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Expansions for phase space coordinates for the plane pendulum

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Abstract

A paper by Bradley Klee on the expansion of the phase space solution of the plane pendulum in an energy variable is discussed, and an alternative expansion of the exact solution is proposed involving the Jacobi nome as energy variable and trigonometric functions.

The phase space solution for the plane mathematical pendulum for librational motion is known exactly in terms of *Jacobi*'s elliptic *sn* nd *cn* functions.

In a recent paper B. Klee [8] proposed an expansion of the phase space coordinates in terms of the energy variable $\alpha = \sin^2\left(\frac{\theta_0}{2}\right)$ with the initial maximal deflection $\theta_0 \in \left[0, \frac{\pi}{2}\right]$ (libration mode, with no motion for $\theta_0 = 0$). With dimensionless quantities one has $\alpha = \frac{E}{2}$, where $E = \frac{\hat{E}}{\omega_0\,\hbar}$ with the energy $\hat{E}$. For the plane pendulum $(L, m, g) =$ (length, mass, acceleration) the action is measured in $\hbar = L^2\,m\,\omega_0$ with $\omega_0 = 2\,\pi T_0 = \sqrt{\frac{g}{L}}$, and the *Hamilton*ian becomes

$$H = H(q,p) = \frac{\hat{H}}{\hbar\,\omega_0} = \frac{1}{2}p^2 + (1 - \cos(q)), \tag{1}$$

with the phase space variables $q = \theta$ (deflection) and $p = \dot{\theta}$, where the dot indicates differentiation w.r.t. the dimensionless time τ (the original dimensionful variables are $\hat{q} = L\,q$ and $\hat{p} = \sqrt{m\,\hbar\,\omega_0}\,p$). For $\dot{p} = 0$ conservation of energy yields

$$E = 2\,\sin^2\left(\frac{\theta_0}{2}\right) = 2\,\alpha\,. \tag{2}$$

The equation of motion obtained from the *Hamilton-Jacobi* differential equations becomes

$$\ddot{q} + \sin(q) = 0\,. \tag{3}$$

For given maximal deflection θ_0 the initial condition for $\tau = \omega_0\,t = 0$ is $(q(0),\,p(0)) = (0,\,2\sqrt{\alpha})$.

In *Klee*'s approach (here written in dimensionless variables) one uses polar coordinates with ϕ dependent radius.

$$q = R(\alpha,\phi)\,\cos(\phi), \quad p = R(\alpha,\phi)\,\sin(\phi)\,. \tag{4}$$

For $r(\alpha,\phi) = \frac{R(\alpha,\phi)}{2\,\sqrt{\alpha}}$ the following series expansion in α is used.

$$r(\alpha,\phi) = \sum_{n=0}^{\infty} r_n(\phi)\,\alpha^n\,, \tag{5}$$

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with $r_0(\phi) = 0$.
Putting this, with *Klee*, into the *Hamilton*ian eq. (1) leads to an implicit equation for $x := r^2(\alpha,\phi)$

$$1 = x\,(1 + h(x)) = x\,g(x), \text{ with } h(x) = \sum_{n=1}^{\infty} c(n)\,x^n, \tag{6}$$

where

$$c(n) = c(\alpha,\,\phi,\,n) = (-1)^n\,(2\,\cos(\phi))^{2\,(n+1)}\,\frac{1}{(2\,(2\,n+1))!}\,\alpha^n. \tag{7}$$

This can be solved for $x = r^2(\alpha,\phi)$ by *Lagrange*'s inversion

$$x = 1 + \sum_{j=1}^{\infty} \frac{1}{(j+1)!}\,\frac{d^j}{dt^j}\,\frac{1}{g(t)^{j+1}}\bigg|_{t=0}. \tag{8}$$

From this one finds $r_n(\phi)$ using the convolution

$$r^2(\alpha,\phi) = \sum_{n=0}^{\infty} \alpha^n \left(\sum_{j=0}^{n} r_n(\phi)\,r_{n-j}(\phi)\right). \tag{9}$$

These $r_n(\phi)$ are functions of $(\cos\phi)^2$ and the rational coefficients are given by $r(n,\,m) =$ A273506(n,m) / A273507(n,m). Thus,

$$r_n(\phi) = \sum_{m=1}^{n} r(n,\,m)\,(\cos\phi)^{2\,(2\,n-(m-1))}, \quad \text{for } n \ge 1, \text{ and } r_0(\phi) = 1\,. \tag{10}$$

In *Klee*'s paper the (dimensionless) period $f(\alpha) = \dfrac{T(\alpha)}{T_0}$ is computed *via* the formula $\hat{T}(E) = \frac{d}{d\hat{E}}\hat{S}(\hat{E})$ with the area $\hat{S}$ for closed phase space trajectories. This formula is given in *V. Arnold*'s book [3] as a problem on p. 20. With dimensionless quantities $S = \dfrac{\hat{S}}{L^2\,\omega_0\,m}$, $E = \dfrac{\hat{E}}{L^2\,m\,\omega_0^2}$ and $R(\alpha,\,\phi) = 2\sqrt{\alpha}\,r(\alpha,\,\phi)$. Inserting the $r(\alpha,\,\phi)$ series, this becomes

$$f(\alpha) = \frac{1}{4\,\pi}\,\alpha^n\,f_n, \quad \text{with} \quad f_n(\alpha) = (n+1)\sum_{j=0}^{n} \frac{1}{2\,\pi}\int_0^{2\,\pi} d\phi\,r^2(\alpha,\,\phi)\;. \tag{11}$$

With the convolution given above this leads to

$$f(\alpha) = \sum_{n=0}^{\infty} \alpha^n\,f_n, \text{ with } f_n = (n+1)\sum_{j=0}^{n} \frac{1}{2\,\pi}\int_0^{2\,\pi} d\phi\,r(j,\,\phi)\,r(n-j,\,\phi)\;. \tag{12}$$

These f_n expansion coefficients coincide with the ones obtained from the known result (see, *e.g.*, *Landau* and *Lifschitz* [10], vol. I, p. 30)

$$f(\alpha) = \frac{T(\alpha)}{T_0} = \frac{2}{\pi}\,K(\sqrt{\alpha}) = {}_2F_1\left(\frac{1}{2},\frac{1}{2},1,\alpha\right), \tag{13}$$

with the complete elliptic integral of first order (also the real quarter period of elliptic functions) $K(k)$. The rational coefficients f_n in the expansion of $k^2 = \alpha$ are, from the expansion of the hypergeometric ${}_2F_1$ function,

$$f_n = \left(\frac{(2\,n)!}{2^{2\,n}\,n!^2}\right)^2, \tag{14}$$

and they are given in OEIS [13] A038534 / A056982, starting with $[1, \frac{1}{4}, \frac{9}{64}, \frac{25}{256}, \frac{1225}{16384}, ...]$.

