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On the Average Activity of a Spanning Tree of a Rooted Map

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ABSTRACT

Using Tutte's definition of internal activity, the author determines the average value of this quantity for a tree in a rooted map with n edges. The number of rooted maps with n edges and a distinguished spanning forest with two components is also determined.

1. INTRODUCTION

In his paper, "A Contribution to the Theory of Chromatic Polynomials" [4], Tutte makes the following definitions:

Let G be a finite connected graph with at least one edge. Enumerate the edges of G as $A_1, A_2, \ldots, A_m$. Consider any spanning tree T of G. Suppose A_j is an edge of T. Then $T'_{A/j}$ (the graph obtained by removing the edge A_j from T) has two components, C and D say. Each has one end of A_j as a vertex. We say A_j is *internally active* in T if each edge A_k of G other than A_j which has one end in C and one end in D satisfies $k < j$. Edges of G with one end in C and one end in D are said to be edges of the bond determined by T and A_j. Now suppose A_j is not an edge of T. Denote its ends by C and D. There is a unique simple path in T from a to b. We say A_j is externally active in T if each edge A_k in this path satisfies $k < j$. Edges in this path together with A_j are said to be edges of the polygon determined by T and A_j.

The *internal activity* of a *tree* in a map whose edges are ordered is the number of edges which are internally active in that tree under that ordering. Let us denote the activity of the spanning tree T under the ordering p by $A(T, p)$. Let $\mathscr{T}_M$ denote the set of spanning trees in the map M. It is an interesting property of activity that for any map M,

$$A(M) = \sum_{T \in \mathscr{T}_M} A(T, p) \tag{1.1}$$

is independent of the ordering p.

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A rooted map is a connected plane graph and its embedding in the plane, in which an edge is distinguished as root edge. This edge is oriented, and left and right sides are specified for it. The negative end of the root edge is referred to as the root vertex; the face on the left of the root edge is called the root face. If a map has no edges, it is the vertex map. In this case the vertex is root vertex, and the bounding face is root face, despite the lack of a root edge.

In this paper we determine the average value of $A(M)$ over the class of all rooted maps of n edges. In the process we determine the number of twin-tree-rooted maps with n edges, where a twin-tree-rooted map is a rooted map in which a spanning forest of two components is distinguished as root forest. The root forest may or may not contain the root edge of the map.

Let us further remark that, since the set of rooted maps with n edges is self-dual, our results on bonds and internal activity are also valid for the dual concepts of polygons and external activity.

2. Labeling of Rooted Maps

Let us denote the class of all rooted plane maps with n edges by $M(n)$. Let us fix an ordering $A_1, A_2, \ldots, A_n$ of the edges, arbitrarily in each map. Each map, and therefore each tree-rooted map, now appears as a labeled map. (A tree-rooted map is a rooted map in which a spanning tree is distinguished as root tree. The root tree may or may not include the root edge.) Each permutation of the index set $\{A_1, A_2, \ldots, A_n\}$ may be viewed as a relabeling of each map.

3. Average Internal Activity

We define the average internal activity $\bar{T}(n)$ of a spanning tree in a rooted map with n edges by

$$\bar{T}(n) \sum_{M \in M(n)} |\mathscr{T}_M| = \sum_{M \in M(n)} A(M), \tag{3.1}$$

where $|\mathscr{T}_M|$ is the cardinal number of $\mathscr{T}_M$. Let us denote the sum on the left of (3.1) by $F(n)$, that is,

$$F(n) = \sum_{M \in M(n)} |\mathscr{T}_M|.$$

Then it is shown in (3) that

$$F(n) = \frac{(2n)!\,(2n+2)!}{n!\,[(n+1)!]^2 (n+2)!}. \tag{3.2}$$

Since $A(M)$ is independent of the particular ordering on M, we may assume we are dealing with the orderings assigned to the maps in Section 2. Let p be the ordering of the maps in $T(n)$, the class of tree-rooted maps on n edges, induced by the labeling in Section 2. Let $U(n)$ be the set of distinguished (root) trees in $T(n)$. Then

$$\sum_{M \in M(n)} A(M) = \sum_{T \in U(n)} A(T, p). \tag{3.3}$$

Let B be an edge of the tree T. Then let $\lambda(T, p, B) = 1$ or 0 according as B is or is not internally active in the tree T under the ordering p. Then

$$\sum_{T \in U(n)} A(T, p) = \sum_{T \in U(n)} \sum_{B \in T} \lambda(T, p, B), \tag{3.4}$$

that is,

$$\bar{T}(n)\, F(n) = \sum_{T \in U(n)} \sum_{B \in T} \lambda(T, p, B). \tag{3.5}$$

Let us denote the set of permutations of $(A_1, A_2, \ldots, A_n)$ by P. Then

$$n!\, \bar{T}(n)\, F(n) = \sum_{p \in P} \sum_{T \in U(n)} \sum_{B \in T} \lambda(T, p, B) \tag{3.6}$$

Let us define a doubly rooted tree map as a tree-rooted map in which an edge of the root tree is distinguished as the secondary root edge. (If the distinguished tree of a tree-rooted map is a vertex tree, no corresponding doubly rooted tree map will exist). It may happen that the original root edge is also the secondary root edge.

