Notes on generalized Riordan arrays

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A Riordan array is a triangular array with column generating functions of the form $f(x)F(x)^k$, $k = 0, 1, 2, ...$, where $f(x)$ and $F(x)$ are formal power series subject to certain restrictions. If we drop these restrictions and allow $f(x)$ and $F(x)$ to be arbitrary power series we arrive at the concept of a generalized Riordan array as defined in [4]. We show that some of the algebraic structure of Riordan arrays carries over to generalized Riordan arrays and give several examples involving arrays in the OEIS.

1. Riordan Arrays.

An infinite array is a Riordan array if the ordinary generating function of the k-th column of the array is

$$f(x)F(x)^k$$

for $k = 0, 1, 2, 3, ...$, where $f(x)$ and $F(x)$ are formal power series in the indeterminate x of the form

$$f(x) = 1 + a_1x + a_2x^2 + \cdots \tag{1}$$
$$F(x) = x^r + b_1x^{r+1} + b_2x^{r+2} + \cdots, \quad r \geq 1. \tag{2}$$

The coefficients a_i and b_i could be real or complex but for examples drawn from the OEIS they will be integers. We denote the Riordan array by $(f(x), F(x))$.

If $r = 1$, the Riordan array is called a proper Riordan array and is a lower unit triangular array. If $r \geq 2$, the Riordan array is a lower ttriangular array called a stretched Riordan array [1, Section 2].

Much of the interest in Riordan arrays comes from the fact that the set of Riordan arrays has an algebraic structure: the matrix product of two Riordan arrays is a Riordan array, the multiplication law being given by

$$(f(x), F(x))\,(g(x), G(x)) = (f(x)g(F(x)), G(F(x)))\,. \tag{3}$$

The proper Riordan array $(1, x)$ is the infinite identity matrix. Thus the set of Riordan arrays forms a semigroup with identity, i.e. a monoid, under matrix multiplication.

A proper Riordan array $(f(x), F(x))$ is a lower unit triangular array and is thus invertible. The inverse array is also a proper Riordan array given by

$$(f(x), F(x))^{-1} = \left(\frac{1}{f(\overline{F}(x))}, \overline{F}(x)\right), \tag{4}$$

where $\overline{F}$ denotes the compositional inverse of F, that is,

$$F(\overline{F}(x)) = \overline{F}(F(x)) = x.$$

The subset of proper Riordan arrays forms a group under matrix multiplication called the Riordan group.

One approach to establishing (3) is to use the Fundamental Theorem of Riordan Arrays (FTRA), which examines the action of a Riordan array on a column vector - see for example [3, Section 2].

FTRA. Let $(f(x), F(x))$ be a Riordan array. Then

$$\begin{aligned} (f(x), F(x)) \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix} &= \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{bmatrix} \\ \iff f(x)A(F(x)) &= B(x), \end{aligned}$$

where $A(x) = \sum_{n\geq=0} a_n x^n$ and $B(x) = \sum_{n\geq=0} b_n x^n$ are the generating functions of the sequences a_n and b_n, respectively.

The multiplication law (3) follows easily from the FTRA by considering the action of the Riordan array $(f(x), F(x))$ on the each of the column generating functions $g(x)G(x)^k$ of the array $(g(x), G(x))$ in turn, and then putting the results together to obtain the right-hand side of (3).

Remark. Notice that the previous sketch-proof of (3) only required $(g(x), G(x))$ to be an array whose k-th column generating function was of the form $g(x)G(x)^k$; it wasn't necessary for the power series $g(x)$ and $G(x)$ to satisfy the restrictions imposed by (1) and (2).

2. Generalized Riordan Arrays.

Definition 1. Let $f(x)$ and $F(x)$ be a pair of formal power series (with, say, integer coefficients). A ***generalized Riordan array*** is an infinite array whose k-th column is given by the coefficients in the expansion of the formal power series $f(x)F(x)^k$, $k = 0, 1, 2, ...$ We denote the generalized Riordan array by $(f(x), F(x))$.

Wilson studied the asymptotics of generalized Riordan arrays in [4].

Examples

(i) The symmetric square array of binomial coefficients $\left(\binom{n+k}{k}\right)_{n,k\geq 0}$, known as the Pascal matrix, is the generalized Riordan array $\left(\frac{1}{1-x}, \frac{1}{1-x}\right)$.

(ii) Pascal's triangle A007318 is the proper Riordan array $(\frac{1}{1-x}, \frac{x}{1-x})$ and has row generating polynomials $(1+x)^n, n \geq 0$. It follows that the transposed Pascal triangle is the generalized Riordan array $(1, 1+x)$.

(iii) The square array of Delannoy numbers, A008288, is the generalized Riordan array $(\frac{1}{1-x}, \frac{1+x}{1-x})$.

(iv) The square array of Whitney numbers, A004070, is the generalized Riordan array $(\frac{1}{1-x}, 1+x)$.

(v) The transpose of triangle A034870 is the generalized Riordan array $(1, 1+2x+x^2)$.