For given energy, *i.e.*, α, the time dependence of the phase space solution $(q(\tau), p(\tau))$ is obtained in *Klee*'s paper from one of the *Hamilton-Jacobi* equations $\frac{d\,q}{d\,\tau} = p$ by

$$d\tau = \frac{1}{p}\,dq = \frac{1}{R\,\sin(\phi)}\left(R'\,\cos(\phi) - R\,\sin(\phi)\right)\,d\phi = ((\log\,r)'\,\cot(\phi) - 1)\,d\phi, \tag{15}$$

where the prime indicates differentiation *w.r.t.* ϕ, and $R = R(\alpha, \phi) = 2\sqrt{\alpha}\,r(\alpha, \phi)$. $(log\ r)'$ can be expanded in α up to a given order α^N using multinomial coefficients from the expansion $\log(1 + \sum_{n=1}^{\infty} r_n\,\alpha^n)$. The results are found in *Klee*'s OEIS contributions

$$dt = d\phi\,\sum_{m=1}^{n}\alpha^n\sum_{m=1}^{n}\frac{A274076(n,m)}{A274078(n,m)}\,(\cos(\phi))^{2\,(n+m)}\,. \tag{16}$$

Integration led *Klee* to

$$\tau = -\left(\frac{2}{\pi}\,K(\sqrt{\alpha})\,\phi + \sum_{n=1}^{\infty}\alpha^n\sum_{m=1}^{2\,n}\frac{A274130(n,m)}{A274131(n,m)}\,\sin(2\,m\,\phi)\right). \tag{17}$$

In an approximation the first term is expanded up to the desired order.
One is interested in the time dependence of ϕ, and this can in principle be done again by *Lagrange* inversion. Finally, one will find the α expansion of the phase space coordinates $q(\tau, \sqrt{\alpha})$ and $p(\tau, \sqrt{\alpha})$.
One should compare this approximation with the well known exact result which is re-derived here for convenience with dimensionless quantities. The exact solution is obtained from the energy eqs. (1) and (2) by $\dot{\theta} = \sqrt{2}\,\sqrt{(1-\cos(\theta_0)) - (1-\cos(\theta))}$, using initial conditions for vanishing $\tau = \omega_0\,t$ as $\theta(0) = 0$ (not θ_0) and $\dot{\theta}(0) = 2\left|\sin\left(\frac{\theta}{2}\right)\right|$. With the variable $s = \frac{\sin\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta_0}{2}\right)} = \frac{1}{\sqrt{\alpha}}\,\sin\left(\frac{\theta}{2}\right)$ and $\frac{ds}{d\theta} = \frac{1}{2\sqrt{\alpha}}\,\cos\left(\frac{\theta}{2}\right)$ one obtains $\dot{\theta} = 2\sqrt{\alpha}\,\sqrt{1 - s^2}$

$$d\tau = \frac{1}{2\sqrt{\alpha}}\,\frac{1}{\sqrt{1 - s^2}}\,d\theta = \frac{1}{\sqrt{1 - s^2}\,\sqrt{1 - \alpha\,s^2}}\,ds. \tag{18}$$

Integrated, with $s = 0$ for $\tau = 0$, this is

$$\tau = \int_0^s ds'\,\frac{1}{\sqrt{(1 - s'^2)\,(1 - \alpha\,s'^2)}} =: F(s, \sqrt{\alpha}). \tag{19}$$

The inverse function is $F^{[-1]}(\tau, \sqrt{\alpha}) = sn(\tau, \sqrt{\alpha})$, with *Jacobi*'s elliptic function sn. Note that our notation differs from *e.g.*, *Abramowitz-Stegun* [1], pp. 569 ff, where the parameter m, our α, is used in the argument of the elliptic functions if it is written at all (like in the *Landen* transformations), but here we use $\sqrt{\alpha}$ which is the modulus k of elliptic functions (see [1] , p. 590). Our notation agrees, *e.g.*, with the one of Whittaker-Watson [15], p. 492, and this is also used in Maple [12] (but not in Mathematica). From $s = \frac{1}{\sqrt{\alpha}}\,\sin\left(\frac{\theta}{2}\right)$ we obtain, for given energy $E = 2\,\alpha$, $\theta(\tau) = 2\,\arcsin(\sqrt{\alpha}\,sn(\tau, \sqrt{\alpha}))$, hence for the rescaled phase space coordinates $\tilde{q} = \frac{\theta}{2\sqrt{\alpha}}$ and $\tilde{p} = \frac{\dot{\theta}}{2\sqrt{\alpha}}$ the exact solutions are

$$\tilde{q}(\tau, \sqrt{\alpha}) = \frac{1}{\sqrt{\alpha}}\,arcsin(\sqrt{\alpha}\,sn(\tau, \sqrt{\alpha}))\,, \tag{20}$$

$$\tilde{p}(\tau, \sqrt{\alpha}) = cn(\tau, \sqrt{\alpha})\,. \tag{21}$$

Remember the initial conditions used in the derivation: $\tilde{q}(0, \sqrt{\alpha}) = 0$ and $\tilde{p}(0, \sqrt{\alpha}) = \frac{\dot{\theta}(0)}{2\sqrt{\alpha}} = 1$.
For the exact solution see, *e.g.*, Lawden [11] ch. 5.1, p. 114 and [2] ch. 1.
The phase space trajectories look like figure $9-2$ on p. 321 of *Goldstein*'s book on classical mechanics [7]. The following figures show such trajectories for $[\tilde{q}, \tilde{p}]$ for $\alpha = \frac{1}{4}, \frac{1}{2}$ and $\frac{3}{4}$.

