Random Triangles V

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We defined the d-dimensional Poisson(λ) point process in an earlier essay [1] and exhibited moment formulas for various cellular parameters of the corresponding Voronoi tessellation. Many of these formulas are analytically intractible; numerical integration is sometimes necessary. For example, when $d = 2$, the probability that a typical cell (a convex polygon) is a triangle is 0.01124001.... Monte Carlo simulation often provides the only window for study. For example, when $d = 2$ or $d = 3$, the value of the density function f_V for the cellular volume V tends to 0 for small arguments [2, 3, 4], although no workable expression for either density is known.

Any $d+1$ particles from the point process define almost surely an open ball which contains the $d+1$ particles on its boundary. If no other particles from the process are contained in the ball, then let C denote the convex polyhedron with vertices at the $d+1$ particles. The collection of all such cells C constitute almost surely a subdivision of $\mathbb{R}^d$, called the **Poisson-Delaunay tessellation**. This can be regarded as dual to what we discussed earlier. Formulas here are more accessible than in [1]. When $d = 2$ and $d = 3$, the cells are almost surely triangles and tetrahedra, respectively [5, 6]. The value of the density f_V tends to 0 for small arguments in both cases [7, 8]. Since our interest is in random triangles, we will focus on the scenario $d = 2$.

The shortest distance between a line L and the origin is the length $|r|$ of the perpendicular segment from $(0, 0)$ to L. If the intersection point is $(r\cos(\theta), r\sin(\theta))$, then clearly the equation for L is

$$x\cos(\theta) + y\sin(\theta) = r.$$

There is a one-to-one correspondence between the set of points

$$Q = \{(r, \theta) : -\infty < r < \infty \text{ and } 0 \le \theta < \pi\}$$

and the set of all lines L. For arbitrary $\lambda > 0$, the Poisson point process of intensity λ in Q induces the **Poisson line process** of intensity λ. The resulting subdivision of $\mathbb{R}^2$ is called the **Goudsmit-Miles tessellation** of the plane. Formulas here are again more accessible than before. The probability that a typical cell (again a convex

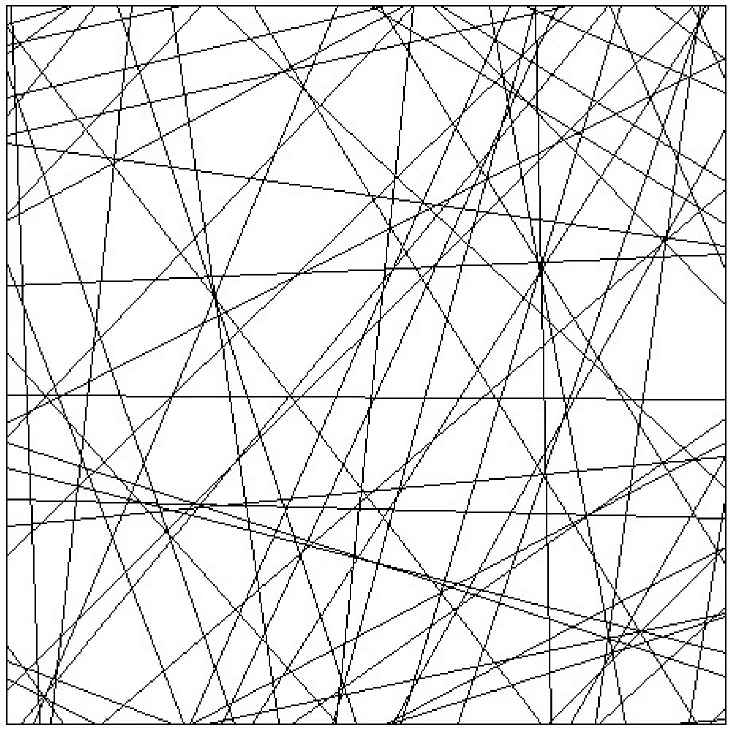

Figure 1: A Poisson line process of intensity 17 in the unit square.

polygon) is a triangle is $2 - \pi^2/6 = 0.3550659331...$ [9, 10, 11]. The value of the density f_V tends to ∞ (not 0) for small arguments; more precisely [12, 13, 14],

$$\begin{aligned}\lim_{x\to 0^+} \sqrt{x} f_V(x) &= \lambda \frac{2\sqrt{2}}{12-\pi^2} \int_0^{\pi} \int_0^{\pi-\varphi} \sqrt{\sin(\varphi)\sin(\psi)\sin(\varphi+\psi)}\, d\psi\, d\varphi \\ &= \frac{6\pi}{12-\pi^2}(0.3231100260...)\lambda.\end{aligned}$$

One of our goals is to explain why this interesting constant arises! It turns out that "small" polygons of the tessellation are almost all triangles, therefore knowledge about triangular areas carries over to limiting polygonal areas. We shall discuss both.

