\alpha_3\alpha_4\alpha_5\alpha_4\alpha_3\alpha_5 \equiv \alpha_3^3\alpha_4^3\alpha_5^3\alpha_2,$$

and we find that the functions

$$(431^3),\ (3^310)$$

occur with the same coefficient in Π_5.

Note that (431^3)

is $\{(4-0),\ 4-1,\ (4-3)^3\}$,

and (3^310)

is $\{(4-1)^3,\ 4-3,\ 4-4\}$;

and, in general, to derive, from a partition, the complementary, we have merely to substitute, for the part p, the part $n-p-1$. Hence every partition has either a complementary with the same coefficient or it is self-complementary.

This circumstance may be regarded either as nearly halving the labour of calculation or as supplying a very important check upon the accuracy of the calculations.

The two complementary partitions involve the same repetitional exponents, and consequently involve the same number of terms.

Hence the product of the coefficient derived from the expansion of Π_n and the number of terms in the function is the same for both symmetric functions. We thus see that the number

$$C \times \frac{n!}{\pi_1!\,\pi_2!\ldots\pi_s!}$$

which arises in the tournament problem is the same for each.

Again, an examination of the product Π_n shews that all terms of Π_n, which involve one zero part, are derivable from the terms of Π_{n-1} by increasing each part (including zero when it occurs) by unity. *E.g.*, from

$$\Pi_4 = (3210) + 2\,(31^3) + 2\,(2^30) + 4\,(2^21^2)$$

we derive

$$\Pi_5 = (43210) + 2\,(42^30) + 2\,(3^310) + 4\,(3^22^20)$$

+ other terms which do not involve a zero part.

4. I now write

$$(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)\ldots(\alpha_1+\alpha_n) = \alpha_1^{n-1} + (1)'\alpha_1^{n-2} + (1^2)'\alpha_1^{n-3} + \ldots + (1^{n-1})',$$

where the dashed partitions denote symmetric functions of

$$\alpha_2,\ \alpha_3,\ \ldots,\ \alpha_n.$$

In Π_n the factor of

$$(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)\ldots(\alpha_1+\alpha_n),$$

when, in it we write α_{s-1} for α_s, is Π_{n-1}.

A portion of the result of multiplication is therefore

$$(1^m)'\alpha_1^{n-m-1}\Pi'_{n-1},$$

where Π'_{n-1} is the same function of $\alpha_2, \alpha_3, \ldots, \alpha_n$ that Π_{n-1} is of $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$.

Hence if D_k be Hammond's well-known differential operator of order k

$$D_{n-m}\Pi_n = (1^{m-1})\,\Pi_{n-1},$$

and in particular $\quad D_{n-1}\Pi_n = \Pi_{n-1}$.

D_k as an operator performed upon a symmetric function, represented by a partition, deletes the number k from such partition and causes it to vanish when the number k is not present. Further, as is well known,

$$D_{p_1}^{\pi_1} D_{p_2}^{\pi_2} \ldots D_{p_s}^{\pi_s}\,(p_1^{\pi_1} p_2^{\pi_2} \ldots p_s^{\pi_s}) = 1.$$

Now, $n-1$ is the highest part in any partition which occurs in Π_n. This leading number is thus struck out by D_{n-1} from all partitions in which it presents itself. This process was named by Cayley 'Decapitation'. Thus

$$D_4\Pi_5 = (3210) + 2\,(31^3) + 2\,(2^30) + 4\,(2^21^2) = \Pi_4.$$

The converse process of 'Capitation' may thus be employed, and from

$$\Pi_4 = (3210) + 2\,(31^3) + 2\,(2^30) + 4\,(2^21^2)$$

we at once proceed to

$$\Pi_5 = (43210) + 2\,(431^3) + 2\,(42^30) + 4\,(42^21^2)$$

+ other terms, no one of which contains the highest possible part 4.

Thus from Π_{n-1} we find directly all terms of Π_n (i) which involve the highest part $n-1$; (ii) which involve the part zero; (iii) the complementary terms of (i) and (ii).

If a term in the development be

$$C\alpha_1^{k_1}\alpha_2^{k_2}\ldots\alpha_n^{k_n},$$

it is clear that in C of the possible $2^{\frac{1}{2}n(n-1)}$ events the players denoted by 1, 2, 3, ..., n will be successful in

$$k_1,\ k_2,\ k_3,\ \ldots,\ k_n$$

games respectively out of the $n-1$ games engaged in by each.