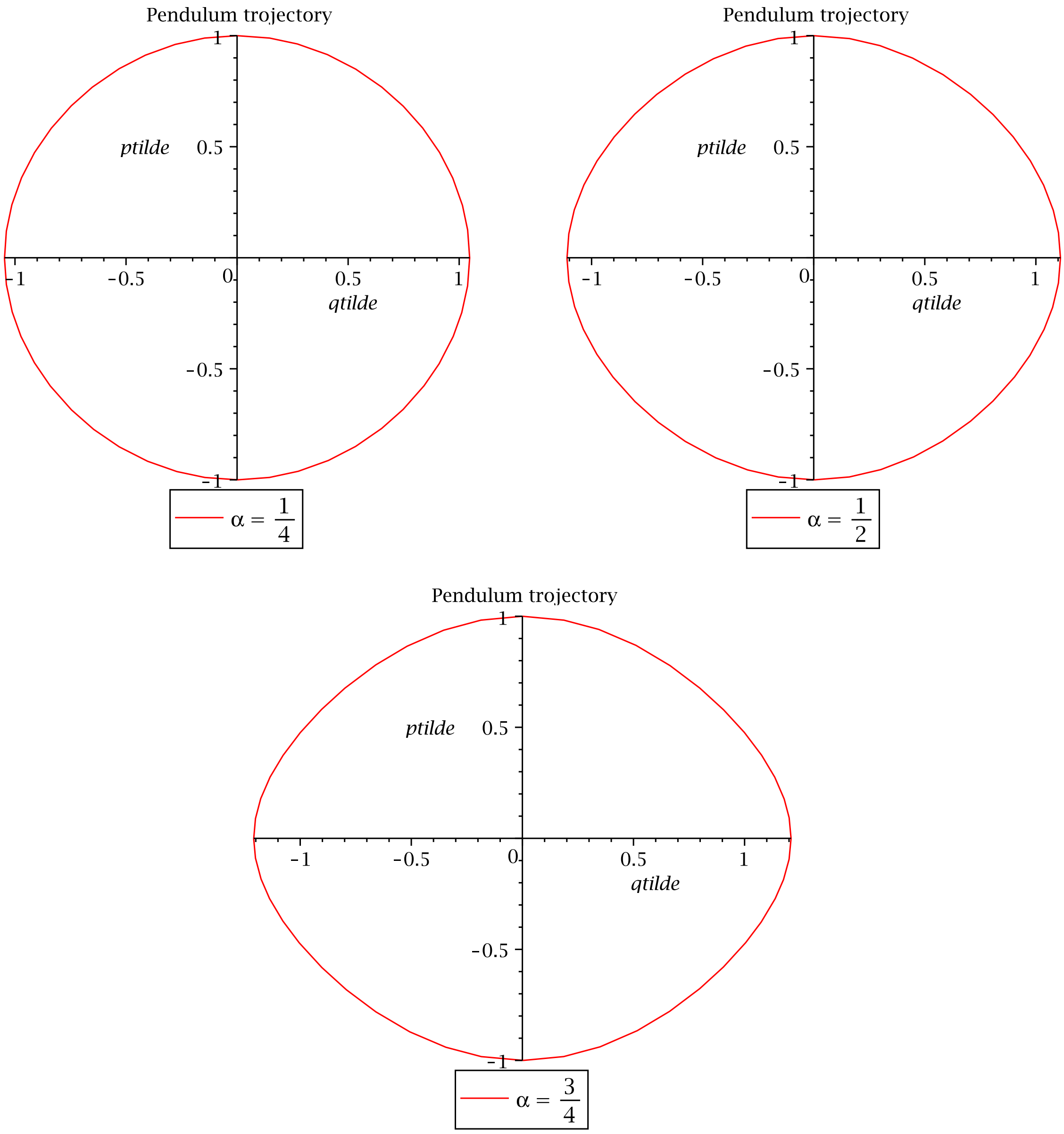

Figure 1: Exact phase space trajectories $[\tilde{q}, \tilde{p}]$ for $\alpha = \frac{1}{4}, \frac{1}{2}$ and $\frac{3}{4}$.

The question arises if *Klee*'s expansion in the energy variable α coincides with the expansion of this exact solution, as it should. However, the *Jacobi* elliptic functions $sn(\tau, \sqrt{\alpha})$ and $cn(\tau, \sqrt{\alpha})$ do not have a simple expansion in $\sqrt{\alpha}$. Therefore, if one is interested in approximations using expansions in an energy variable it is advisable not to use $\alpha = 2\,E$ but the variable which arises in the *Fourier* expansion of sn and cn or their representation by quotients of *Jacobi*'s theta functions. Then the new energy variable is the *Jacobi* nome q (no confusion with the phase space coordinate should arise) and instead of the dimensionless time τ one uses a variable v. These new variables are, with $k^2 = \alpha$,

$$q = q(k^2) = exp\left(-\pi\,\frac{K'(k)}{K(k)}\right) \quad \text{and} \quad v = v(\tau,\, k^2) = \frac{\tau}{\frac{2}{\pi}\,K(k)}. \tag{22}$$

Here K and $i\,K'$ are the real and imaginary quarter periods of *Jacobi*'s elliptic functions, which depend on the parameter k^2. $\hat{K} := \frac{2}{\pi}\,K = {}_2F_1\left(\frac{1}{2},\frac{1}{2},1,k^2\right)$ has been encountered already above in eq. (13), and the rational expansion coefficients have been given there. The formula for $\hat{K}' := \frac{2}{\pi}\,K'$ as function of k^2 could also be given (see a comment and the Fricke references [4] and [5] in $\frac{\text{A274653}}{\text{A274654}}$, but here only the quotient $\frac{K'}{K}$ is relevant, and the expansion of q in powers of $\left(\frac{k}{4}\right)^2$ is given in A005797, or [1], p. 591, 17.3.21, starting with [1, 8, 84, 992, 12514, 164688, ...]. However, a better convergent series (which will be used later on) has been proposed by *Kneser* [9], p. 218 (treated also in Tricomi, [14], p. 176). This is shown in A002103, and it gives an expansion in certain powers of $\varepsilon = \frac{1}{2}\,\frac{1\,-\,(1\,-\,k^2)^{\frac{1}{4}}}{1\,+\,(1\,+\,k^2)^{\frac{1}{4}}}$, *viz* $q = \varepsilon\,(1\,+\,\sum_{n=1}^{\infty}\text{A002103}\,(n)\,\varepsilon^{4\,n})$. The coefficients begin (offset 0) with [1, 2, 15, 150, 1707, 20910, ...]. (There is another slower convergent series, given by *Kneser* on p. 218 (attributed to *Lindelöf*), involving odd powers of $\frac{1\,-\,\sqrt{1\,-\,k^2}}{1\,+\,\sqrt{1\,+\,k^2}}$. See A274344, beginning with [1, 4, 34, 360, 4239 ...]).