There are effectively no stationary line processes except the one with Poisson structure. This is somewhat exaggerated – mixtures of the λ parameter in Poisson line processes lead to Cox line processes and there are pathological examples with many parallel lines – but essentially it is not worthwhile to consider any other tessellation based on a line process [15, 16].

Characteristics of **Delaunay triangles** and **Miles triangles** will dominate this essay. There is a third type of random triangle – called **Miles intriangles** – that we shall touch upon as well. Let us review: The largest circle inscribed in a given convex polygon is called the incircle. This circle will almost surely be tangent to three sides of the polygon. Let T denote the triangle determined by these three sides, extended as far as required. Given a typical cell C in a planar Goudsmit-Miles tessellation, the intriangle T might be considerably larger than C (the prefix "in" refers to the fact that C and T share an incircle, *not* that T is inscribed in anything). Clearly no other line is allowed to intersect the incircle. However, it is possible that one or more lines might hit the intriangle elsewhere. It can be shown that both the area V of T and the perimeter S of T have infinite expectation [17].

Buried in an appendix to [17], we find these three types of random triangles listed in a table. Clarifying Miles' table is the second goal of this essay. For simplicity, we shall assume $\lambda = 1$ henceforth.

0.1. Delaunay Triangles. Unlike the examples in [18], it is easier to start with angles than with sides. The bivariate density for arbitrary angles α, β in a typical (triangular) cell of a Poisson-Delaunay tessellation is [5, 17, 19]

$$\frac{2}{3\pi}\left[\sin(2x)+\sin(2y)-\sin(2x+2y)\right]=\frac{8}{3\pi}\sin(x)\sin(y)\sin(x+y)$$

where $x>0$, $y>0$, $x+y<\pi$. Integrating out y, we obtain the density $g(x)$ for α:

$$\frac{2}{3\pi}\left[(\pi-x)\sin(2x)-\cos(2x)+1\right]=\frac{4}{3\pi}\left[(\pi-x)\cos(x)+\sin(x)\right]\sin(x)$$

and

$$\mathrm{E}(\alpha)=\frac{\pi}{3}=1.0471975511..., \qquad \mathrm{E}(\alpha^2)=\frac{2\pi^2}{9}-\frac{5}{6}=1.3599120891....$$

Corresponding to the density of $\max\{\alpha,\beta,\gamma\}$, the expression $3g(x)$ holds when $\pi/2<x<\pi$; the expression when $\pi/3<x<\pi/2$ is [20, 21]

$$\frac{2}{\pi}\left[(3x-\pi)\sin(2x)-\cos(2x)+\cos(4x)\right]=\frac{4}{\pi}\left[(3x-\pi)\cos(x)-\sin(3x)\right]\sin(x).$$

It thus follows that

$$\mathrm{P}(\text{a typical Delaunay triangle is acute})=\frac{1}{2}=0.5.$$

As in [22], the cross-correlation coefficient $\rho(\alpha,\beta)=-1/2$, hence

$$\mathrm{E}(\alpha\,\beta)=\frac{\pi^2}{18}+\frac{5}{12}=0.9649780222....$$

The density for an arbitrary side a is [6, 23, 24]

$$\frac{\pi\,x}{3}\left[x\exp\left(-\frac{\pi\,x^2}{4}\right)+\operatorname{erfc}\left(\frac{\sqrt{\pi}x}{2}\right)\right]$$

where $x>0$ and

$$\operatorname{erfc}(w)=\frac{2}{\sqrt{\pi}}\int_w^\infty \exp(-t^2)\,dt=1-\operatorname{erf}(w)$$

is the complementary error function; also

$$\mathrm{E}(a)=\frac{32}{9\pi}=1.1317684842\ldots,\qquad \mathrm{E}(a^2)=\frac{5}{\pi}=1.5915494309....$$

No such simple density formula exists for perimeter $S=a+b+c$. Muche [7] gave

$$\frac{\pi\,x^3}{12}\int_0^{2\pi}\int_0^{2\pi-\varphi}\frac{\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\psi}{2}\right)\sin\left(\frac{\varphi+\psi}{2}\right)}{\left[\sin\left(\frac{\varphi}{2}\right)+\sin\left(\frac{\psi}{2}\right)+\sin\left(\frac{\varphi+\psi}{2}\right)\right]^4}\exp\left\{\frac{-\pi\,x^2}{4\left[\sin\left(\frac{\varphi}{2}\right)+\sin\left(\frac{\psi}{2}\right)+\sin\left(\frac{\varphi+\psi}{2}\right)\right]^2}\right\}d\psi\,d\varphi$$

where $x>0$ and

$$\mathrm{E}(S)=\frac{32}{3\pi}=3.3953054526\ldots,\qquad \mathrm{E}(S^2)=\frac{125}{3\pi}=13.2629119243....$$