(vi) The transpose of triangle A260492 is the generalized Riordan array $(1, 1+x^2)$.

In general, matrix multiplication of generalized Riordan arrays may not be well-defined. However, it is possible to multiply a Riordan array and a generalized Riordan array and by the remark at the end of section 1 we have the following result:

Proposition 1. Let $(f(x), F(x))$ be a Riordan array and let $(g(x), G(x))$ be a generalized Riordan array. Then the matrix product of $(f(x), F(x))$ and $(g(x), G(x))$ is well-defined and the product is a generalized Riordan array given by

$$(f(x), F(x))\,(g(x), G(x)) \quad = \quad (f(x)g(F(x)), G(F(x)))\,. \tag{5}$$

Another result of this type involves a subset of the set of generalized Riordan arrays.

Definition 2. A ***generalized Riordan array of polynomial type*** is a generalized Riordan array $(p(x), P(x))$ where $p(x)$ and $P(x)$ are polynomials in x.

In a similar way that we used the FTRA to prove the multiplication law (3) of Riordan arrays we can prove the following result:

Proposition 2. Let $(f(x), F(x))$ be a generalized Riordan array and let $(p(x), P(x))$ be a generalized Riordan array of polynomial type. Then the matrix product of $(f(x), F(x))$ and $(p(x), P(x))$ is well-defined and the product is a generalized Riordan array given by

$$(f(x), F(x))\,(p(x), P(x)) \quad = \quad (f(x)p(F(x)), P(F(x)))\,. \tag{6}$$

In particular, if $f(x)$ and $F(x)$ are both polynomial functions in x then we see from (6) that the set of generalized Riordan arrays of polynomial type form a semigroup under the operation of matrix multiplication, with identity element $(1, x)$.

Examples

(i) The generalized Riordan array of polynomial type $(1, a-x)$ is an involution.

$$(1, a-x)\,(1, a-x) = (1, x)\,.$$

$$\begin{bmatrix} 1 & a & a^2 & a^3 & \cdots \\ 0 & -1 & -2a & -3a^2 & \cdots \\ 0 & 0 & 1 & 3a & \cdots \\ 0 & 0 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(ii) Let P denote Pascal's triangle

$$\mathrm{P} = \left(\frac{1}{1-x}, \frac{x}{1-x}\right).$$

The transposed Pascal triangle is a generalized Riordan array of polynomial type

$$\mathrm{P}^T = (1, 1+x).$$

Using (5) we find

$$\begin{aligned} \mathrm{PP}^T &= \left(\frac{1}{1-x}, \frac{x}{1-x}\right)(1, 1+x) \\ &= \left(\frac{1}{1-x}, \frac{1}{1-x}\right), \end{aligned}$$

the square symmetric Pascal matrix. This calculation gives rigorous mathematical meaning to the remarks in [2, p.10].

(iii) The square array of Delannoy numbers, A008288, is the generalized Riordan array $(1/(1-x),(1+x)/(1-x))$. Using (5) and (6) we can find the LDU factorization of the square array of Delannoy numbers.

$$\begin{aligned}\left(\frac{1}{1-x},\frac{1+x}{1-x}\right) &= \left(\frac{1}{1-x},\frac{x}{1-x}\right)(1,1+2x)\\ &= \left(\frac{1}{1-x},\frac{x}{1-x}\right)(1,2x)\,(1,1+x)\\ &= \mathrm{PDP}^T\end{aligned}$$

where $\mathrm{D}=\mathrm{diag}(1,2,2^2,2^3,...)$.

A similar calculation provides the LDU factorization of A081577 through A081580 (all viewed as square arrays).

(iv) A055248, read as a square array, is the generalized Riordan array $\left(\frac{1}{1-2x},\frac{1}{1-x}\right)$. Using (5) we find the matrix factorization

$$\begin{aligned}\left(\frac{1}{1-2x},\frac{1}{1-x}\right) &= \left(\frac{1}{1-x},\frac{x}{1-x}\right)\left(\frac{1}{1-x},x\right)(1,1+x)\\ &= \mathrm{PUP}^T,\end{aligned}$$

where U denotes the lower unit triangular array with 1's on or below the main diagonal.

(v) A260492 is defined as Pascal's triangle aerated with columns of zeros. Equivalently, the n-th row polynomial of A260492 is $(1+x^2)^n$. Let us write $p(x)=1+x^2$. Then it follows that

$$\mathrm{A260492}=(1,p(x))^T.$$

Hence by (6)

$$\mathrm{A260492}^2=(1,p(p(x)))^T=(1,2+2x^2+x^4)^T,$$

$$\mathrm{A260492}^3=(1,p(p(p(x))))^T=(1,5+8x^2+8x^4+4x^6+x^8)^T$$

and so on.

Setting $x=1$, we see that row sums of A260492 are equal to 2^n, the row sums of $\mathrm{A260492}^2$ are equal to 5^n, the row sums of $\mathrm{A260492}^3$ are equal to 26^n, and so on.

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