The inverse transformation $k^2 = k^2(q)$ is obtained from $\sqrt{k} = \frac{\vartheta_2(0,\,q)}{\vartheta_3(0,\,q)}$ with *Jacobi*'s theta functions $\vartheta_i(z,\,q))$ evaluated at $z = 0$:

$$\vartheta_2(0,\,q) \;=\; 2\,q^{\frac{1}{4}}\sum_{n=0}^{\infty}q^{n\,(n+1)} \;=\; 2\,q_0\,q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1\,+\,q^n)^2\,, \tag{23}$$

$$\vartheta_3(0\,q) \;=\; 1\,+\,2\sum_{n=1}^{\infty}q^{n^2} \;=\; q_0\prod_{n=1}^{\infty}(1\,+\,q^{2\,n-1})^2\,, \tag{24}$$

where in the product representation

$$q_0 := \prod_{n=1}^{\infty}(1\,-\,q^{2\,n}) = \frac{1}{q^{\frac{1}{12}}}\,\eta(2\,z), \quad \text{with} \quad q = e^{2\,\pi\,i\,z},\; Im(z) > 0. \tag{25}$$

Dedekind's $\eta-$function entered with z in the upper half plane. See, *e.g.*, [1], p. p. 576, sect. 16.27, [2], p.104 (there θ instead of ϑ is used), [4], p. 418, [14], p. 156 and p. 167. For the result given for $\sqrt{k}$ as a quotient of θ functions see, *e.g.*, [2], p. 83, eq. (4,25), [4], p. 419, eq. (22), [9], p. 213 and [14], p. 173, eq. $(3\cdot 81)$. Therefore, with the sum representation,

$$\left(\frac{k}{4}\right)^2 = \frac{q\,(\sum_{n=0}^{\infty}q^{n\,(n+1)})^4}{(1\,+\,2\,\sum_{n=1}^{\infty}q^{n^2})^4} = q\sum_{n=0}^{\infty}\text{A005798}(n+1)\,q^n\;. \tag{26}$$

A005798 is the *Lagrange* inversion (also called reversion) of A005797 and starts with (offset 1) [1, −8, 44, −192, 718, −2400, ...]. The product version is

$$\left(\frac{k}{4}\right)^2 = q\left(\frac{\prod_{n=1}^{\infty}(1\,+\,q^{2\,n})}{\prod_{n=1}^{\infty}(1\,+\,q^{2\,n-1})}\right)^8\;. \tag{27}$$

For $\tau = \tau(v,\,q) = \hat{K}\,v$ one uses the result, see *e.g.*, [14], p. 175. eq. $(3\cdot 86)$, or [15], p. 486.

$$\hat{K} = \frac{2}{\pi}\,K = \vartheta_3^2(0,\,q), \tag{28}$$

which has expansion coefficients shown in A004018, starting with [1, 4, 4, 0, 4, 8, 0, 0, 4, 4, 8, ...]

A) Expansion using the Fourier series

As a first expansion in the new time and energy variables v and q we consider the *Fourier* series for the *Jacobi* elliptic functions. See e.g., Whittaker-Watson [15], p. 510, sect. 22 · 6, [2], pp. 51-56, sect. 2.9 and Exercises 2.9, and [1], p. 575, sect. 16.24. For the sn function this is

$$sn(\tau,\,\sqrt{\alpha}) \;=\; \frac{4\,q^{\frac{1}{2}}}{\hat{K}\,\sqrt{\alpha}}\sum_{n=0}^{\infty}\frac{q^n}{1\,-\,q^{(2\,n+1)}}\,\sin((2\,n+1)\,v)\;. \tag{29}$$

Of course, this should be read as an identity if on the $r.h.s.$ one uses $v = v(\tau,\,\alpha)$ and $q\,=\,q(\alpha)$ as given above with $k^2\,=\,\alpha$.

The prefactor is a function of q, obvious when expressed in terms of theta functions, *viz* the reciprocal of (we use $k\,=\,\sqrt{\alpha}$)

$$\frac{k}{4\,q^{\frac{1}{2}}}\,\hat{K}\;=\;\left(\frac{\frac{1}{2\,q^{\frac{1}{4}}}\,\vartheta_2(0,\,q)}{\vartheta_3(0,\,q)}\right)^2\,\vartheta_3^2(0,\,q)\;=\;\left(\frac{1}{2\,q^{\frac{1}{4}}}\vartheta_2(0,\,q)\right)^2\;=\;\left(\sum_{n=0}^{\infty}q^{n\,(n+1)}\right)^2\;. \tag{30}$$

The expansion of the prefactor is then given by

$$\frac{1}{\left(\sum_{n=0}^{\infty}\,q^{n\,(n+1)}\right)^2}\;=\;\sum_{n=0}^{\infty}a(n)\,q^n,\;\text{with}\;\;a(n)\;=\;\begin{cases}0 & \text{if } n\,=\,2\,k+1\;,\\ \text{A274621(k)} & \text{if }\;n\,=\,2\,k\;.\end{cases} \tag{31}$$

The sequence A274621 begins [1, −2, 3, −6, 11, −18, 28, −44, 69, −104, 152, −222, 323 ...].
The sum in eq. (29) can be reorganized as power series in q as

$$\sum_{n=0}^{\infty}\frac{q^n}{1\,-\,q^{(2\,n+1)}}\,\sin((2\,n+1)\,v)\;=\;\sum_{m=0}^{\infty}\,q^m\,b(m,\,v)\,. \tag{32}$$

with

$$b(m,\,v)\;=\;\sum_{k\,|\,(2\,m+1)}\sin(k\,v)\;=\;\sum_{l=1}^{\text{A099774}(m+1)}\sin(\text{A274658}(m,\,l)\,v)\,. \tag{33}$$

For example, $b(4,\,v)\;=\;\sin(1\,v)\;+\;\sin(3\,v)\;+\;\sin(9\,v)$, with A099774$(4+1)\;=\;3$, and the row $m=4$ of A274658 is [1, 3, 9], the divisors of $2\,\cdot\,4+1\;=\;9$.