In contrast, for area $V=(1/4)\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$, we have a formula due to Rathie [8]:

$$\frac{8\,\pi\,x}{9}K_{1/6}\left(\frac{2\,\pi\,x}{3\sqrt{3}}\right)^2$$

where $x>0$ and

$$K_{1/6}(w)=\sum_{i=0}^{\infty}\frac{\pi}{i!\Gamma\left(\frac{5}{6}+i\right)}\left(\frac{w}{2}\right)^{-\frac{1}{6}+2i}-\sum_{j=0}^{\infty}\frac{\pi}{j!\Gamma(\frac{7}{6}+j)}\left(\frac{w}{2}\right)^{\frac{1}{6}+2j}$$

is the modified Bessel function of the second kind; also

$$\mathrm{E}(V)=\frac{1}{2}=0.5,\qquad \mathrm{E}(V^2)=\frac{35}{8\pi^2}=0.4432801784....$$

0.2. Miles Triangles. Cells of a Goudsmit-Miles tessellation are sampled until we obtain a triangular one. The bivariate density for arbitrary angles α, β in such a typical triangle is [17, 25]

$$\frac{2}{12-\pi^2}\left[\cos(x)+\cos(y)-\cos(x+y)-1\right]=\frac{8}{12-\pi^2}\sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

where $x>0$, $y>0$, $x+y<\pi$. Integrating out y, we obtain the density for α:

$$\frac{2}{12-\pi^2}\left[2\sin(x)-(\pi-x)\left(1-\cos(x)\right)\right]$$

and

$$\mathrm{E}(\alpha)=\frac{\pi}{3}=1.0471975511..., \qquad \mathrm{E}(\alpha^2)=\frac{8}{12-\pi^2}+\frac{\pi^2}{6}-4=1.4001051740....$$

No one seems to have computed the density of $\max\{\alpha,\beta,\gamma\}$ yet. As in [22], $\rho(\alpha,\beta)=-1/2$, hence

$$\mathrm{E}(\alpha\,\beta)=-\frac{4}{12-\pi^2}+\frac{\pi^2}{12}+2=0.9448814798....$$

Miles [17] gave the trivariate density of sides a, b, c:

$$\frac{1}{12-\pi^2}\frac{(x+y+z)(-x+y+z)(x-y+z)(x+y-z)}{x^2y^2z^2}\exp\left(-(x+y+z)\right)$$

if $|x-y|<z<x+y$, and we shall verify this later. The condition $|x-y|<z<x+y$ is equivalent to $|x-z|<y<x+z$ and to $|y-z|<x<y+z$ via the Law of Cosines. As a consequence,

$$\mathrm{E}(a)=\frac{1}{3}=0.3333333333..., \qquad \mathrm{E}(a^2)=-\frac{96}{\pi^2\left(12-\pi^2\right)}+4+\frac{8}{\pi^2}=0.2448288927...,$$

$$\mathrm{E}(a\,b)=\frac{48}{\pi^2\left(12-\pi^2\right)}-\frac{5}{3}-\frac{4}{\pi^2}=0.2109188869....$$

The cross-correlation coefficient

$$\rho(a,b)=\frac{48/(12-\pi^2)-16\pi^2/9-4}{-96/(12-\pi^2)+35\pi^2/9+8}=0.7464061592...$$

is quite large, indicating strong positive dependency. Integrating out z gives the bivariate density of a, b:

$$\frac{\exp\left(-2(x+y)\right)}{(12-\pi^2)x^2y^2}\left\{\left(x^2-y^2\right)^2\exp\left(x+y\right)\left[\mathrm{Ei}\left(-(x+y)\right)-\mathrm{Ei}\left(-(y-x)\right)\right]+\pi\sum_{k=0}^{3}x^kh_k(x,y)\right\}$$

for $0 < x < y$, where

$$h_0(x,y) = (-1+\exp(2x))\left(-y^3+y^2-2y-2\right), \qquad h_2(x,y) = (-1+\exp(2x))(y+1),$$
$$h_1(x,y) = (1+\exp(2x))\left(-y^2+2y+2\right), \qquad h_3(x,y) = 1+\exp(2x)$$

and Ei is the exponential integral

$$\operatorname{Ei}(w) = \int_{-\infty}^{w} \frac{\exp(t)}{t}\,dt, \qquad w < 0.$$

For $0 < y < x$, simply use symmetry. Unlike Delaunay triangles, a closed-form expression for the univariate density of a seems out of reach.

Perimeter S is exponentially distributed, with density

$$\exp(-s), \qquad s > 0$$

and moments $\operatorname{E}(S) = 1$, $\operatorname{E}(S^2) = 2$. A starting point for area V was provided by Miles [25]: if $U = \sqrt{V}$ and

$$\chi = 2\sqrt{\cot\left(\tfrac{\alpha}{2}\right)\cot\left(\tfrac{\beta}{2}\right)\tan\left(\tfrac{\alpha+\beta}{2}\right)} = 2\sqrt{\frac{\sin(\alpha)+\sin(\beta)+\sin(\alpha+\beta)}{\cos(\alpha)+\cos(\beta)-\cos(\alpha+\beta)-1}}$$

