Proof of a conjecture regarding A268251

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Theorem 1 *Let c_n for $n \geq 0$ be the nonnegative integers m, in increasing order, such that $\lfloor m/2 \rfloor \lfloor m/3 \rfloor$ is a square. Then the generating function for this sequence is*

$$g(x) = \sum_{n=0}^{\infty} c_n x^n = \frac{x(1 + 2x - 96x^2 - 148x^3 + 45x^4 + 50x^5 + 2x^6)}{1 - 99x^2 + 99x^4 - x^6}$$

and the exponential generating function is

$$e(x) = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!} = -50 - 2x + \frac{3}{4}e^x + \frac{1}{4}e^{-x} + \left(\frac{147}{8} - \frac{15}{2}\sqrt{6}\right) e^{(5+2\sqrt{6})x}$$
$$+ \left(\frac{147}{8} + \frac{15}{2}\sqrt{6}\right) e^{(5-2\sqrt{6})x} + \left(\frac{49}{8} + \frac{5}{2}\sqrt{6}\right) e^{(-5+2\sqrt{6})x} + \left(\frac{49}{8} - \frac{5}{2}\sqrt{6}\right) e^{(-5-2\sqrt{6})x}$$

Proof Let S be the set of nonnegative integers m such that $\lfloor m/2 \rfloor \lfloor m/3 \rfloor$ is a square. Write $m = 5j + i$ with $0 \leq i \leq 5$, $j \geq 0$.

If $i = 0$ or 1, $\lfloor m/2 \rfloor \lfloor m/3 \rfloor = 6j^2$ is a square if and only if $j = 0$. This gives the terms $c_0 = 0$ and $c_1 = 1$.

If $i = 2$, $\lfloor m/2 \rfloor \lfloor m/3 \rfloor = (3j+1)(2j) = 6j^2 + 2j = y^2$ is equivalent to $(6j+1)^2 = 6y^2 + 1$. Thus $m = 6j+2$ is included in S if and only if $(X = m-1, Y = y)$ is a nonnegative solution of the Pell equation

$$X^2 - 6Y^2 = 1 \tag{1}$$

such that $X \equiv 1 \bmod 6$.

If $i = 3$, $\lfloor m/2 \rfloor \lfloor m/3 \rfloor = (3j+1)(2j+1) = 6j^2 + 5j + 1 = y^2$ is equivalent to $(12j+5)^2 = 24y^2 + 1$. Thus $m = 6j + 3$ is included in S if and only if $(X = 12j + 5 = 2m - 1, Y = 2y)$ is a nonnegative solution of (1) with Y even and $X \equiv 5 \bmod 12$.

If $i = 4$ or 5, $\lfloor m/2 \rfloor \lfloor m/3 \rfloor = (3j+2)(2j+1) = 6j^2 + 7j + 2$ is equivalent to $(12j+7)^2 = 24y^2 + 1$. This will correspond to solutions of (1) with Y even and $X \equiv 7 \bmod 12$ (which, as we shall see, do not exist).

Let $M = \begin{pmatrix} 5 & 2 \\ 12 & 5 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 0 \\ 0 & -6 \end{pmatrix}$. The matrix M leaves invariant the quadratic form corresponding to matrix Q, i.e. $MQM^T = Q$. The nonnegative solutions of (1) are given by $(X_k, Y_k) = (1, 0)M^k$ for nonnegative integers k, and $X_{k+1} > X_k$.

Now it is easy to show by induction that Y_k is always even, while $X_k \equiv 1 \bmod 12$ for even k and $X_k \equiv 5 \bmod 12$ for odd k. The even $k \geq 2$ produce the cases with $i = 2$, the odd k produce the cases with $i = 3$, and we never have $i = 4$ or $i = 5$. Thus for $j \geq 0$

$$c_{2j+2} = X_{2j} + 1 = (M^{2j})_{1,1} + 1$$
$$c_{2j+3} = \frac{X_{2j+1} + 1}{2} = \frac{(M^{2j+1})_{1,1} + 1}{2}$$

Now the generating function

$$\begin{aligned} g(x) &= x + \sum_{j=0}^{\infty} (M^{2j})_{1,1} + 1) x^{2j+2} + \sum_{j=0}^{\infty} \frac{M_{1,1}^{2j+1} + 1}{2} x^{2j+3} \\ &= x + x^2((I - x^2 M^2)^{-1})_{1,1} + \frac{x^2}{1-x^2} + \frac{x^3}{2}(M(1 - x^2 M^2)^{-1})_{1,1} + \frac{x^3}{2(1-x^2)} \end{aligned}$$

and using

$$(I - x^2 M^2)^{-1} = \frac{1}{1 - 98x^2 + x^4} \begin{pmatrix} 1 - 49x^2 & 20x^2 \\ 120x^2 & 1 - 49x^2 \end{pmatrix}$$

we find, after some simplification, that

$$g(x) = \frac{x(1 + 2x - 96x^2 - 148x^3 + 45x^4 + 50x^5 + 2x^6)}{1 - 99x^2 + 99x^4 - x^6}$$

Similarly, we calculate the exponential generating function using an explicit form for $\exp(tM)$.