Let $B(n)$ be the set of secondary root edges in the class of all doubly rooted tree maps with n edges. Then

$$n!\, \bar{T}(n)\, F(n) = \sum_{B \in B(n)} \sum_{p \in P_B} \lambda(T, p, B), \tag{3.7}$$

where P_B is the set of all permutations of the ordering of the edges of the doubly rooted tree maps whose secondary root edge is B and T is the root tree of the map containing B.

Let $B(n, k)$ be the set of secondary root edges of the class of doubly rooted tree maps in which the bond β (determined by some T and B) contains $k+1$ edges. Then

$$\sum_{B\in B(n)} \sum_{p\in P_B} \lambda(T, p, B) = \sum_{k=0}^{n-1} \sum_{B\in B(n,k)} \sum_{p\in P_B} \lambda(T, p, B) \tag{3.8}$$

Suppose $B \in B(n, k)$. Then

$$\sum_{p\in P_B} \lambda(T, p, B) = \frac{n!}{k+1} \tag{3.9}$$

Indeed, let us consider $P(n_1, n_2, ..., n_{k+1})$, the class of orderings in which the edges of the bond β have the labels $A_{n_1}, A_{n_2}, ..., A_{n_{k+1}}$. In $k!$ of these orderings, B is labeled with the largest element. Thus

$$\sum_{p\in P(n_1,n_2,...,n_{k+1})} \lambda(T, p, B) = k!$$

But there are $\binom{n}{k+1}$ n-k-1! such classes of orderings; this establishes (3.9). Thus

$$\bar{T}(n)\, F(n) = \sum_{k=0}^{n-1} \sum_{B\in B(n,k)} \frac{1}{k+1}, \tag{3.10}$$

that is,

$$\bar{T}(n)\, F(n) = \sum_{k=0}^{n-1} \frac{1}{k+1} \mid B(n, k) \mid, \tag{3.11}$$

where $\mid B(n, k) \mid$ is the cardinal number of $B(n, k)$.

For the convenience of the reader, Figures 5, 6, 7, and 8 exhibit the maps of $M(n)$, $T(n)$, $B(n)$, and $H(n, k)$ (yet to be defined) for $n = 1$ and 2, and $k = 1, 2$.

4. Tree-Rooted Maps and Hamiltonian Trivalent Maps

Let us note that any tree-rooted map M can be made correspond to a trivalent map M^* with a distinguished Hamiltonian polygon, one of whose residual domains is distinguished as root domain. This correspondence can be produced as follows:

A simple closed curve C is drawn around the distinguished tree T of M in such a fashion that it meets no edge or vertex of T and meets every other edge exactly twice. Through every edge D of T an arc X_D is drawn such that D and X_D have exactly one point in common, X_D has its ends in C, and X_D meets no other vertex or edge of M.

Let us denote by K the residual domain of C which does not contain T.

All edges of M which are not edges of T will be called the co-tree edges of M relative to T, or merely the co-tree or distinguished co-tree of M

Every edge D of the co-tree $\bar{T}$ of T contains an arc Y_D which crosses K. Then

$$H = \bigcup_{D\in T} X_D \bigcup_{D\in \bar{T}} Y_D$$

defines a trivalent map M^* in which C is a Hamiltonian polygon and K is root domain.

Let K_1 be the residual domain of C which contains T. Every face of K_1 in M^* corresponds in an obvious manner to a vertex of M; every edge crossing K_1 in M^* corresponds to an edge T in M.

Let us assume that M is a rooted map. We can induce a rooting on M^* as follows:

If the root edge E of M is a member of the co-tree $\bar{T}$ of M, we use the residual arc Y_E in M^* with the orientation and left and right sides induced by the orientation and left and right sides of E. If E is the root tree T of M, the corresponding edge X_E of M is oriented such that X_E crosses E from left to right and is such that the face corresponding to the positive end of E is on the left of X_E. The oriented edge X_E or Y_E is then taken as the root edge of M^*. This rooted map with its distinguished domain will be called a Hamiltonian-rooted trivalent map for purposes of this paper.

Let us note that any other edge X_D or Y_D of M^* may be oriented in terms of the orientation on the root edge of M^* by the following rule:

Let x and y be, respectively, the negative and positive ends of the root edge E of M^*. The vertices x and y divide the curve C into two arcs R and S, the arc R being chosen such that the closure of $R \cup E$ bounds the region on the left of E. Suppose that F is any other edge of

$$F = \bigcup_{D\in T} X_D \bigcup_{D\in \bar{T}} Y_D,$$

then if F has one end in R and one end in S, we direct F from S to R. If, however, both ends of F lie on the same residual arc, Z, where Z is either R or S, then the ends of F also divide C into two arcs, one of which, W, contains x and y. Then the closure of $(Z \cap W) \cup E \cup F$ is a polygon through E and F. We orient every edge of this consistently with the orientation on E. This provides an orientation for F. Left and right sides are assigned to F consistently with the assignment of left and right sides to E. The assignment of the secondary orientation above has the following property: Suppose any edge F with the orientation induced by E is taken as root edge of map M^*. Then the secondary orientation induced on E

relative to the root F is precisely that as was on E originally. This observation is useful in the following section.