then the conditional density for U, given α and β, is

$$\chi\exp(-\chi\,u), \qquad u > 0.$$

Therefore the unconditional density for U is

$$\int_0^{\pi}\int_0^{\pi-\alpha} \chi\exp(-\chi\,u)\,\frac{2}{12-\pi^2}\left[\cos(\alpha)+\cos(\beta)-\cos(\alpha+\beta)-1\right]d\beta\,d\alpha$$

$$= \frac{4\sqrt{2}}{12-\pi^2}\int_0^{\pi}\int_0^{\pi-\alpha} \sqrt{\sin(\alpha)\sin(\beta)\sin(\alpha+\beta)}\exp(-\chi\,u)\,d\beta\,d\alpha$$

and, transforming to V, we obtain

$$\frac{2\sqrt{2}}{12-\pi^2}\frac{1}{\sqrt{v}}\int_0^{\pi}\int_0^{\pi-\alpha} \sqrt{\sin(\alpha)\sin(\beta)\sin(\alpha+\beta)}\exp\left(-\chi\sqrt{v}\right)d\beta\,d\alpha$$

as the area density. Integrating first with respect to v over $(0,\infty)$, it follows that

$$\begin{aligned} \mathrm{E}(V) &= \frac{\sqrt{2}}{12-\pi^2}\int_0^{\pi}\int_0^{\pi-\alpha}\left(\frac{2}{\chi}\right)^3\sqrt{\sin(\alpha)\sin(\beta)\sin(\alpha+\beta)}d\beta\,d\alpha \\ &= \frac{\pi\,(25-36\ln(2))}{12-\pi^2} = 0.0688684716..., \end{aligned}$$

$$\begin{aligned} \mathrm{E}(V^2) &= \frac{3\sqrt{2}}{12-\pi^2}\int_0^{\pi}\int_0^{\pi-\alpha}\left(\frac{2}{\chi}\right)^5\sqrt{\sin(\alpha)\sin(\beta)\sin(\alpha+\beta)}d\beta\,d\alpha \\ &= \frac{3\,(15\pi^2-148)}{2(12-\pi^2)} = 0.0310266433.... \end{aligned}$$

The density formula for V also serves to motivate the constant 0.3231100260... at the beginning of this essay (asymptotics for polygonal cells as $v\to 0^+$).

0.3. A Verification. Well-known formulas give angles α, β in terms of sides a, b, c:

$$\begin{aligned} \sin\left(\frac{\alpha}{2}\right)^2 &= \frac{\left(\frac{S}{2}-b\right)\left(\frac{S}{2}-c\right)}{b\,c} = \frac{(S-2b)(2a+2b-S)}{4b(S-a-b)}, \\ \sin\left(\frac{\beta}{2}\right)^2 &= \frac{\left(\frac{S}{2}-a\right)\left(\frac{S}{2}-c\right)}{a\,c} = \frac{(S-2a)(2a+2b-S)}{4a(S-a-b)} \end{aligned}$$

where $S=a+b+c$. To compute the Jacobian determinant J of $(a,b,S)\to(\alpha,\beta,S)$, we differentiate $\sin(\alpha/2)^2$:

$$\begin{aligned} \sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\frac{\partial\alpha}{\partial a} &= \frac{\partial}{\partial a}\frac{(S-2b)(2a+2b-S)}{4b(S-a-b)}, \\ \sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\frac{\partial\alpha}{\partial b} &= \frac{\partial}{\partial b}\frac{(S-2b)(2a+2b-S)}{4b(S-a-b)}, \\ \sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\frac{\partial\alpha}{\partial S} &= \frac{\partial}{\partial S}\frac{(S-2b)(2a+2b-S)}{4b(S-a-b)} \end{aligned}$$

and likewise differentiate $\sin(\beta/2)^2$. Additional formulas

$$\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right) = \sin\left(\frac{\alpha}{2}\right)\sqrt{\frac{\frac{S}{2}\left(\frac{S}{2}-a\right)}{b\,c}} = \frac{\sqrt{S(S-2a)(S-2b)(2a+2b-S)}}{4b(S-a-b)},$$

$$\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)=\sin\left(\frac{\beta}{2}\right)\sqrt{\frac{\frac{S}{2}\left(\frac{S}{2}-b\right)}{a\,c}}=\frac{\sqrt{S(S-2a)(S-2b)(2a+2b-S)}}{4a(S-a-b)}$$

permit the expression of $\partial\alpha/\partial a$, $\partial\beta/\partial a$, $\ldots$ entirely in terms of a, b, S. We find that

$$J=\frac{S}{a\,b(S-a-b)}=\frac{S}{a\,b\,c}.$$

Let $\gamma=\pi-\alpha-\beta$, then $\sin(\gamma/2)=\cos((\alpha+\beta)/2)$. The conditional density of a and b, given S, is thus

$$\begin{aligned}\frac{8}{12-\pi^2}\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)J &= \frac{8}{12-\pi^2}\frac{\left(\frac{S}{2}-a\right)\left(\frac{S}{2}-b\right)\left(\frac{S}{2}-c\right)}{a\,b\,c}\frac{S}{a\,b\,c}\\ &= \frac{S}{12-\pi^2}\frac{S-2a}{a^2}\frac{S-2b}{b^2}\frac{S-2c}{c^2}\end{aligned}$$

where $\max\{2a,2b,2c\}<S$. Our formula corrects an error in [17], which inexplicably gives $4/S^2$ as the first factor. The inequality $2c<S$ implies $S<2a+2b$. It follows that the unconditional density of a, b, S is

$$\frac{S}{12-\pi^2}\frac{S-2a}{a^2}\frac{S-2b}{b^2}\frac{S-2c}{c^2}\exp\left(-S\right)$$

as was to be shown.