The *Fourier* expansion coefficients of the q powers for sn are now obtained by the convolution of the sequences $\{b\}$ and $\{a\}$ from eq. (31).

$$sn(\tau,\,\sqrt{\alpha})\;\;=\;\;\sum_{n=0}^{\infty}\,q^n\,s(n,\,v)\;\;\text{with} \tag{34}$$

$$s(n,\,v)\;\;=\;\;\sum_{j=0}^{n}\,b(j,\,v)\,a(n-j)\,. \tag{35}$$

This convolution can be read off the number triangle A274659. The row polynomials $P(n,\,x)\;=\;\sum_{m=0}^{n}$ A274659$(n,\,m)\,x^m$ give $s(n,\,v)$ by replacing x^j by $\sin((2\,j+1)\,v)$. *E.g.*, row $n=4$, [2, 1, −2, 0, 1] leads to $P(4,\,x)=2\,x^0+1\,x^1-2\,x^2+1\,x^4$, *i.e.*, $s(4,\,v)\;=\;2\,\sin(1\,v)+1\,\sin(3\,v)-2\,\sin(5\,v)+1\,\sin(9\,v)$. The row sums $P(n,\,1)$ give A273166, [1, 2, 0 , −2, 2, 4, −4, −6, 6, 10, −8 ...].

As an example for the approximation we compare the exact result for $\tau = 1$ and $\alpha = \frac{1}{2}$, which is $sn\left(1, \sqrt{\frac{1}{2}}\right) \approx 0.8030018249$ (Maple [12], $JacobiSN\left(1, \sqrt{\frac{1}{2}}\right)$) with the first 11 terms of eqs. (34) and (35), *i.e.*, $\sum_{n=0}^{10} q^n\, s(n,\, v)$ with the corresponding values $v = \frac{1}{\frac{2}{\pi}\, K(\sqrt{\frac{1}{2}})}$ which is from eq. (13) $v \approx 0.8472130848$, and $q \approx 0.04321391815$ obtained already with the first 2 terms of the above mentioned expansion of *Kneser* involving A002103. This approximation gives 0.8030018241.

. The *arcsin* series (cf.A055786/A002595 without zero numerator entries) is now employed. This yields in a first step

$$\frac{1}{\sqrt{\alpha}}\, arcsin(\sqrt{\alpha}\, sn(\tau,\, \sqrt{\alpha})) = \sum_{n=0}^{\infty} \frac{1}{2\,n+1} \binom{2\,n}{n} \left(\frac{\alpha}{4}\right)^n \left(sn(\tau,\, \sqrt{\alpha})\right)^{2\,n+1} . \tag{36}$$

Now sn in the new variables v and q from eqs. (34) and (35) is inserted, and for $\frac{\alpha}{4} = 4\left(\frac{k}{4}\right)^2 = 4\,q \sum_{n=0}^{\infty}$ A005798$(n+1)\, q^n$ from eq. (26) is used. Here one cuts off the infinite sums in order to achieve an approximation up to and including powers of q^N. The main sum is cut off at $n = N$ because of the q^n term obtained from the $\left(\frac{\alpha}{4}\right)^n$ factor. The other factor involving A005798 and the sum on sn of eq. (34) is also cut off at $n = N$ because both sums start with 1. Thus one finally finds

$$\tilde{q}(\tau(v,\, q),\, \sqrt{\alpha}(q)) =: \hat{q}(v,\, q) = \sum_{n=0}^{\infty} S(n,\, v)\, q^n , \tag{37}$$

with $S(n,\, v)$ sums of powers of sin functions of multiple v arguments. See the *Table* for $S(n,\, v)$ for $n = 0,\, 1,\, ...,\, 6$.

In *Figure 2* the exact $\hat{q}(v\, q)$ and the expansion up to and including q^5 are shown, for three q values.

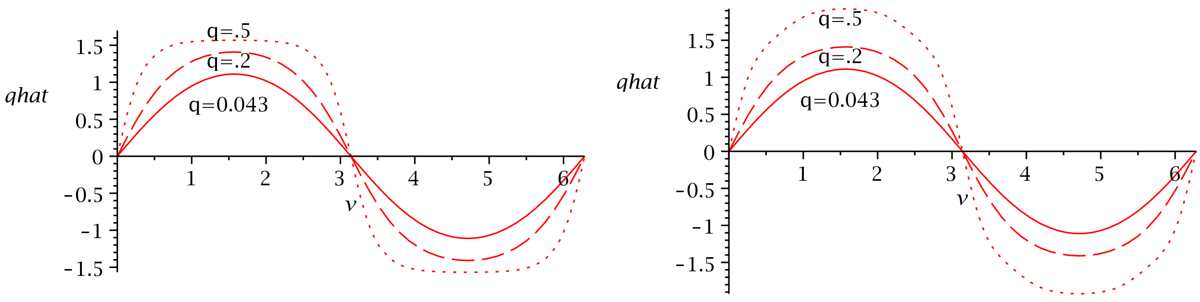

Figure 2 a) and b): The exact $\hat{q}(v,\, q)$ compared with the approximation up to and including q^5, for $q = 0.043,\, .2,\, .5$ and $v = [0,\, 2\,\pi]$.

For the other phase space coordinate the *Fourier* series for *cn* is use (see [15], p. 511, Example 1, [2]. p. 55, eq. (2.55), [1] p. 575, 16.23.2)

$$cn(\tau,\, \sqrt{\alpha}) \;=\; \frac{4\, q^{\frac{1}{2}}}{\hat{K}\, \sqrt{\alpha}} \sum_{n=0}^{\infty} \frac{q^n}{1 \,+\, q^{(2\,n+1)}}\, \cos((2\,n+1)\,v)\ . \tag{38}$$

The prefactor is the same as above, given in eq. (31), and the sum is expanded in q to give

$$\sum_{n=0}^{\infty} \frac{q^n}{1 \,+\, q^{(2\,n+1)}}\, \cos((2\,n+1)\,v) \;=\; \sum_{n=0}^{\infty} q^n\, c(n,\, v)\,. \tag{39}$$

with

$$\begin{aligned} c(n,\, v) &= \textstyle\sum_{(2\,k+1)\,|\,(2\,n+1)} (-1)^{n+k}\, \cos((2\,k+1)\,v) \\ &= \textstyle\sum_{m=1}^{A099774(n+1)} sign(A274660(n,\, m))\, \cos(|\,A274660(n,\, m)\,|\,v)\,. \end{aligned} \tag{40}$$

For example $c(4,\, v) \;=\; +\cos(1\,v) \;-\; \cos(3\,v) \;+\; \cos(9\,v)$ from the row $n = 4$ of A274660: $[1\; -3,\, 9]$, and A099774$(5) \;=\; 3$.