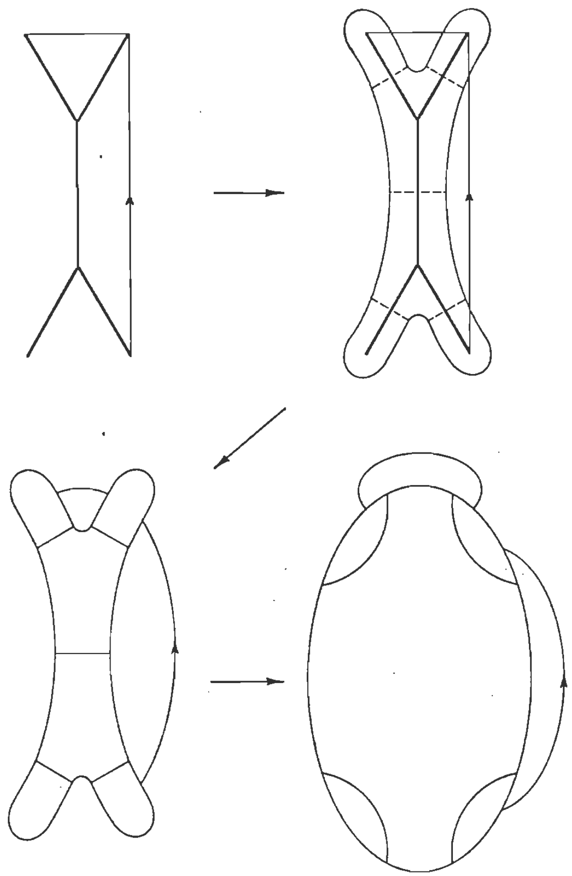

FIGURE 1

5. Doubly Rooted Hamiltonian Trivalent Maps and Doubly Rooted Tree Maps

In view of the foregoing analysis we note that the number of doubly rooted tree maps with n edges is n times the number of tree-rooted maps in which the root edge is contained in the spanning tree. Indeed, let M be a doubly rooted tree map. Under the construction of Section 4, M becomes a rooted Hamiltonian-rooted trivalent map M^* in which one

of the edges $X_{D'}$ corresponding to the secondary root edge D' in T of M, is distinguished as secondary root edge of M^*. Such a map will be called a doubly rooted Hamiltonian-rooted trivalent map. If $X_{D'}$ is already the primary root of M^* it is already oriented. Otherwise $X_{D'}$ can be secondarily oriented with respect to the orientation on the root edge of M^*. The edge $X_{D'}$ can then be taken as root edge for M^*. Each such map M^* can be restored to n doubly rooted Hamiltonian maps by choosing any edge E of M^* as primary root for M^* and using the orientation on E induced by the orientation on $X_{D'}$ to give the rooting orientation on the primary root. This established the fact that $G(n) = nH(n)$, where $G(n)$ is the number of doubly rooted tree maps with n edges and $H(n)$ is the number of tree-rooted maps in which the root edge is contained in the root tree.

6. The Number of Doubly Rooted Tree Maps

To compute $\bar{T}(n)$, the average internal activity of a tree in a map with edges, we need determine $|B(n, k)|$, or equivalently the number of doubly rooted tree maps with secondary root B in which the bond determined by B and T, the root tree, contains $k + 1$ edges. By the construction of Section 5, this is n times the number of tree-rooted maps on n edges in which the root edge B belongs to the root tree T and the bond determined by B and T has $k + 1$ edges. Let us denote the class of such maps as $E(n, k)$. $|E(n, k)|$ is most readily determined in terms of Hamiltonian-rooted trivalent maps. Let us denote the class of Hamiltonian-rooted maps corresponding to $E(n, k)$ by $D(n, k)$. Suppose M^* belongs to $D(n, k)$. Let B be the root edge of M^*. Since the root edge of M belongs to the root tree of T, B does not cross the distinguished domain K of M^*. Let x and y be, respectively, the negative and positive ends of B. As before, x and y divide the distinguished polygon C into two open arcs, R and S, R being such that the region bounded by the closure of $R \cup B$ lies on the left of B. Since B crosses K_1, the non-root domain of M^*, no edge crossing D_1 has one end in R and the other in S. There are precisely k edges crossing K with one end in R and one end in S; these edges of M^* correspond to the non-root edges of the bond determined by B and T in M. These edges of M^* form the set H. An arc F can be drawn across K from x to y such that F meets every edge of H in exactly one point, and meets no other edge of M^* except at x and y (see Fig. 2).

The configuration in Figure 2 may be described as follows: Let C be a simple closed curve on a 2-sphere and let K be one of its residual domains. Let $P_1, P_2, \ldots, P_{2n}$ be $2n$ distinct points chosen on C. By a cross-connection of these points in K we understand a set of n non-intersecting open arcs in K joining the $2n$ points P_i in pairs. Further, let x and y be two points of C distinct from the points P_i and each other. Points x and y separate C into two distinct arcs R and S. Let us suppose that precisely k of the

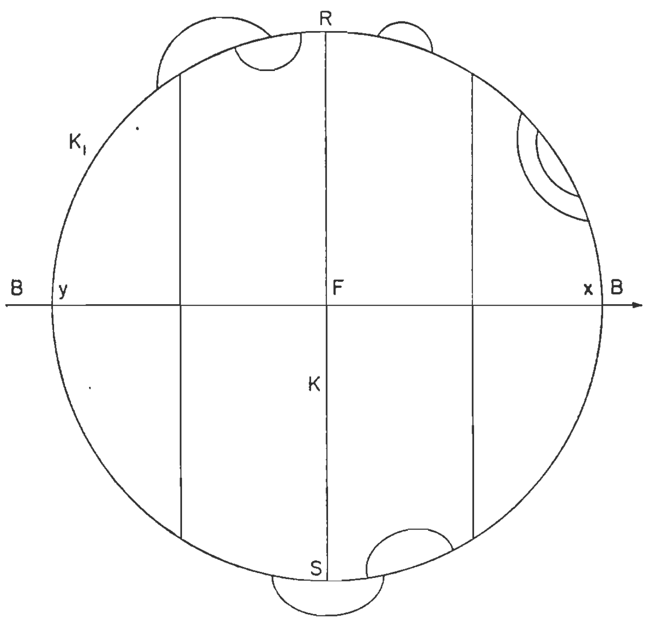

FIGURE 2

cross-connecting arcs of K have one end in R and one end in S. Let us denote $n - k$ by l and refer to the above configuration as a (k, l)-dissection. The arc F joining x and y across K is added merely for convenience in the following enumerative argument and is not an essential part of the