0.4. Miles Intriangles. Three sides (tangential to the incircle) of a typical cell in a Goudsmit-Miles tessellation are extended until they intersect. Not much is known about the triangle so formed; we shall be brief. The bivariate density for arbitrary angles α, β in such a typical triangle is [12, 17]

$$\frac{1}{3\pi}\left[\sin(x)+\sin(y)+\sin(x+y)\right]=\frac{4}{3\pi}\cos\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right)\sin\left(\frac{x+y}{2}\right)$$

where $x>0$, $y>0$, $x+y<\pi$. Integrating out y, we obtain the density for α:

$$\frac{1}{3\pi}\left[(\pi-x)\sin(x)+2\left(1+\cos(x)\right)\right]$$

and $\mathrm{E}(\alpha)=\pi/3$. A density for side a is not known, but $\mathrm{E}(a)=\infty$ as implied earlier. We mention two results:

$$\begin{aligned}\mathrm{P}(\text{a typical intriangle is hit by just one line}) &= 2-\pi^2\left(\tfrac{17}{2}-12\ln(2)\right)\\ &= 0.2014241570...,\end{aligned}$$

$$\begin{aligned}\mathrm{P}(\text{a typical intriangle is hit by at least two lines}) &= -3+\pi^2\left(\tfrac{26}{3}-12\ln(2)\right)\\ &= 0.4435099098....\end{aligned}$$

0.5. Goudsmit-Miles Cells. We move away from triangles and talk about convex polygons with N vertices. Goudsmit [26] proved that a typical such cell, determined via a Poisson line process of intensity $\lambda = 1$, has the following mean values:

$$\mathrm{E}(N) = 4, \qquad \mathrm{E}(S) = 2, \qquad \mathrm{E}(V) = \frac{1}{\pi} = 0.3183098861....$$

Miles [12, 13] proved that

$$\mathrm{P}(\text{a typical cell has } N = 3) = 2 - \frac{\pi^2}{6} = 0.3550659331...,$$

$$\mathrm{E}(N^2) = \frac{\pi^2}{2} + 12 = 16.9348022005...,$$

$$\mathrm{E}(S^2) = \frac{\pi^2}{2} + 2 = \frac{68.4437543191...}{\pi^2} = 6.9348022005...,$$

$$\mathrm{E}(V^2) = \frac{1}{2} = \frac{48.7045455170...}{\pi^4} = 0.5$$

and announced that D. G. Kendall (unpublished) had obtained

$$\mathrm{E}(V^3) = \frac{4\pi}{7} = \frac{1725.8818444438...}{\pi^6} = 1.7951958020....$$

Cross-moments between N, S, V were also given.

Let us focus on N-results first. Tanner [27, 28] computed that

$$\mathrm{P}(\text{a typical cell has } N = 4) = -\frac{1}{3} - \frac{7\pi^2}{36} + \pi^2 \ln(2) - \frac{7}{2}\zeta(3) = 0.3814662248...$$

where $\zeta(3)$ is Apéry's constant [29],

$$\begin{aligned}\mathrm{E}(N^3) &= \frac{232}{7} + \frac{39\pi^2}{14} + \frac{\pi^4}{21} + \frac{12\pi^2}{7}\ln(2) - 6\zeta(3) - \frac{192}{7}\int_0^{\pi/2} x^2 \tan(x)\ln(\sin(x))dx \\ &= 76.0364049460...,\end{aligned}$$

and $\mathrm{E}(N^4) = 362.08446....$ The fourth moment can be expressed as an elaborate quadruple integral and deserves more attention. Simulation [11, 14, 30, 31] suggests that $\mathrm{P}(N = 5) \approx 0.196$ and $\mathrm{P}(N = 6) \approx 0.062$. The function $\mathrm{P}(N = k)$ is apparently maximized when $k = 4$ and falls off for $k \geq 5$; it is known that asymptotically [32]

$$\mathrm{P}(N = k) \sim \frac{8}{3k} \frac{2\left(4\pi^2\right)^{k-1}}{(2k)!}$$

as $k \to \infty$.