Write now

$$(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)\ldots(\alpha_{n-1}+\alpha_n)=\Sigma C(k_1k_2k_3\ldots k_n),$$

where $$(k_1k_2k_3\ldots k_n)=\Sigma\alpha_1^{k_1}\alpha_2^{k_2}\alpha_3^{k_3}\ldots\alpha_n^{k_n},$$

and we conclude that the players in any assigned order are successful in $k_1, k_2, k_3, \ldots, k_n$ games in C out of the possible $2^{\frac{1}{2}n(n-1)}$ events.

As we shall have to consider equalities between the numbers $k_1, k_2, k_3, \ldots, k_n$, we now write

$$(k_1k_2k_3\ldots k_n)=(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots p_s^{\pi_s}),$$

and writing the literal product, briefly, Π_n we have

$$\Pi_n=\Sigma C(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots p_s^{\pi_s}).$$

When the symmetric function

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots p_s^{\pi_s})$$

is written out at length it involves

$$\frac{n!}{\pi_1!\,\pi_2!\,\pi_3!\ldots\pi_s!}\ \text{terms,}$$

so that, disregarding the final order of the players, the numbers of games to their credit at the conclusion of the tournament will constitute the partition

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots p_s^{\pi_s})\ \text{ of }\ \tfrac{1}{2}n(n-1)$$

in $$C\times\frac{n!}{\pi_1!\,\pi_2!\,\pi_3!\ldots\pi_s!}$$

events out of the possible $2^{\frac{1}{2}n(n-1)}$, and since the number of terms in Π_n is $2^{\frac{1}{2}n(n-1)}$ we must have

$$\Sigma C\cdot\frac{n!}{\pi_1!\,\pi_2!\,\pi_3!\ldots\pi_s!}=2^{\frac{1}{2}n(n-1)},$$

where the summation is for every partition

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots p_s^{\pi_s})$$

that occurs in the development of Π_n.

This relation supplies a ready means of verifying our results.

3. We have therefore, for study, the development of Π_n as a linear function of monomial symmetric functions, since such linear function completely analyses the events of the tournament.

We commence by noting simple properties of the partition

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots p_s^{\pi_s})$$

that is before us. We have

$$\Sigma\pi p=\binom{n}{2},\quad \Sigma\pi=n.$$

Since no player can win more than $n-1$ games, and, moreover, two players cannot both win $n-1$ games,

$$p_1 \not> n-1,\quad \pi_1\not>1,\ \text{ if }\ p_1=n-1.$$

Since one player, but not more than one, can lose $n-1$ games, p_s may be zero but $\pi_s\not>1$ if $p_s=0$.

Including a single zero as a possible part, every partition has n parts.

If n be uneven and equal to $2m+1$, no partition can have a greatest part $<m$.

If n be even and equal to $2m$, no partition can have a greatest part $<m$.

In order to make other properties clear it is convenient to have before us the early calculations

$\Pi_2=(10)$,

$\Pi_3=(210)+2(1^3)$,

$\Pi_4=(3210)+2(31^3)+2(2^30)+4(2^21^2)$,

$$\Pi_5=(43210)+2(431^3)+2(3^310)+2(42^30)+4(42^21^2)$$
$$+4(3^22^20)+8(3^221^2)+14(32^31)+24(2^5).$$

Consider the structure of Π_n.

In performing the multiplication we obtain a term by selecting the first or second term of each factor. We obtain a corresponding term by selecting the second or first term of each factor. The consequence is that the symmetric functions,

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PURE AND APPLIED MATHEMATICS.

AN AMERICAN TOURNAMENT TREATED BY THE CALCULUS OF SYMMETRIC FUNCTIONS.

By Major P. A. MacMahon.

Part I.

1. In a tournament of n players, where each player plays every other player, there are $\frac{1}{2}n(n-1)$ games. Since each game may be won or lost there are $2^{\frac{1}{2}n(n-1)}$ events and I propose to analyse them by means of the powerful calculus of symmetric functions. The final result of the play is that the players are arranged in a definite order, each with a certain number of games to his credit. These numbers constitute a partition of the number $\frac{1}{2}n(n-1)$, and we may ask how many of the $2^{\frac{1}{2}n(n-1)}$ events will yield a given partition of $\frac{1}{2}n(n-1)$ when the players are or are not in an assigned order.

2. Consider the symmetric function

$$(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)\ldots(\alpha_{n-1}+\alpha_n)$$

of the n quantities $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$.

It involves $\frac{1}{2}n(n-1)$ factors, and the terms, after carrying out the multiplication, are grouped together in monomial symmetric functions.

Consider a factor of the product

$$\alpha_p+\alpha_q, \quad p<q,$$

and let us agree to form a factor of a term in the development by selecting α_p or α_q, according as a player, denoted by p, defeats or is defeated by a player, denoted by q.