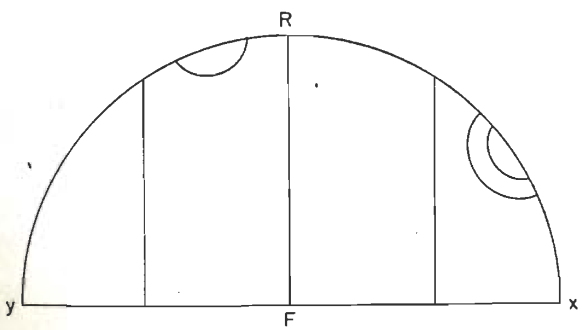

FIGURE 3

configuration. The (k, l) dissection of C induces a cross-connection of K', the interior of $R \cup F$, which is shown in Figure 3. It is characterized by the fact that its boundary contains a distinguished arc (the arc F) in which there are exactly k end-points of cross-connecting arcs, and no two of these end-points are joined across K'. We will call such a figure a (k, l) *configuration*. Clearly the total number of cross-connecting edges in a (k, l) configuration is $k + l$. Figure 4 illustrates the fact that each

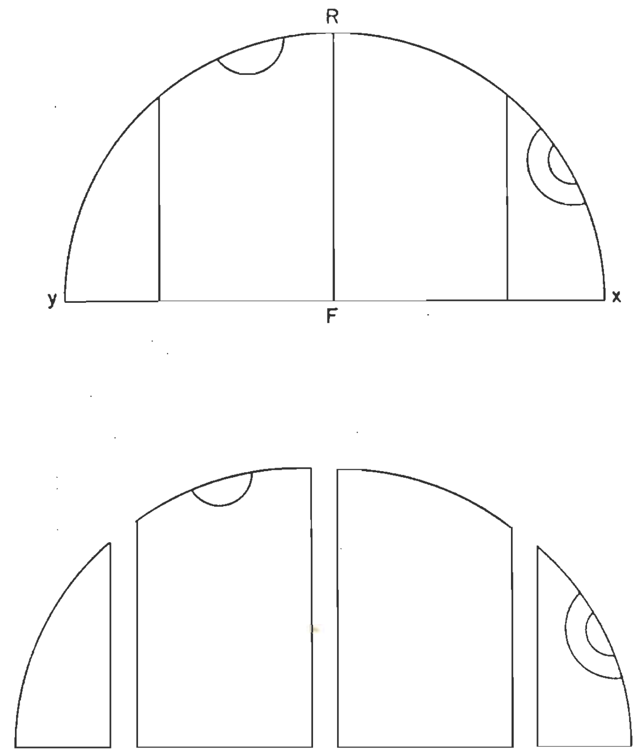

FIGURE 4

(k, l) configuration can be considered as an ordered set of $(0, l_i)$-configuration $i = 1, 2, \dots, k + l$, such that $l_1 + l_2 + \cdots + l_{k+1} = l$. It was shown by Tutte [6, p. 412] that there are $2l!/l!(l + 1)!$ topologically distinct $(0, l)$ configurations. Hence it is shown in [2, p. 142] that if $A_{l,k}$ is the number of topologically distinct (k, l) configurations,