The convolution with the prefactor then yields

$$cn(\tau,\, \sqrt{\alpha}) \;=\; \sum_{n=0}^{\infty} q^n\, C(n,\, v)\,, \text{ with} \tag{41}$$

$$C(n,\, v) \;=\; \sum_{j=0}^{n} c(j,\, v)\, a(n-j)\,, \tag{42}$$

where the $\{a_n\}$ sequence is the one from eq. 31. The other phase space coordinate expressed in the new variables can thus be writes as

$$\tilde{p}(\tau(v,\, q),\, \sqrt{\alpha}(q)) \;=:\; \hat{p}(v,\, q) \;=\; \sum_{n=0}^{\infty} C(n,\, v)\, q^n\ , \tag{43}$$

with $C(n,\, v)$ sums of powers of cos functions of multiple v arguments which are given by

$$C(n,\, v) \;=\; \sum_{m=0}^{n} A274661(n,\, m)\, \cos((2\,m+1)\,v). \tag{44}$$

For example $C(4,\, v) \;=\; +2\, \cos(1\,v) \;-\; \cos(3\,v) \;-\; 2\, \cos(5\,v) \;+\; \cos(9\,v)$ from the row $n = 4$ of A274661: $[2,\, -1,\, -2,\, 0,\, 1]$.

In *Figure 3* the exact $\hat{p}(v,\, q)$ is compared with the approximation up to and including q^6 for three q values.

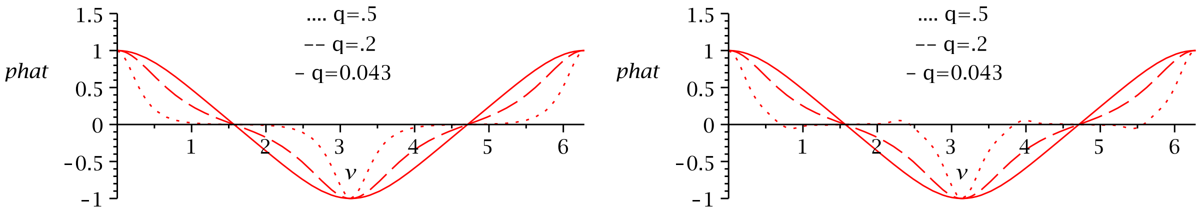

Figure 3 a) and b): The exact $\hat{p}(v,\,q)$ compared with the approximation up to and including q^6, for $q\,=\,0.043,\ .2,\ .5$ and $v\,=\,[0,\,2\,\pi]$.

B) Expansion using the quotient representation in terms of ϑ functions

Next we consider the expansions based on the formulae for sn and cn in the new variable v and q as quotients of ϑ functions (as above, $k\,=\,\sqrt{\alpha}$).

$$sn(\tau,\,k) \;=\; \frac{\vartheta_3(0,q)}{\vartheta_2(0,q)}\,\frac{\vartheta_1(v,q)}{\vartheta_4(v,q)}\,, \tag{45}$$

$$cn(\tau,\,k) \;=\; \frac{\vartheta_4(0,q)}{\vartheta_2(0,q)}\,\frac{\vartheta_2(v,q)}{\vartheta_4(v,q)}\,. \tag{46}$$

with the four ϑ functions either in a sum or product version (see, *e.g.*, [14], p. 176, eq. $(3\cdot 87)$, p. 156, eq. $(3\cdot 51)$, p. 167, eq. $(3\cdot 71)$ with $(3\cdot 71')$, p. 173, eq. $(3\cdot 81)$); [?], p. 418, eqs. (17) and (18) (he used ϑ_0 for ϑ_4); [1], p. 576, 16.27.1.-4.; [2] p. 78-86; [15] p. 464, pp.469-470, sect. $21\cdot 3$) That is,

$$sn(\tau,\,k) \;=\; \frac{\sum_{n=0}^{\infty}(-1)^n\,q^{n\,(n+1)}\,\sin((2\,n+1)\,v)\Big/\sum_{n=0}^{\infty}\,q^{n\,(n+1)}}{(1+2\,\sum_{n=1}^{\infty}(-1)^n\,q^{n^2}\,\cos(2\,n\,v))\Big/(1+2\,\sum_{n=1}^{\infty}\,q^{n^2})} \tag{47}$$

$$cn(\tau,\,k) \quad \frac{\sum_{n=0}^{\infty}\,q^{n\,(n+1)}\,\cos((2\,n+1)\,v)\Big/\sum_{n=0}^{\infty}\,q^{n\,(n+1)}}{(1+2\,\sum_{n=1}^{\infty}(-1)^n\,q^{n^2}\,\cos(2\,n\,v))\Big/(1+2\,\sum_{n=1}^{\infty}\,(-1)^n\,q^{n^2})} \tag{48}$$

or the product version

$$sn(\tau,\,k) \;=\; \sin(v)\,\frac{\prod_{n=1}^{\infty}(1-2\,q^n\,\cos(2\,v)+q^{4\,n})\Big/(1+q^{2\,n})^2}{\prod_{n=1}^{\infty}(1+2\,q^{2\,n-1}\,\cos(2\,v)+q^{2\,(2\,n-1)})\Big/(1+q^{2\,n-1})^2} \tag{49}$$

$$cn(\tau,\,k) \;=\; \cos(v)\,\frac{\prod_{n=1}^{\infty}(1+2\,q^n\,\cos(2\,v)+q^{4\,n})\Big/(1+q^{2\,n})^2}{\prod_{n=1}^{\infty}(1-2\,q^{2\,n-1}\,\cos(2\,v)+q^{2\,(2\,n-1)})\Big/(1-q^{2\,n-1})^2}\,. \tag{50}$$

Like in part **A**) these formulae turn into identities if one uses on the *r.h.s.* $v\,=\,v(\tau,\,k^2)$ and $q\,=\,q(k^2)$ given in eq. (22) (but for q we use the *Kneser* expansion in ε instead of k^2 mentioned earlier).