Results for perimeter S include [28]

$$\mathrm{E}(S^3) = \mathrm{E}(N^3) - \frac{3\pi^2}{2} - 28 = \frac{1030.4005353057...}{\pi^3} = 33.2319983444...,$$

$$\mathrm{E}(S^4) = \mathrm{E}(N^4) - 2\,\mathrm{E}(N^3) - \frac{\pi^2}{2} - 4 = \frac{19586.7132...}{\pi^4} = 201.07685...$$

but nothing comparable is known for the fourth moment of area V. Simulation [11, 14, 31] suggests that $\mathrm{E}(V^4) \approx 11.4$. No formulas for the density of either S or V are known.

Our study has been devoted to "typical" cells C; an alternative is the Crofton cell C_0, which is the unique polygon containing the origin. The Crofton cell is *not* typical for Goudsmit-Miles (unlike Poisson-Voronoi, for which typicality *does* hold [33, 34]). Even less is known here. Matheron [35, 36] proved that

$$\mathrm{E}(N_0) = \mathrm{E}(S_0) = \frac{\pi^2}{2} = 4.9348022005...$$

and Miles [17] proved that

$$\mathrm{P}(\text{the Crofton cell has } N_0 = 3) = \frac{\pi^2\,(25 - 36\ln(2))}{6} = 0.0768208880....$$

The following moments

$$\mathrm{E}(V_0) = \frac{\pi}{2} = 1.5707963267..., \qquad \mathrm{E}(V_0^2) = \frac{4\pi^2}{7} = 5.6397739434...$$

are listed in [31], but a reference cannot be found. Simulation [30, 31] suggests that $\mathrm{P}(N_0 = 4) \approx 0.297$, $\mathrm{P}(N_0 = 5) \approx 0.341$ and $\mathrm{P}(N_0 = 6) \approx 0.196$. The function $\mathrm{P}(N_0 = k)$ is apparently maximized when $k = 5$ and falls off for $k \geq 6$; it is known that asymptotically [32]

$$\mathrm{P}(N_0 = k) \sim \frac{2k}{3}\frac{2\,(4\pi^2)^{k-1}}{(2k)!}$$

as $k \to \infty$. The distribution of N_0 has a thicker tail (greater weight for large k) than N does. Simulation [31] further suggests that $\mathrm{E}(N_0^2) \approx 25.72$, $\mathrm{E}(S_0^2) \approx 30.51$, $\mathrm{E}(V_0^3) \approx 36.03$ and $\mathrm{E}(V_0^4) \approx 357.8$. Again, no density formulas are known.

Hilhorst & Calka [32] wrote about cloud chamber experiments in physics motivating the work of Goudsmit [26]: The problem was "to calculate the probability for three independent lines to nearly pass through the same point, or, put differently, for a typical triangular cell to have an area less than ε in the limit of very small

ε". It would seem that the solution came almost twenty years later [12, 13], with asymptotics

$$\int_0^\varepsilon f_V(x)dx \sim \frac{12\pi}{12-\pi^2}(0.3231100260...)\lambda\sqrt{\varepsilon}$$

valid as $\varepsilon \to 0^+$.

0.6. Addendum. Let Ω be a planar convex region with area V and perimeter S. A Poisson point process yields K points in Ω satisfying $\mathrm{E}(K) = \mathrm{Var}(K) = \lambda V$ whereas a Poisson line process yields L lines hitting Ω satisfying $\mathrm{E}(L) = \mathrm{Var}(L) = \lambda S$. The total length M of the line segments crossing Ω is a sum of L independent identically distributed chord lengths and hence is approximately normally distributed with $\mathrm{E}(M) = \pi\lambda V$ and

$$\mathrm{Var}(M) = \begin{cases} \frac{2}{3}\pi\lambda & \text{if } \Omega \text{ is a disk of unit diameter,} \\ \frac{4}{3}\left(1-\sqrt{2}+3\operatorname{arcsinh}(1)\right)\lambda & \text{if } \Omega \text{ is a square of unit side,} \\ \frac{3}{4}\ln(3)\lambda & \text{if } \Omega \text{ is an equilateral triangle of unit side} \end{cases}$$

for suitably large λ. Studies on such chord lengths for regular polygons and ellipses include [37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48].

The number I of intersection points (between the $\binom{L}{2}$ pairs of lines) in Ω satisfies $\mathrm{E}(I) = \pi\,\lambda^2 V$ in general and

$$\mathrm{E}(I) = \frac{1}{4}\pi^2\lambda^2, \qquad \mathrm{Var}(I) = \frac{1}{4}\pi^2\lambda^2 + \frac{8}{3}\pi\,\lambda^3$$

in the special case when Ω is a disk of unit diameter. How is the general formula proved? Under the condition that $L = \ell$ is fixed, we have [9, 49]

$$\mathrm{E}(I \mid L = \ell) = \ell(\ell-1)\pi\frac{V}{S^2}$$

thus, allowing L to vary,

$$\begin{aligned} \mathrm{E}(I) &= \left[\mathrm{E}(L^2) - \mathrm{E}(L)\right]\pi\frac{V}{S^2} \\ &= \left[\mathrm{Var}(L) + \mathrm{E}(L)^2 - \mathrm{E}(L)\right]\pi\frac{V}{S^2} \\ &= \pi\,\mathrm{E}(L)^2\frac{V}{S^2} = \pi\,\lambda^2 S^2\frac{V}{S^2} = \pi\,\lambda^2 V. \end{aligned}$$

Finding the variance expression for the disk is more complicated; it is possible to do likewise for the square and equilateral triangle.