$$A_{l,k} = \frac{(k + 1)(2l + k)!}{l![(l + k) - 1]!}.$$

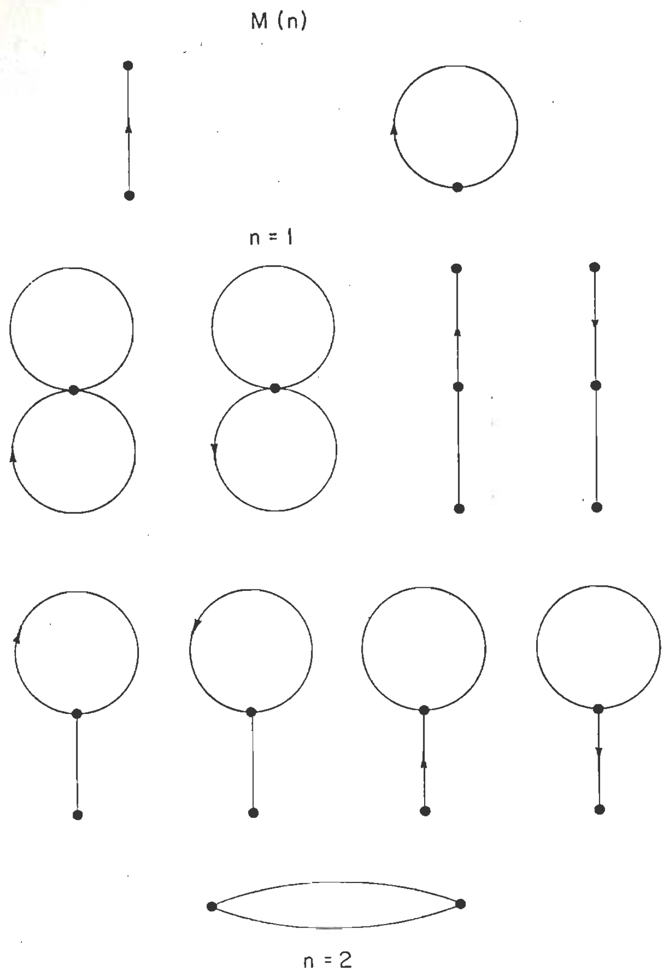

FIGURE 5

Returning to M^*, we see that the closure of $B \cup F$ is a simple closed curve which separates the map M^* into two residual domains, P and Q, P being that which contains the arc R. R divides P into two domains, one of which, bounded by the closure $R \cup F$, is a (k, s) configuration and the other, bounded by the closure of $R \cup B$, is an $(0, t)$ configuration for some values of s and t. The arc B does not contain any end-point of any edge which crosses P, and F is the distinguished arc of the (k, s) configuration above. Such a configuration will be called a k-half-map. Clearly any map of $D(n, k)$ corresponds to a pair of k-half-

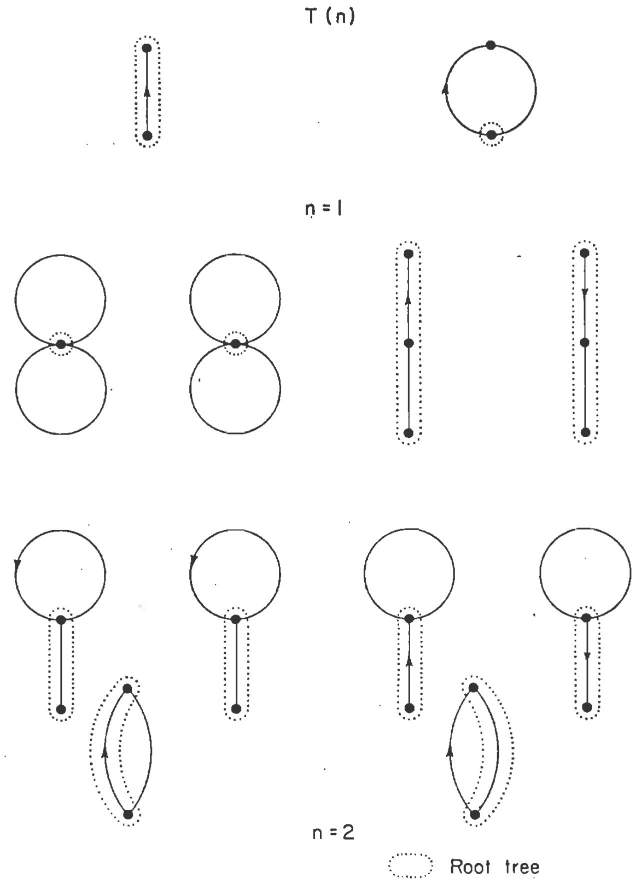

FIGURE 6

maps. Let H be any k-half-map in which there are precisely $k + u$ edges which have at least one end in the open arc R. Let us enumerate such configurations by the following method: Let G be a simple closed curve in the closed plane and let J be one of its residual domains. Let x and y be two distinct points of G and let R be a simple arc crossing J with ends x and y and directed from x to y. Distinguish the left and right sides of R. Let the residual arcs of the points x and y in G be F and B, F being such that the region bounded by the closure of $F \cup R$ lies to the right of R. Then $2u + k$ points are distinguished in the interior of R, and k points are distinguished in the interior of F. Then $2t$ points are chosen on R and joined across the region of J on the left of R to form a $(0, t)$ configuration. The remaining points are joined across the region on the right of R to yield a (k, s) configuration where $s = u - t$. Thus there are

$$\sum_{t=0}^{u}\binom{2u+k}{2t}\frac{(2t)!}{t!\,(t+1)!}\,\frac{(2u-2t+k)!\,(k+1)}{(u-t)!\,(u-t+k+1)!} \tag{6.1}$$

such k-half maps. This may be written as

$$\frac{(k+1)(2u+k)!}{u!\,(u+k+2)!}\sum_{t=0}^{u}\frac{u!}{t!\,(u-t)!}\,\frac{(u+k+2)!}{(t+1)!\,(u-t+k+1)!}$$

$$=\frac{(k+1)(2u+k)!}{u!\,(u+k+2)!}\sum_{t=0}^{u}\binom{u}{t}\binom{u+k+2}{u+k+1-t} \tag{6.2}$$