We start by rewriting the $sn(\tau,\,k)/\sin(v)$ sum representation in terms of even powers of $2\,\cos(v)$ with the help of *Chebyshev* polynomials S and T (for their coefficient triangles see A049310 and A053120).

$$\frac{sn(\tau,\,k)}{\sin(v)} \;=\; \frac{\sum_{n=0}^{\infty}(-1)^n\,q^{n\,(n+1)}\,S(2\,n,\,2\,\cos(v))\Big/\sum_{n=0}^{\infty}\,q^{n\,(n+1)}}{(1\,+\,2\,\sum_{n=1}^{\infty}(-1)^n\,q^{n^2}\,T(2\,n,\,\cos(v)))\Big/(1\,+\,2\,\sum_{n=1}^{\infty}\,q^{n^2})}\,. \tag{51}$$

or more directly in the product version (using $\cos(2\,v) \;=\; -1 \,+\, 2\,\cos^2(v))$)

$$\frac{sn(\tau,\,k)}{\sin(v)} \;=\; \frac{\prod_{n=1}^{\infty}\left(1\,-\,\frac{q^{2\,n}}{(1+q^{2\,n})^2}\,(2\,\cos(v))^2\right)}{\prod_{n=1}^{\infty}\left(1\,-\,\frac{q^{2\,n-1}}{(1+q^{2\,n-1})^2}\,(2\,\cos(v))^2\right)}\,. \tag{52}$$

Both versions lead to (identity theorem for powers series)

$$sn(\tau,\,k) \;=\; \sin(v)\sum_{n=0}^{\infty} q^n \sum_{m=0}^{n} TS(n,\,m)\;(2\,\cos(v))^{2\,m}\;, \tag{53}$$

with the number triangle $TS(n,\,m) \;=$A274662$(n,\,m)$, which begins $[1]$, $[0,1]$, $[0,-3,1]$, $[0,4,-5,1]$, $[0,-3,13,-7,1]$, ...

The phase space coordinate $\hat{q}(v,\,q)$ is obtained as in part **A)** by inserting the r.h.s. of eq. (47) into eq. (36) where $\alpha \;=\; k^2$ is also expanded in powers of q. Here the even powers $sin^{2\,n}(v)$ are replaced by $(1\,-\,cos^2(v))^n$ and the left over $sin(v)$ is put in front. Also the least common multiple of $\{1,\,3,\,...,\,2\,n+1\} \;=$A025547$(n+1) \;=:\; L(n)$ is extracted to avoid a noninteger triangle The result is

$$\hat{q}(\tau(v,\,q),\,\sqrt{\alpha}(q)) \;=:\; \hat{q}(v,\,q) \;=\; sin(v)\sum_{n=0}^{\infty} S2(n,\,v)\,\frac{q^n}{L(n)}\;, \tag{54}$$

where $S2(n,\,v)$ are integer polynomials of of $(2\,cos(v))^2$ of degree n.

$$S2(n,\,v) \;=\; \sum_{m=0}^{n} TS2(n,\,m)\,(\cos(v))^{2\,m}\,, \tag{55}$$

with the number triangle $TS2(n,\,m) \;=$A275790$(n,\,m)$, starting with $[1]$, $[8,\,1]$, $[-32,\,11,\,3]$, $[-736,\,-92,\,9,\,15]$, $[2816,\,-593,\,-249,\,-65\,,35]$, ...

As an example take the contribution to q^4 to $\hat{q}(v,\,q)$: $sin(v)\,\frac{1}{315}\,(2816 - 593\,(2\,cos(v))^2 - 249\,(2\,cos(v))^4 - 65\,(2\,cos(v))^6 \,+\, 35\,(2\,cos(v))^8$. A check for $\tilde{q}\left(\tau = 1,\,\sqrt{\alpha} = \sqrt{\frac{1}{2}}\right) \approx 0.8539594802$ (Maple, 10 digits) with the expansion up to and including q^4, with the corresponding values $v \approx 0.8472130848$ and $q \approx 0.04321391815$ yields 0.8539578317. Going up to and including q^6 gives 0.8539594865. One can plot $\hat{q}(v,\,q)$ for various q values and $v \,\in\, [0,\,2\,\pi]$ up to certain powers of q. For powers up to and including q^5 one finds no difference in the figures for $q \;=\; .043,\,.2,\,.5$ with those given in *Figure* 2 b) from the approximation used in part **A)**.

The $cn(\tau,\,k)/\cos(v)$ sum representation in terms of even powers of $2\,\cos(v) \;=\; x$ with the help of *Chebyshev* T polynomials is

$$\frac{cn(\tau,\,k)}{\cos(v)} \;=\; \frac{\left(\sum_{n=0}^{\infty} q^{n\,(n+1)}\,T(2\,n+1,\,\cos(v))/\cos(v)\right)\Big/\sum_{n=0}^{\infty} q^{n\,(n+1)}}{\left(1\,+\,2\,\sum_{n=1}^{\infty}(-1)^n\,q^{n^2}\,T(2\,n,\,\cos(v))\right)\Big/\left(1\,+\,2\,\sum_{n=1}^{\infty}(-1)^n\,q^{n^2}\right)}\,. \tag{56}$$

One could use *Chebyshev* identities to see the even powers of $2\,cos(v)$ directly (see A084930)

$$\frac{T(2\,n+1,\,x)}{x} \;=\; (-1)^n\,S(2\,n,\,2\,i\,\sqrt{x^2\,-\,1}) \;=\; S(n,\,2\,(2\,x^2\,-\,1)) \,-\, S(n-1,\,2\,(2\,x^2\,-\,1)). \tag{57}$$

In the product version the even powers of $(2\,\cos(v))$ are obvious.

$$\frac{cn(\tau,\,k)}{\cos(v)} \;=\; \frac{\prod_{n=1}^{\infty}\left((1\,-\,q^{2\,n-1})^2\,((1\,-\,q^{2\,n})^2\,+\,q^{2\,n}\,(2\,\cos(v))^2\right)}{\prod_{n=1}^{\infty}\left((1\,+\,q^{2\,n})^2\,((1\,+\,q^{2\,n-1})^2\,-\,q^{2\,n-1}\,(2\,\cos(v))^2\right)}\,. \tag{58}$$

Both versions lead to (again due to the identity theorem for power series)

$$cn(\tau,\,k)\;=\;\cos(v)\,\sum_{n=0}^{\infty}q^n\,\sum_{m=0}^{n}TC(n,\,m)\;(2\,\cos(v))^{2\,m}\;, \tag{59}$$

with the number triangle $TC(n,\,m)\;=$ A275791$(n,\,m)$, which begins [1], [−4, 1], [4, −5, 1], [0, 12, −7, 1], [4, −21, 25, −9, 1], ...