0.7. Acknowledgement. I am grateful to Richard Cowan for a helpful discussion about the Poisson line process and Miles' fundamental work on the resulting tessellations. Thanks are also due to Adrian Baddeley and Rolf Turner for writing an R package *spatstat* [50], which enables testing of numerical predictions in this essay via simulation [51].

References

[1] S. R. Finch, Poisson-Voronoi tessellations, unpublished note (2005).

[2] A. L. Hinde and R. E. Miles, Monte-Carlo estimates of the distributions of the random polygons of the Voronoi tessellation with respect to a Poisson process, *J. Statist. Comput. Simul.* 10 (1980) 205–223.

[3] M. Tanemura, Random Voronoi cells of higher dimensions, http://www.mi.sanu.ac.rs/vismath/proceedings/tanemura.htm.

[4] E. Pineda , P. Bruna and D. Crespo, Cell size distribution in random tessellations of space, *Phys. Rev. E* 70 (2004) 066119.

[5] A. Okabe, B. Boots, K. Sugihara and S. N. Chiu, *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams*, 2^{nd} ed., Wiley, 2000, pp. 389–400; MR1210959 (94a:52033) and MR1770006 (2001c:52013).

[6] D. Stoyan, W. S. Kendall and J. Mecke, *Stochastic Geometry and its Applications*, 2^{nd} ed., Wiley, 1995, pp. 248–255, 307–315, 324–333; MR0895588 (88j:60034a).

[7] L. Muche, Distributional properties of the three-dimensional Poisson Delaunay cell, *J. Stat. Phys.* 84 (1996) 147–167; MR1401254 (97j:60027).

[8] P. N. Rathie, On the volume distribution of the typical Poisson-Delaunay cell, *J. Appl. Probab.* 29 (1992) 740–744; MR1174447 (93f:60016).

[9] L. A. Santaló, *Integral Geometry and Geometric Probability*, Addison-Wesley, 1976, pp. 27–30, 45–58; MR0433364 (55 #6340).

[10] H. Solomon, *Geometric Probability*, SIAM, 1978, pp. 39–60; MR0488215 (58 #7777).

[11] E. I. George, Sampling random polygons, *J. Appl. Probab.* 24 (1987) 557–573; MR0900299 (88g:60030).

[12] R. E. Miles, Random polygons determined by random lines in a plane, *Proc. Nat. Acad. Sci. U.S.A.* 52 (1964) 901–907; MR0168000 (29 #5265).

[13] R. E. Miles, Random polygons determined by random lines in a plane. II, *Proc. Nat. Acad. Sci. U.S.A.* 52 (1964) 1157–1160; MR0169258 (29 #6510).

[14] I. K. Crain and R. E. Miles, Monte Carlo estimates of the distributions of the random polygons determined by random lines in the plane, *J. Statist. Comput. Simul.* 4 (1976) 293–325.

[15] L. Breiman, The Poisson tendency in traffic distribution, *Annals of Math. Statist.* 34 (1963) 308–311; MR0144785 (26 #2326).

[16] T. Thedéen, A note on the Poisson tendency in traffic distribution, *Annals of Math. Statist.* 35 (1964) 1823–1824; MR0169688 (29 #6933).

[17] R. E. Miles, The various aggregates of random polygons determined by random lines in a plane, *Adv. Math.* 10 (1973) 256–290; MR0319232 (47 #7777).

[18] S. R. Finch, Random triangles III, unpublished note (2010).

[19] R. E. Miles, On the homogeneous planar Poisson point process, *Math. Biosci.* 6 (1970) 85–127; MR0279853 (43 #5574).

[20] K. V. Mardia, R. Edwards and M. L. Puri, Analysis of central place theory, Proc. 41st Session of the International Statistical Institute, New Delhi, 1977, v. 2, *Bull. Inst. Internat. Statist.* 47 (1977) 93–110, 138–147; MR0617573 (82i:52014).

[21] B. N. Boots, Using angular properties of Delaunay triangles to evaluate point patterns, *Geographical Analysis* 18 (1986) 252–259.

[22] S. R. Finch, Random triangles, unpublished note (2010).

[23] R. Collins, A geometrical sum rule for two-dimensional fluid correlation functions, *J. Phys. C* 1 (1968) 1461–1471.

[24] R. Sibson, The Dirichlet tessellation as an aid in data analysis, *Scand. J. Statist.* 7 (1980) 14–20; MR0565960 (81b:62007).