But $\binom{u}{t}$ is the coefficient of x^t in the expansion of $(1+x)^u$ and $\binom{u+k+2}{u+k+1-t}$ is the coefficient of $x^{u+k+1-t}$ in the expansion of $(1+x)^{u+k+2}$. Thus

$$\sum_{t=0}^{u}\binom{u}{t}\binom{u+k+2}{u+k+1-t}=\frac{(2u+k+2)!}{(u+k+1)!\,(u+1)!}\,; \tag{6.3}$$

the coefficient of x^{u+k+1} in the expansion of $(1+x)^u\,(1+x)^{u+k+2}$. Thus the formula of (6.1) may be written as

$$\frac{(k+1)(2u+k)!\,(2u+k+2)!}{u!\,(u+k+2)!\,(u+k+1)!\,(u+1)!} \tag{6.4}$$

which we denote by $b_{k,u}$. By assembling pairs of k-half maps with $2u+k$ and $2v+k$ points on the directed arcs corresponding to R in the preceding construction, we obtain trivalent rooted maps in which $u+v+k$ non-root edges cross the residual domains of the distinguished polygons. These correspond to a tree-rooted map with $u+v+k+1$ edges, counting the root edge. Thus

$$|\,E(n,k)\,| = \sum_{u=0}^{n-k-1} b_{k,u}\,b_{k,n-k-1-u}\,. \tag{6.5}$$

But $|\,B(n,k)\,| = n\,|\,E(n,k)\,|$. Thus by (3.11),

$$F(n)\,\bar{T}(n) = \sum_{k=0}^{n-1}\sum_{u=0}^{n-1-k}\frac{(k+1)(2u+k)!\,(2u+k+2)!}{u!\,(u+k+2)!\,(u+k+1)!\,(u+1)!}$$

$$\times\frac{(2n-2u-k-2)!\,(2n-2u-k)!}{(n-k-1-u)!\,(n+1-u)!\,(n-u)!\,(n-k-u)!}$$

which is an explicit formula to $\bar{T}(n)$.

7. Conjugate Members of $B(n, k)$

A rooted map M is said to be twin-tree-rooted if a spanning forest with two connected components is distinguished as root forest of M. Let M be a twin-tree-rooted map with n edges whose spanning forest has components T_1 and T_2. Let us assume that there are $k+1$ edges of M which have one end in T_1 and one end in T_2. This set of edges is called the bond determined by T_1 and T_2, or the root bond of M.

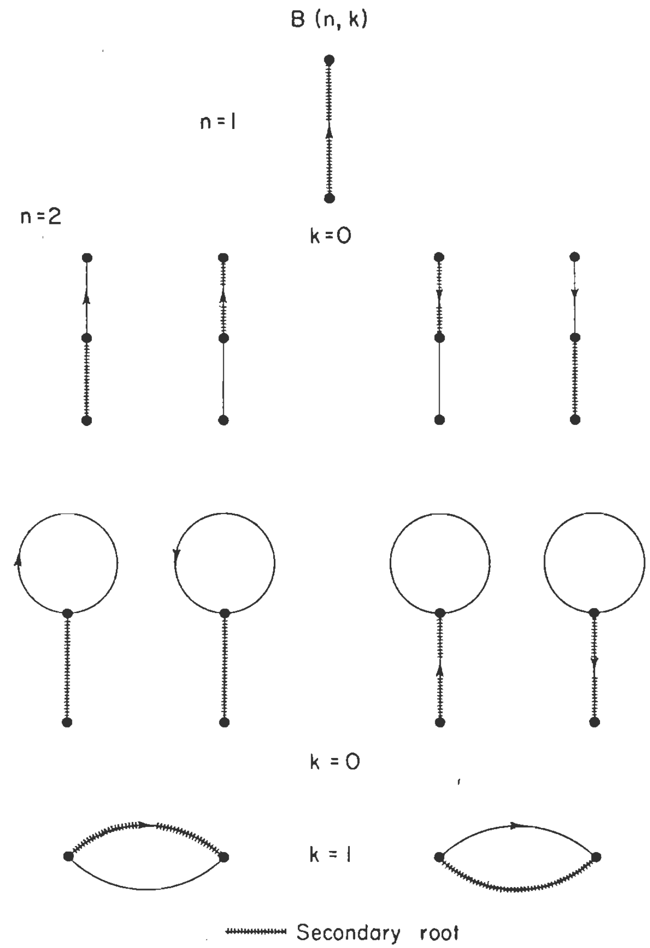

Figure 7

Let E be an edge of K, the bond determined by T_1 and T_2. If E is distinguished and adjoined to T_1 and T_2, the result is a double-rooted

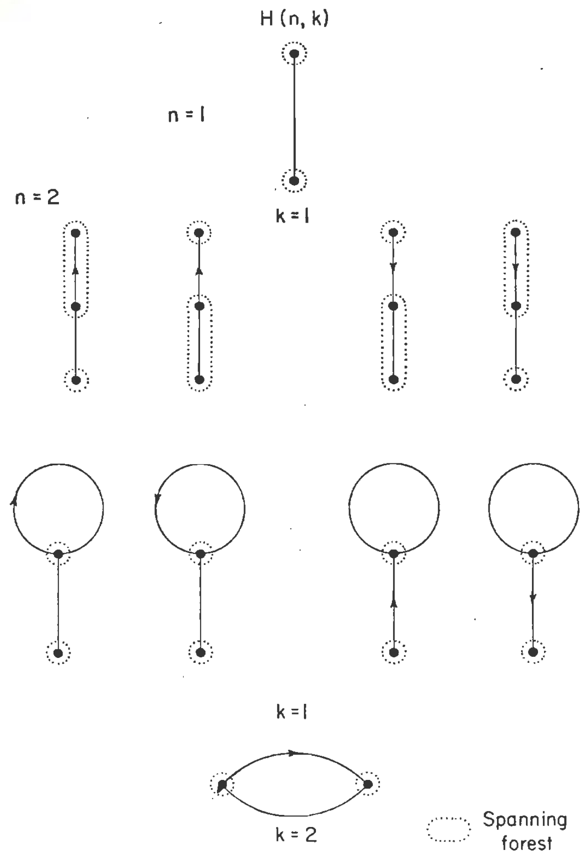

FIGURE 8

tree map in which E is the secondary root edge. Thus each such twin-tree-rooted map generates $k + 1$ double-rooted tree maps, and hence $k + 1$ members of $B(n, k)$. These members of $B(n, k)$ are called conjugate edges, and $B(n, k)$ is thus partitioned into equivalence classes of conjugate edges, each of which contains $k + 1$ edges. The number of such classes is $| H(n, k + 1) |$, where $H(n, k + 1)$ is the set of twin-tree-rooted maps with n edges, whose root bond contains $k + 1$ edges. Thus