From eq. (21) one has for the other phase space coordinate in the new variables v and q the expansion

$$\hat{p}(v,\,q)\;=\;\cos(v)\,\sum_{n=0}^{\infty}q^n\,\sum_{m=0}^{n}TC(n,\,m)\;(2\,\cos(v))^{2\,m}\;. \tag{60}$$

For example the contribution to $\hat{p}(v,\,q)$ from the term q^4 in this expansion is $\cos(v)\,(4-21\,(2\,\cos(v))^2+25\,(2\,\cos(v))^4-9\,(2\,\cos(v))^6+(2\,\cos(v))^8)$. If one takes all terms up to q^6 inclusive one obtains for $\hat{p}(v,\,q)$ for $v\approx 0.8472130848$ and $q\approx 0.04321391815$ corresponding to $\tau\;=\;1$ and $k\;=\;\sqrt{\frac{1}{2}}$ the value 0.5959765640. This should be compared with $cn\left(1,\,\sqrt{\frac{1}{2}}\right)\approx 0.5959765676$ (Maple 10 digits).

One can compare the expansions of parts **A)** and **B)** by plotting $\hat{p}(v,\,q)$ like in *Figure* 3*b*) for $q\;=\;.043,\,.2$ and .5, expanding up to the power q^6 inclusive. One will find coinciding plots.

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Table: $\tilde{q}(\tau, \sqrt{\alpha})$ expansion in q^n, $n \geq 0$ and v.

n	expansion in $\sin^m(k\,v)$ written as k^m
0	1^1
1	$1^1 + (8/3)\cdot 1^3 + 3^1$, 3 terms
2	$-1^1 + 8\cdot 1^2\,3^1 - (40/3)\cdot 1^3 + (96/5)\cdot 1^5 + 5^1$, 5 terms
3	$-1^1 + (160/3)\cdot 1^3 - 48\cdot 1^2\,3^1 + 8\cdot 1^2\,5^1 + 8\cdot 1^1\,3^2 - (1056/5)\cdot 1^5 + 96\cdot 1^4\,3^1 + (1280/7)\cdot 1^7$ $-2\cdot 3^1 + 7^1$, $10 terms$
4	$2\cdot 1^1 - (544/3)\cdot 1^3 + 200\cdot 1^2\,3^1 - 48\cdot 1^2\,5^1 + 8\cdot 1^2\,7^1 - 56\cdot 1^1\,3^2 + 16\cdot 1^1\,3^1\,5^1 + 192\cdot 1^3\,3^2$ $-1152\cdot 1^4\,3^1 + 96\cdot 1^4\,5^1 + (7392/5)\cdot 1^5 + 1280\cdot 1^6\,3^1 - (21760/7)\cdot 1^7 + (17920/9)\cdot 1^9 + 3^1$ $+(8/3)\cdot 3^3 - 2\cdot 5^1 + 9^1 + 7680\cdot 1^5\,3^1\,5^1 + 6400\cdot 1^4\,3^3 + 192\cdot 1^3\,5^2 + (5358080/9)\cdot 1^9$ $+71680\cdot 1^7\,3^2 - 430080\cdot 1^8\,3^1 + 17920\cdot 1^8\,5^1$, 25 terms
5	$2\cdot 1^1 + (1648/3)\cdot 1^3 - 696\cdot 1^2\,3^1 + 200\cdot 1^2\,5^1 - 48\cdot 1^2\,7^1 + 8\cdot 1^2\,9^1 + 248\cdot 1^1\,3^2 - 112\cdot 1^1\,3^1\,5^1$ $+16\cdot 1^1\,3^1\,7^1 + 8\cdot 1^1\,5^2 - (40608/5)\cdot 1^5 + 8448\cdot 1^4\cdot 3^1 - 1152\cdot 1^4\,5^1 + 96\cdot 1^4\,7^1 - 384\cdot 1^3\,3^1\,5^1$ $+192\cdot 1^2\,3^3 - 2496\cdot 1^3\,3^2 + (217600/7)\cdot 1^7 + 3840\cdot 1^5\,3^2 - 23040\cdot 1^6\,3^1 + 1280\cdot 1^6\,5^1$ $-(412160/9)\cdot 1^9 + (258048/11)\cdot 1^{11} + 3\cdot 3^1 - (64/3)\cdot 3^3 + 8\cdot 3^2\,5^1 + 17920\cdot 3^1\,1^8$ $-2\cdot 7^1 + 11^1$, 27 terms
6	$-4\cdot 1^1 - (4592/3)\cdot 1^3 + 2144\cdot 1^2\,3^1 - 704\cdot 1^2\,5^1 + 200\cdot 1^2\,7^1 - 48\cdot 1^2\,9^1 + 8\cdot 1^2\,11^1 - 888\cdot 1^1\,3^2$ $-56\cdot 1^1\,5^2 + 496\cdot 1^1\,3^1\,5^1 - 112\cdot 1^1\,3^1\,7^1 + 16\cdot 1^1\,3^1\,9^1 + 16\cdot 1^1\,5^1\,7^1 + (189984/5)\cdot 1^5 - 47904\cdot 1^4\,3^1$ $+8448\cdot 1^4\,5^1 - 1152\cdot 1^4\,7^1 + 96\cdot 1^4\,9^1 + 19200\cdot 1^3\,3^2 - 4992\cdot 1^3\,3^1\,5^1 + 384\cdot 1^3\,3^1\,7^1 - 2688\cdot 1^2\,3^3$ $576\cdot 1^2\,3^2\,5^1 + 96\cdot 1^1\,3^4 - (1640960/7)\cdot 1^7 + 239360\cdot 1^6\,3^1 - 23040\cdot 1^6\,5^1 + 1280\cdot 1^6\,7^1 - 72960\cdot 1^5\,3^2$ $+7680\cdot 1^5\,3^1\,5^1 + 6400\cdot 1^4\,3^3 + 192\cdot 1^3\,5^2 + (5358080/9)\cdot 1^9 + 71680\cdot 1^7\,3^2 - 430080\cdot 1^8\,3^1$ $+17920\cdot 1^8\,5^1 - (7483392/11)\cdot 1^1 1 + 258048\cdot 1^1 0\,3^1 + (3784704/13)\cdot 1^1 3 - 2\cdot 3^1 - 64\cdot 3^2\,5^1 + 8\cdot 3^2\,7^1$ $+(304/3)\cdot 3^3 + 8\cdot 3^1\,5^2 + 3\cdot 5^1 - 2\cdot 9^1 + 13^1$, 47 terms
$\vdots$	

Example: n = 2, coefficient of q^2: $-\sin(v) + 8\,\sin(v)^2\,\sin(3\,v) - \dfrac{40}{3}\,\sin(v)^3 + \dfrac{96}{5}\,\sin(v)^5 + \sin(5\,v)$.