[25] R. E. Miles, A heuristic proof of a long-standing conjecture of D. G. Kendall concerning the shapes of certain large random polygons, *Adv. Appl. Probab.* 27 (1995) 397–417; MR1334821 (96h:60017).

[26] S. Goudsmit, Random distribution of lines in a plane, *Rev. Mod. Phys.* 17 (1945) 321–322.

[27] J. C. Tanner, The proportion of quadrilaterals formed by random lines in a plane, *J. Appl. Probab.* 20 (1983) 400–404; MR0698543 (84e:60023).

[28] J. C. Tanner, Polygons formed by random lines in a plane: some further results, *J. Appl. Probab.* 20 (1983) 778–787; MR0720469 (84k:60020).

[29] S. R. Finch, Apéry's constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 40–53.

[30] P. Calka, Precise formulae for the distributions of the principal geometric characteristics of the typical cells of a two-dimensional Poisson-Voronoi tessellation and a Poisson line process, *Adv. Appl. Probab.* 35 (2003) 551–562; MR1990603 (2004k:60021).

[31] J. Michel and K. Paroux, Empirical polygon simulation and central limit theorems for the homogeneous Poisson line process, *Methodol. Comput. Appl. Probab.* 9 (2007) 541–556; MR2404742 (2009c:60026).

[32] H. J. Hilhorst and P. Calka, Random line tessellations of the plane: statistical properties of many-sided cells, *J. Stat. Phys.* 132 (2008) 627–647; MR2429698 (2009k:60029).

[33] J. Møller, *Lectures on Random Voronoi Tessellations*, Lect. Notes in Statist. 87, Springer-Verlag, 1994, p. 112; MR1295245 (96b:60028).

[34] J. Mecke, On the relationship between the 0-cell and the typical cell of a stationary random tessellation, *Pattern Recognition* 32 (1999) 1645–1648.

[35] G. Matheron, Ensembles fermés aléatoires, ensembles semi-markoviens et polyèdres poissoniens, *Adv. Appl. Probab.* 4 (1972) 508–541; MR0348819 (50 #1314).

[36] G. Matheron, *Random Sets and Integral Geometry*, Wiley, 1975 pp. 177–185; MR0385969 (52 #6828).

[37] R. Sulanke, Die Verteilung der Sehnenlangen an ebenen und raumlichen Figuren, *Math. Nachr.* 23 (1961) 51–74; MR0150720 (27 #707).

[38] R. Coleman, Random paths through convex bodies, *J. Appl. Probab.* 6 (1969) 430–441; MR0254890 (40 #8097).

[39] A. M. Kellerer, Considerations on the random traversal of convex bodies and solutions for general cylinders, *Radiation Research* 47 (1971) 359–376.

[40] F. Piefke, The chord length distribution of the ellipse, *Litovsk. Mat. Sb.* v. 19 (1979) n. 3, 45–54; Engl. transl. *Lithuanian Math. J.* 19 (1979) 325–332; MR0551926 (80m:60019).

[41] P. W. Kuchel and R. J. Vaughan, Average lengths of chords in a square, *Math. Mag.* 54 (1981) 261–269; MR0644078 (83f:60026).

[42] K. Voss, Powers of chords for convex sets, *Biometrical J.* 24 (1982) 513–516; MR690489 (84d:52006).

[43] W. Gille, Chord length distribution density of an infinitely long circular hollow cylinder, *Math. Comput. Modelling* 34 (2001) 423–431; MR1835837.

[44] N. G. Agaronyan and V. K. Oganyan, Chord length distribution functions for polygons, *Izv. Nats. Akad. Nauk Armenii Mat.* v. 40 (2005) n. 4, 42–57; Engl. transl. *J. Contemp. Math. Anal.* v. 40 (2005) n. 4, 43–56; MR2310110 (2007k:52015).

[45] E. Geciauskas, Recurrent approach to Blaschke's problem, *Geom. Dedicata* 121 (2006) 9–18; MR2276232 (2007k:52017).

[46] L. Heinrich, On lower bounds of second-order chord power integrals of convex discs, *Rend. Circ. Mat. Palermo, Suppl.* 81 (2009) 213–222; MR2809426 (2012d:60024).

[47] H. S. Harutyunyan and V. K. Ohanyan, The chord length distribution function for regular polygons, *Adv. Appl. Probab.* 41 (2009) 358–366; MR2541181 (2011b:60039).

[48] U. Bäsel, Random chords and point distances in regular polygons, *Acta Math. Univ. Comenian.* 83 (2014) 1–18; MR3178155.

[49] L. A. Santaló, Random lines and tessellations in a plane, *Stochastica* 4 (1980) 3–13; MR573721 (81e:60012).

[50] A. Baddeley, E. Rubak and R. Turner, *Spatial Point Patterns: Methodology and Applications with R*, Chapman and Hall/CRC Press, 2015; http://spatstat.github.io/.

[51] S. R. Finch, Simulations in R involving triangles and tetrahedra, http://www.people.fas.harvard.edu/~sfinch/csolve/rsimul.html.