$$| B(n, k) | = (k + 1) \, | H(n, k + 1) |, \tag{7.1}$$

and, by (3.11),

$$F(n) \, \bar{T}(n) = \sum_{k=0}^{n-1} H(n, k + 1) = \sum_{k=1}^{n} H(n, k). \tag{7.2}$$

But the right side of (7.2) is the number of twin-tree-rooted maps with n edges. Let h_n be the number of such twin-tree-rooted maps. That is,

$$\bar{T}(n) = \frac{n! \, [(n + 1)!]^2 (n + 2)!}{(2n + 2)! \, (2n)!} \, h_n . \tag{7.3}$$

8. Asymptotic Estimates

We may obtain asymptotic estimates of $\bar{T}(n)$ and h_n from the following considerations: By (3.11)

$$\bar{T}(n) \, F(n) = \sum_{k=0}^{n-1} \frac{1}{k + 1} \, | B(n, k) | \leqslant \sum_{k=0}^{n-1} | B(n, k) | = | B(n) |. \tag{8.1}$$

But $| B(n) |$ may be determined as follows. Recall that $T(n)$ denotes the set of tree-rooted maps on n edges. For $M \in T(n)$, let $e(M)$ be the number of edges in the root tree of M. Every M in $T(n)$ can produce $e(M)$ members of $B(n)$. Thus

$$| B(n) | = \sum_{M \in T(n)} e(M). \tag{8.2}$$

Let us pair each map M with its dual $\bar{M}$. Then, since $e(M) + e(\bar{M}) = n$,

$$2 \, | B(n) | = \sum_{M \in T(n)} [e(M) + e(\bar{M})] = n \sum_{M \in T(n)} 1 = nF(n). \tag{8.3}$$

That is,

$$| B(n) | = \frac{n}{2} F(n), \tag{8.4}$$

and by (8.1)

$$\bar{T}(n) \leqslant \frac{n}{2}. \tag{8.5}$$

On the other hand, consider the terms of (6.6) corresponding to $k = 0$, $u = 0$ and $k = 0$, $u = n - 1$. These terms are equal, let us denote them by $C(n)$. Then for $n \geqslant 2$

$$\bar{T}(n) \geqslant 2 \frac{C(n)}{F(n)} = \frac{(n + 2)(n + 1)(n)}{2(n + 1)(2n - 1)} > \frac{n}{8}. \tag{8.6}$$

We now show that $T(n)/n$ approaches a limit as n approaches infinity. To this end, we define

$$A(n, k, u) = \frac{b_{k,u} \, , \, b_{k,n-k-1-u}}{(k + 1) \, F(n)}, \tag{8.7}$$

where $b_{k,u}$ is as defined in Section 6. Then define $S(n)$ by

$$S(n) = T(n)/n = \sum_{k=0}^{n-1} \sum_{u=0}^{n-k-1} A(n, k, u). \tag{8.8}$$

Since

$$A(n, k, u) = A(n, k, n - k - 1 - u),$$

we may write

$$S(n) = 2 \sum_{k=0}^{n-1} \sum_{u=0}^{[\frac{n-k-1}{2}]} A(n, k, u) - \sum_{k=0}^{n-1} \theta(n, k), \tag{8.9}$$

where $[x]$ indicates the greatest integer contained in x, and, $\theta(n, k) = 0$ if $n \not\equiv k \bmod 2$ and

$$\theta(n, k) = A\left(n, k, \frac{n-k-1}{2}\right),$$

otherwise. Let us note that we may also write

$$S_n = 2 \sum_{k=0}^{n-2} \sum_{u=0}^{[\frac{n-k-2}{2}]} A(n, k, u) + \sum_{k=0}^{n-1} \theta(n, k). \tag{8.10}$$

Let us define $Q(n)$ and $R(n)$ by

$$Q(n) = 2 \sum_{k=0}^{n-1} \sum_{u=0}^{[\frac{n-k-1}{2}]} A(n, k, u), \tag{8.11}$$

$$R(n) = 2 \sum_{k=0}^{n-2} \sum_{u=0}^{[\frac{n-k-2}{2}]} A(n, k, u).$$

Now for $0 \leqslant k \leqslant n - 1$ and

$$0 \leqslant u \leqslant \left(\frac{n-k-1}{2}\right),$$

$$\frac{A(n+1, k, u)}{A(n, k, u)} = \frac{1}{4} \frac{n+3}{2n+3} \frac{2(n-u)+1-k}{n-u+2} \cdot \frac{n+2}{2n+1} \frac{2(n-u)-1-k}{n-u+1}$$
$$\times \frac{2(n-u)-k+2}{n-u+k+1} \cdot \frac{2(n-u)-k}{n-u+k}. \tag{8.12}$$

But

$$\frac{n+3}{2n+3} \cdot \frac{2(n-u)+1-k}{n-u+2} \leqslant \frac{n+3}{2n+3} \cdot \frac{2(n-u)+1}{(n-u)+2}$$
$$= \frac{2n(n-u)+7n-6u+3}{2n(n-u)+7n-3u+6} < 1. \tag{8.13}$$

Similarly,

$$\frac{n+2}{2n+1} \cdot \frac{2(n-u)-1-k}{n-u-k} < 1.$$

Also

$$\frac{2(n-u)-k+2}{n-u+k+1} \leqslant 2,$$

$$\frac{2(n-u)-k}{n-u+k} \leqslant 2.$$

Thus $A(n+1, k, u) < A(n, k, u)$, and hence

$$R(n+1) < Q(n). \tag{8.14}$$

But

$$S(n+1) = R(n+1) + \sum_{k=0}^{n} \theta(n+1, k)$$
$$< Q_n + \sum_{k=0}^{n} \theta(n+1, k)$$
$$= S_n + \sum_{k=0}^{n-1} \theta(n, k) + \sum_{k=0}^{n} \theta(n+1, k). \tag{8.15}$$

Arguments similar to those of statements (8.7) to (8.14) show that

$$\theta(n, k) \leqslant \theta(n, 0) \quad \text{if } n \equiv 1 \bmod 2,$$
$$\theta(n, k) \leqslant \theta(n, 1) \quad \text{if } n \equiv 0 \bmod 2.$$

But by Stirling's formula,

$$\theta(n, 0) = 0(n^{-3}) \quad n \equiv 1 \bmod 2,$$
$$\theta(n, 1) = 0(n^{-3}) \quad n \equiv 0 \bmod 2,$$

Hence

$$\sum_{k=0}^{n} \theta(n+1, k) + \sum_{k=0}^{n} \theta(n, k) = 0(n^{-2}),$$

$$S_{n+1} \leqslant S_n + 0(n^{-2}). \qquad (8.16)$$

This motivates the following lemma, which is, no doubt, well known. Since the author can supply no reference, a proof is included.

CONVERGENCE LEMMA. *Let $S = (S_1, S_2, \ldots)$ be a sequence of real numbers such that*

$$S_{n+1} \leqslant S_n + A_n,$$

where A_n is a sequence of real numbers such that $\sum_{n=1}^{\infty} | A_n |$ is convergent. If $\{S_n\}$ is bounded below, then $\{S_n\}$ is convergent.

PROOF: Given $\epsilon > 0$, there exists n_0 such that

$$\sum_{n=n_0}^{\infty} | A_n | < \frac{\epsilon}{2}.$$

if $g = \text{g.l.b.} \bigcup_{n=n_0}^{\infty} \{S_n\}$, g is finite since S is bounded below, and there exists $n_1 \geqslant n_0$ such that $0 \leqslant S_{n_1} - g < \epsilon/2$. Now if $n > n_1$

$$g \leqslant S_n \leqslant S_{n-1} + A_{n-1} \leqslant S_{n-2} + A_{n-2} + A_{n-1}$$

$$\leqslant S_{n_1} + \sum_{k=n_1}^{n-1} | A_n | \leqslant g + \frac{\epsilon}{2} + \frac{\epsilon}{2} = g + \epsilon.$$

Thus if $m > n_1$, $n > n_1$ $| S_m - S_n | < \epsilon$, and $\{S_n\}$ is convergent.

Thus by (8.16) and (8.6), $\bar{T}(n)/n$ is convergent. Let us denote the limit of this sequence by α. Thus $\bar{T}(n) \sim \alpha n$, and since $h_n = \bar{T}(n) F(n)$, applying Stirling's formula to $F(n)$,

$$h_n \sim \frac{\alpha 2^{4n+2}}{\pi n^2}.$$

Tables I and II exhibit the numbers of maps in many of the aforementioned classes for small values of n.

TABLE I

$\|B(n,k)\|$					$\|H(n,k)\|$				
n/k	0	1	2	3	n/k	1	2	3	4
1	1				1	1			
2	8	2			2	8	1		
3	72	30	3		3	72	15	1	
4	720	380	72	4	4	720	190	24	1

TABLE II

n	$\|M(n)\|$	$\|T(n)\|$	$\|B(n)\|$	h_n	$\bar{T}(n)$
0	1	1	0	0	0
1	2	2	1	1	.5
2	9	10	10	9	.9
3	54	70	105	88	1.26
4	378	588	1196	935	1.59

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REFERENCES

1. R. C. MULLIN, On the Average Number of Trees in Certain Maps, *Can. J. Math.* **18** (1966), 33–41.
2. R. C. MULLIN, The Enumeration of Hamiltonian Polygons in Triangular Maps, *Pacific J. Math.* **16** (1966), 139–145.
3. R. C. MULLIN, On Counting Tree-Rooted Maps, *Can. J. Math.* **19** (1967), 174–183.
4. W. T. TUTTE, A Contribution to the Theory of Chromatic Polynomials, *Can. J. Math.* **6** (1953), 80–91.
5. W. T. TUTTE, A Census of Planar Maps, *Can. J. Math.* **15** (1963), 249–271.
6. W. T. TUTTE, A Census of Hamiltonian Polygons, *Can. J. Math.* **14** (1962), 402–417.

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